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On Ore-Stirling numbers defined by normal ordering in the Ore algebra

Toufik Mansour[a], Matthias Schork[b]

[a]Department of Mathematics, University of Haifa, 3498838 Haifa, Israel
[b]Institute for Mathematics, Würzburg University, Emil-Fischer Str. 40, 97074 Würzburg, Germany

Abstract. Normal ordering in the Weyl algebra is related to the Stirling numbers of the second kind, while normal ordering in the shift algebra is related to the unsigned Stirling numbers of the first kind. The Ore algebra – this name was introduced recently by Patrias and Pylyavskyy – is an algebra closely related to the Weyl algebra and the shift algebra. We consider a two-parameter family of generalized Ore algebras which comprises all algebras mentioned by specializing the parameters suitably. Analogs of the Stirling numbers – called Ore-Stirling numbers – are introduced as normal ordering coefficients in the generalized Ore algebra. In the limit where one parameter vanishes they reduce to the Stirling numbers of the second kind or the unsigned Stirling numbers of the first kind. Choosing the parameters appropriately, a one-parameter family of Ore-Stirling numbers interpolating between Stirling numbers of the second kind and unsigned Stirling numbers of the first kind is found. Several properties of the Ore-Stirling numbers as well as the associated Ore-Bell numbers are discussed.

1. Introduction

The *Weyl algebra* $\mathcal{W}$ is the (complex) unital algebra generated by letters D and U satisfying the commutation relation $DU - UD = I$, where I denotes the identity. A word ω in the letters D, U can always be brought into normal ordered form where all letters D stand to the right of all the letters U. Upon normal ordering, certain combinatorial coefficients appear, the normal ordering coefficients. The most famous example due to Scherk from 1823 [31] (see the discussion in [1, 25]) concerns $\omega = (UD)^n$ where one has

$$(UD)^n = \sum_{k=0}^{n} S(n,k) U^k D^k, \tag{1}$$

where $S(n,k)$ denotes the Stirling numbers of the second kind (A008277 in [37]). The numbers $S(n,k)$ count the number of set partitions of a set of n elements into k nonempty disjoint subsets and are among the most important combinatorial numbers, see, e.g., [12, 24, 25]. In fact, Scherk considered the representation in terms of the multiplication and differentiation operator on smooth functions of a real variable, $U \mapsto X, D \mapsto \hat{D}$, where $(Xf)(x) = xf(x)$ and $(\hat{D}f)(x) = \frac{df}{dx}(x)$. Due to this connection to operational calculus, normal

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Communicated by Paola Bonacini
Email addresses: tmansour@univ.haifa.ac.il (Toufik Mansour), matthias.schork@mathematik.uni-wuerzburg.de (Matthias Schork)

ordering in the Weyl algebra (and certain variants of it) has been studied closely, see, e.g., [1, 25, 33] and the many references therein. A related algebra is the *shift algebra* $\mathcal{S}$ which is the (complex) unital algebra generated by letters D and U satisfying $DU - UD = D$. This algebra has also been considered from the perspective of normal ordering, see, e.g., [25, 33] and the references therein. The analog of (1) is that in $\mathcal{S}$ one has

$$(UD)^n = \sum_{k=0}^{n} |s(n,k)| U^k D^n, \tag{2}$$

where $|s(n,k)|$ denotes the unsigned Stirling numbers of the first kind (see A008275) in [37]). The numbers $|s(n,k)|$ count the number of permutations of n elements with exactly k cycles, see [12].

Recently, Patrias and Pylyavskyy [30] considered the *Ore algebra* $\mathcal{O}$ as the (complex) unital algebra generated by letters D and U satisfying the commutation relation $DU - UD = D + I$. Thus, in a certain sense, it combines the Weyl algebra and the shift algebra. It is natural to introduce two parameters $\lambda, \mu \in \mathbb{C}$ and consider the family $\mathcal{A}_{\lambda,\mu}$ of *generalized Ore algebras* generated by D and U satisfying

$$DU - UD = \lambda D + \mu I. \tag{3}$$

By choosing the parameters appropriately, one recovers $\mathcal{W}, \mathcal{S}$ and $\mathcal{O}$ from above. Now, it is natural to define in $\mathcal{A}_{\lambda,\mu}$ analogs of the Stirling numbers – called Ore-Stirling numbers – as normal ordering coefficients of $(UD)^n$ in $\mathcal{A}_{\lambda,\mu}$ (where D, U satisfy (3)) and consider their properties and their connection to $S(n,k)$ and $|s(n,k)|$. This is what we start in the present paper. Let us point out that Levandovskyy et al. [22] considered the following four-parameter family of algebras given by letters x, y (their notation) and commutation relation

$$yx - qxy = \alpha x + \beta y + \gamma. \tag{4}$$

The algebra satisfying (4) was said to be *of type* $(q, \alpha, \beta, \gamma)$. They showed that there exist exactly five isomorphism classes. The representatives of theses classes were called *model algebras*. The algebras $\mathcal{W}$ and $\mathcal{S}$ are model algebras while $\mathcal{A}_{\lambda,\mu}$ – which is of type $(1, 0, \lambda, \mu)$ – is not a model algebra, but belongs to the isomorphism class of $\mathcal{S}$. In Table 1 of [22], one can find the corresponding isomorphism and some basic normal ordering results, which we will use in the following. Since our main motivation is to study the Ore-Stirling numbers and their connection to the conventional Stirling numbers, we restrict to the family $\mathcal{A}_{\lambda,\mu}$ mentioned above.

By considering the normal ordering coefficients of other words in D, U in the Weyl algebra $\mathcal{W}$, many generalizsations of the Stirling numbers of the second kind have been considered, see [1, 25, 33]. For example, if $r, s \in \mathbb{N}$, then considering $(U^r D^s)^n$ leads to the generalized Stirling numbers of the second kind $S_{r,s}(n,k)$ (with $r = s = 1$ reducing to (1)), considering $(UD)(UD - \lambda) \cdots (UD - (n-1)\lambda)$ leads to degenerate Stirling numbers $S_\lambda(n,k)$ [15], and considering $(mUD + r)^n$ leads to the r-Whitney numbers $W_{m,r}(n,k)$ [23]. Recently, by using normal ordering techniques in the Weyl algebra (represented by boson operators), properties of degenerate r-Stirling numbers [17], degenerate r-Whitney numbers [18] and degenerate r-Bell numbers [19] have been studied. See also [20] for further applications of normal ordering in this context and [16] for further properties of degenerate r-Whitney numbers and their relatives. Basic information and combinatorial interpretations for the numbers mentioned can be found in [29].

In a similar fashion, generalized Stirling numbers of the first kind were introduced recently by one of the authors [35] as normal ordering coefficients of $(U^r D^s)^n$ in the shift algebra $\mathcal{S}$ (with $r = s = 1$ reducing to (2)). Considering the word $(UD)^n$ where D, U satisfy the closely related commutation relation $DU - UD = hU^m$ led to the introduction of generalized Stirling numbers [26, 27] which turned out to be a particular subfamily of the generalized Stirling numbers of Hsu and Shiue [13], see also [2–4]. Using a similar approach, generalized Stirling numbers were defined recently as normal ordering coefficients in the n-th Weyl algebra [6, 8]. A beautiful combinatorial interpretation for the normal ordering coefficients of arbitrary words in the n-th Weyl algebra was given in [9, 10], providing, in particular, a combinatorial interpretation for the generalized Stirling numbers introduced in [6, 8]. Finally, we want to point out that normal ordering in the Weyl algebra has also been studied intensely in the physics literature since the commutation relation is the same as the one of a single bosonic degree of freedom, see [25] and the references therein.

The structure of the paper is as follows. In Section 2, the generalized Ore algebra $\mathcal{A}_{\lambda,\mu}$ is introduced and the Ore-Stirling numbers $S_{\lambda,\mu}(n; j, k)$ as well as the associated Ore-Bell numbers $B_{\lambda,\mu}(n)$ are defined. Also, some more context is given and some basic normal ordering results from [22] are recalled. In Section 3, the Ore-Stirling numbers are examined and basic properties like the recurrence relation and explicit expressions are derived. A connection to rook numbers and file numbers is drawn in Section 4. In the final Section 5, some conclusions and a list of avenues for future research are presented.

2. Definitions and basic results

As mentioned in the Introduction, we are interested in a particular two-parameter family of algebras generalizing the Weyl algebra $\mathcal{W}$ and the shift algebra $\mathcal{S}$ simultaneously.

Definition 2.1 (Generalized Ore algebra). *Let $\mu, \nu \in \mathbb{C}$. The* generalized Ore algebra $\mathcal{A}_{\lambda,\mu}$ *is the complex unital algebra generated by letters D and U satisfying the commutation relation*

$$DU - UD = \lambda D + \mu I, \tag{5}$$

where I denotes the identity (in the following, κI will also be identified with κ).

We will be interested mostly in the case $(\lambda, \mu) \in [0, 1] \times [0, 1]$ since all combinatorial consequences appear already here. Noting that $\mathcal{A}_{0,0} = \mathbb{C}[D, U]$, the commutative ring of polynomials in two variables, the family $\mathcal{A}_{\lambda,\mu}$ with $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$ can be nicely interpreted as a two-parameter deformation of the polynomial ring $\mathbb{C}[D, U]$ (see Figure 1):

- The algebra $\mathcal{A}_{0,1}$ is the first Weyl algebra $\mathcal{W}$ where $DU - UD = I$.
- The algebra $\mathcal{A}_{1,0}$ is the shift algebra $\mathcal{S}$ where $DU - UD = D$.
- The algebra $\mathcal{A}_{1,1}$ is the Ore algebra $\mathcal{O}$ (as defined in [30]) where $DU - UD = D + I$.

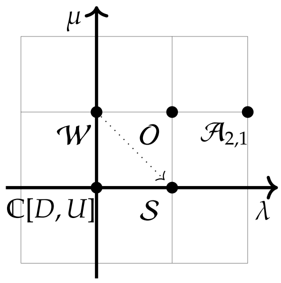

Figure 1: The family of generalized Ore algebras $\mathcal{A}_{\lambda,\mu}$.

Observe that for $\mu = 1 - \lambda$ one has a one-parameter family of algebras $\mathcal{A}_{\lambda,1-\lambda}$ which "interpolates" for $\lambda \in [0, 1]$ between the Weyl algebra $\mathcal{W}$ ($\lambda = 0$) and the shift algebra $\mathcal{S}$ ($\lambda = 1$):

$$\mathcal{W} = \mathcal{A}_{0,1} \longrightarrow \mathcal{A}_{\lambda,1-\lambda} \longrightarrow \mathcal{A}_{1,0} = \mathcal{S}. \tag{6}$$

This corresponds to the dotted arrow in Figure 1.

Recall from the Introduction that the Stirling numbers of the second kind $S(n, k)$ can be defined as normal ordering coefficients of $(UD)^n$ in $\mathcal{W} = \mathcal{A}_{0,1}$, and that the unsigned Stirling numbers of the first kind $|s(n, k)|$ are normal ordering coefficients of $(UD)^n$ in $\mathcal{S} = \mathcal{A}_{1,0}$. This motivates the following definition.

Definition 2.2 (Ore-Stirling numbers). *The* Ore-Stirling numbers $S_{\lambda,\mu}(n;j,k)$ *are defined as normal ordering coefficients of* $(UD)^n$ *in* $\mathcal{A}_{\lambda,\mu}$*, i.e., by*

$$(UD)^n = \sum_{j=0}^{n}\sum_{k=0}^{n} S_{\lambda,\mu}(n;j,k)U^jD^k, \tag{7}$$

where D and U satisfy (5)*. The associated* Ore-Bell numbers *are defined by*

$$B_{\lambda,\mu}(n) = \sum_{j=0}^{n}\sum_{k=0}^{n} S_{\lambda,\mu}(n;j,k).$$

Clearly, for most of the vertices of the unit square of Figure 1 the Ore-Stirling numbers are known:

- $S_{0,0}(n;j,k) = \delta_{j,k}\delta_{n,k}$ (Kronecker delta). The associated Ore-Bell numbers are trivial,
$$B_{0,0}(n) = \sum_{j=0}^{n}\sum_{k=0}^{n} \delta_{j,k}\delta_{n,k} = 1.$$
- $S_{0,1}(n;j,k) = \delta_{j,k}S(n,k)$. The associated Ore-Bell numbers are given by the conventional Bell numbers (A000110 in [37]),
$$B_{0,1}(n) = \sum_{k=0}^{n} S(n,k) = B_n.$$
- $S_{1,0}(n;j,k) = \delta_{k,n}|s(n,j)|$. The associated Ore-Bell numbers are given by factorials (A000142 in [37]),
$$B_{1,0}(n) = \sum_{j=0}^{n} |s(n,j)| = n!$$

The case $S_{1,1}(n;j,k)$ corresponding to the Ore algebra $\mathcal{O}$ remains to be determined. From (6), one obtains a one-parameter family of Ore-Stirling numbers $S_{\lambda,1-\lambda}(n;j,k)$ interpolating between Stirling numbers of the second kind and unsigned Stirling numbers of the first kind:

$$S(n,k)\delta_{j,k} = S_{0,1}(n;j,k) \longrightarrow S_{\lambda,1-\lambda}(n;j,k) \longrightarrow S_{1,0}(n;j,k) = |s(n,j)|\delta_{k,n}. \tag{8}$$

Turning to the associated Ore-Bell numbers, one obtains a one-parameter family of Ore-Bell numbers $B_{\lambda,1-\lambda}(n)$ interpolating between Bell numbers $B_n = B_{0,1}(n)$ and factorials $n! = B_{1,0}(n)$.

Remark 2.3. *Recall from the Introduction that in the Weyl algebra* $\mathcal{W} = \mathcal{A}_{0,1}$ *and in the shift algebra* $\mathcal{S} = \mathcal{A}_{1,0}$ *generalized Stirling numbers are defined as normal ordering coefficients of* $(U^rD^s)^n$ *where* $r,s \in \mathbb{N}$*. Clearly, in the same fashion one can introduce* generalized Ore-Stirling numbers $S^{(r,s)}_{\lambda,\mu}(n;j,k)$ *as normal ordering coefficients of* $(U^rD^s)^n$ *in* $\mathcal{A}_{\lambda,\mu}$*. In particular, the normal ordering coefficients of* $(U^2D)^n$ *in* $\mathcal{W}$ *are given by* (unsigned) Lah numbers *(A008297 in [37]), so the case* $r = 2, s = 1$ *might also be particularly interesting in* $\mathcal{A}_{\lambda,\mu}$*, yielding* Ore-Lah numbers $S^{(2,1)}_{\lambda,\mu}(n;j,k)$*.*

As mentioned in the Introduction, Levandovskyy et al. [22] considered the *algebra of type* (q,α,β,γ) generated by x and y satisfying $yx - qxy = \alpha x + \beta y + \gamma$, see (4). The algebras $\mathcal{A}_{\lambda,\mu}$ are of type $(1,0,\lambda,\mu)$, and we have the correspondence

$$x \longleftrightarrow U, \quad y \longleftrightarrow D, \quad \beta \longleftrightarrow \lambda, \quad \gamma \longleftrightarrow \mu. \tag{9}$$

As mentioned in Table 1 of [22], the algebra of type $(1,0,\lambda,\mu)$, i.e., $\mathcal{A}_{\lambda,\mu}$ is isomorphic to the shift algebra $\mathcal{S}$ (as one of the 5 model algebras), given by $YX = XY + Y$. Indeed, inserting

$$X = \lambda^{-1}U, \quad Y = \lambda D + \mu \tag{10}$$

into $YX = XY + Y$, one recovers (5).

Remark 2.4. *In $\mathcal{S}$, one has $(XY)^n = \sum_{k=1}^n |s(n,k)| X^k Y^n$, see (2). Inserting (10), one finds in $\mathcal{A}_{\lambda,\mu}$ the relation*

$$(UD + \lambda^{-1}\mu U)^n = \sum_{k=1}^{n} |s(n,k)| \lambda^{-k} U^k (\lambda D + \mu)^n.$$

However, UD and U do not satisfy a simple commutation relation in $\mathcal{A}_{\lambda,\mu}$ (i.e, $(UD)U - U(UD) = \lambda UD + \mu D$), so it does not seem to be straightforward to expand the left-hand side and obtain a formula for the powers $(UD)^n$.

For the convenience of the reader, we present in the following proposition some expansions mentioned by Levandoskyy et al. [22] in terms of the variables used here, see (9).

Proposition 2.5 ([22]). *For $m, n \in \mathbb{N}$ and $\lambda \neq 0$, one has in $\mathcal{A}_{\lambda,\mu}$ the following expansions*

$$D^m U = UD^m + mD^{m-1}(\lambda D + \mu), \tag{11}$$

$$DU^n = \frac{1}{\lambda}\left((U+\lambda)^n(\lambda D + \mu) - \mu U^n\right), \tag{12}$$

$$D^m U^n = \frac{1}{\lambda^m}\sum_{j=0}^{m}\binom{m}{j}(-\mu)^{m-j}(U + j\lambda)^n(\lambda D + \mu)^j. \tag{13}$$

By expanding the binomial and rearranging, we can write (12) equivalently as

$$DU^n = U^n D + \sum_{\ell=0}^{n-1}\binom{n}{\ell}\lambda^{n-\ell}U^{\ell}(D + \lambda^{-1}\mu). \tag{14}$$

From this formula, one can read off that the expansion makes sense for $\lambda \to 0$, too.

3. The Ore-Stirling numbers

In this section, we examine the Ore-Stirling numbers closer and derive some of their properties. Note that, for given $n \in \mathbb{N}$, there exist $(n+1)^2$ Ore-Stirling numbers $S_{\lambda,\mu}(n; j, k)$. For small n, they can be determined directly from their definition as normal ordering coefficients. For example,

$$(UD)^2 = UDUD = U(UD + \lambda D + \mu I)D = U^2D^2 + \lambda UD^2 + \mu UD,$$

showing $S_{\lambda,\mu}(2; 2, 2) = 1, S_{\lambda,\mu}(2; 1, 2) = \lambda, S_{\lambda,\mu}(2; 1, 1) = \mu$, and the remaining 6 numbers $S_{\lambda,\mu}(2; j, k)$ vanish. In the same fashion, one finds

$$(UD)^3 = U^3D^3 + 3\lambda U^2D^3 + 2\lambda^2 UD^3 + 3\mu U^2D^2 + 3\lambda\mu UD^2 + \mu^2 UD,$$

and the resulting Ore-Stirling numbers $S_{\lambda,\mu}(3; j, k)$ are displayed in Table 1. Observe that for $(\lambda, \mu) = (0, 1)$ all summands in the expansion of $(UD)^3$ proportional to λ vanish, implying $(UD)^3 = U^3D^3 + 3U^2D^2 + UD$, in accordance with (1) and $S_{0,1}(n; j, k) = S(n, k)\delta_{j,k}$. Thus, in Table 1, we find the Stirling numbers of the second kind on the diagonal (for $\mu = 1$). On the other hand, choosing $(\lambda, \mu) = (1, 0)$, we find $(UD)^3 = U^3D^3 + 3U^2D^3 + 2UD^3$, in accordance with (2) and $S_{1,0}(n; j, k) = |s(n, j)|\delta_{k,n}$. Thus, in Table 1, we find the unsigned Stirling numbers of the first kind in the last column (for $\lambda = 1$).

A slightly more cumbersome calculation gives

$$(UD)^4 = U^4D^4 + 6\lambda U^3D^4 + 11\lambda^2U^2D^4 + 6\lambda^3UD^4 + 6\mu U^3D^3 + 18\lambda\mu U^2D^3 + 12\lambda^2\mu UD^3 + 7\mu^2U^2D^2 + 7\lambda\mu^2UD^2 + \mu^3UD.$$

The $S_{\lambda,\mu}(4; j, k)$ are displayed in Table 2. For $(\lambda, \mu) = (0, 1)$, one finds $(UD)^4 = U^4D^4 + 6U^3D^3 + 7U^2D^2 + UD$, again in accordance with (1) and $S_{0,1}(n; j, k) = S(n, k)\delta_{j,k}$. Similarly, for $(\lambda, \mu) = (1, 0)$, one finds $(UD)^4 = U^4D^4 + 6U^3D^4 + 11U^2D^4 + 6UD^4$, in accordance with (2) and $S_{1,0}(n; j, k) = |s(n, j)|\delta_{k,n}$. In Table 2, one can recognize the Stirling numbers of the second kind on the diagonal and the unsigned Stirling numbers of the first kind in the last column.

From the first few values of the Ore-Stirling numbers, one reads off the following properties.

$j\backslash k$	1	2	3
1	$\mathbf{1}\mu^2$	$3\lambda\mu$	$\mathbf{2}\lambda^2$
2		$\mathbf{3}\mu$	$\mathbf{3}\lambda$
3			$\mathbf{1}$

Table 1: The nonvanishing Ore-Stirling numbers $S_{\lambda,\mu}(3; j,k)$.

$j\backslash k$	1	2	3	4
1	$\mathbf{1}\mu^3$	$7\lambda\mu^2$	$12\lambda^2\mu$	$\mathbf{6}\lambda^3$
2		$\mathbf{7}\mu^2$	$18\lambda\mu$	$\mathbf{11}\lambda^2$
3			$\mathbf{6}\mu$	$\mathbf{6}\lambda$
4				$\mathbf{1}$

Table 2: The nonvanishing Ore-Stirling numbers $S_{\lambda,\mu}(4; j,k)$.

Proposition 3.1. *Let $n \in \mathbb{N}$. The Ore-Stirling numbers satisfy the following properties.*

1. $S_{\lambda,\mu}(n; n, n) = 1$,
2. $S_{\lambda,\mu}(n; 1, 1) = \mu^{n-1}$,
3. $S_{\lambda,\mu}(n; j,k) = 0$, *if* $j > k$,
4. $S_{\lambda,\mu}(n; j,k) = 0$, *if* $n \geq 2$ *and* $jk = 0$.

Proof. We prove these properties by induction, using (14). Since all properties mentioned hold true for $n \leq 3$ we can assume that $n \geq 3$. Writing $(UD)^{n+1} = UD(UD)^n$, using (7) as well as the induction hypothesis for 4, one has

$$(UD)^{n+1} = \sum_{j=1}^{n} \sum_{k=1}^{n} S_{\lambda,\mu}(n; j,k) U(DU^j)D^k.$$

Using (14) for DU^j, this yields

$$(UD)^{n+1} = \sum_{j=1}^{n} \sum_{k=1}^{n} S_{\lambda,\mu}(n; j,k) \left\{ U^{j+1}D^{k+1} + \sum_{\ell=0}^{j-1} \binom{j}{\ell} \lambda^{j-\ell} U^{\ell+1}(D + \lambda^{-1}\mu)D^k \right\}. \tag{15}$$

From this, one reads off that no summands of the form U^jD^0 or U^0D^k appear, showing 4. The coefficient $S_{\lambda,\mu}(n+1; n+1, n+1)$ of $U^{n+1}D^{n+1}$ in (15) is given by $S_{\lambda,\mu}(n; n, n)$ which equals 1 by the induction hypothesis, showing 1. Property 3 holds for $n \leq 3$, and if it holds true for n, then (15) shows that it holds true also for $n+1$. Finally, to show 2 we determine the coefficient $S_{\lambda,\mu}(n+1; 1, 1)$ of UD in (15) to be $S_{\lambda,\mu}(n; 1, 1)\lambda^1(\lambda^{-1}\mu) = \mu S_{\lambda,\mu}(n; 1, 1)$ which by the induction hypothesis equals μ^n, showing 2. □

After having considered some particular Ore-Stirling numbers, we derive their recurrence relation.

Proposition 3.2. *Let $n, m, k \in \mathbb{N}$ with $m, k \leq n$. The Ore-Stirling numbers satisfy the recurrence relation*

$$S_{\lambda,\mu}(n+1; m, k) = S_{\lambda,\mu}(n; m-1, k-1) + \sum_{j=m}^{n} \binom{j}{m-1} \lambda^{j-m} \left\{ \lambda S_{\lambda,\mu}(n; j, k-1) + \mu S_{\lambda,\mu}(n; j,k) \right\}. \tag{16}$$

Proof. Let us consider $(UD)^{n+1}$. From the definition in (7), one has

$$(UD)^{n+1} = \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} S_{\lambda,\mu}(n+1; j,k) U^j D^k.$$

On the other hand, expanding the right-hand side of (15), one obtains for $(UD)^{n+1}$ the sum

$$\sum_{j=1}^{n+1}\sum_{k=1}^{n+1} S_{\lambda,\mu}(n;j-1,k-1)U^jD^k + \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=0}^{j-1}\binom{j}{\ell}S_{\lambda,\mu}(n;j,k)\lambda^{j-\ell}(U^{\ell+1}D^{k+1}+\lambda^{-1}\mu U^{\ell+1}D^k).$$

Rearranging the summands in the second sum, one has

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=0}^{j-1}\binom{j}{\ell}S_{\lambda,\mu}(n;j,k)\lambda^{j-\ell}U^{\ell+1}D^{k+1} = \sum_{m=1}^{n}\sum_{k=1}^{n+1}\sum_{j=m}^{n}\binom{j}{m-1}S_{\lambda,\mu}(n;j,k-1)\lambda^{j-m+1}U^mD^k.$$

In a similar fashion, one may write

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=0}^{j-1}\binom{j}{\ell}S_{\lambda,\mu}(n;j,k)\lambda^{j-\ell}U^{\ell+1}D^{k} = \sum_{m=1}^{n}\sum_{k=1}^{n}\sum_{j=m}^{n}\binom{j}{m-1}S_{\lambda,\mu}(n;j,k)\lambda^{j-m+1}U^mD^k.$$

Comparing the two expressions for $(UD)^{n+1}$, one finds

$$\begin{aligned}\sum_{m=1}^{n+1}\sum_{k=1}^{n+1} S_{\lambda,\mu}(n+1;m,k)U^mD^k =& \sum_{m=1}^{n+1}\sum_{k=1}^{n+1} S_{\lambda,\mu}(n;m-1,k-1)U^mD^k\\ &+\sum_{m=1}^{n}\sum_{k=1}^{n+1}\sum_{j=m}^{n}\binom{j}{m-1}S_{\lambda,\mu}(n;j,k-1)\lambda^{j-m+1}U^mD^k\\ &+\lambda^{-1}\mu\sum_{m=1}^{n}\sum_{k=1}^{n}\sum_{j=m}^{n}\binom{j}{m-1}S_{\lambda,\mu}(n;j,k)\lambda^{j-m+1}U^mD^k.\end{aligned}$$

Thus, by comparing coefficients, one obtains the recurrence relation

$$S_{\lambda,\mu}(n+1;m,k) = S_{\lambda,\mu}(n;m-1,k-1) + \sum_{j=m}^{n}\binom{j}{m-1}\lambda^{j-m}\left\{\lambda S_{\lambda,\mu}(n;j,k-1)+\mu S_{\lambda,\mu}(n;j,k)\right\},$$

as asserted. □

Remark 3.3. *Let us check* (16) *by considering* $(\lambda,\mu)=(0,1)$*. The recurrence* (16) *reduces to*

$$S_{0,1}(n+1;m,k) = S_{0,1}(n;m-1,k-1) + mS_{0,1}(n;m,k)$$

since only the summand $j=m$ *does not vanish. Recalling* $S_{0,1}(n;j,k)=\delta_{j,k}S(n,k)$*, this equals* $\delta_{m,k}S(n+1,k) = \delta_{m,k}S(n,k-1)+m\delta_{m,k}S(n,k)$*, thereby recovering the recurrence relation of the Stirling numbers of the second kind,* $S(n+1,k)=S(n,k-1)+kS(n,k)$*. On the other hand, letting* $(\lambda,\mu)=(1,0)$*, the recurrence* (16) *reduces to*

$$S_{1,0}(n+1;m,k) = S_{1,0}(n;m-1,k-1) + \sum_{j=m}^{n}\binom{j}{m-1}S_{1,0}(n;j,k-1).$$

Recalling $S_{1,0}(n;j,k)=\delta_{k,n}|s(n,j)|$ *(and cancelling the factor* $\delta_{k,n+1}$ *on both sides), one finds*

$$|s(n+1,m)| = |s(n,m-1)| + \sum_{j=m}^{n}\binom{j}{m-1}|s(n,j)|.$$

This is not the expected recurrence relation $|s(n+1,m)| = |s(n,m-1)| + n|s(n,m)|$ *of the unsigned Stirling numbers of the first kind, but in the form*

$$|s(n+1,m)| = \sum_{j=m-1}^{n} \binom{j}{m-1} |s(n,j)| \tag{17}$$

a well-known identity, see, e.g., (6.16) in [12].

From the above explicit values for $S_{\lambda,\mu}(n;j,k)$ with $n \leq 4$ we see that the dependence on (λ,μ) can be separated from a numerical coefficient. This holds true in general.

Proposition 3.4. *Let* $n \in \mathbb{N}$*. The Ore-Stirling numbers can be written, for all* $j \leq k \leq n$*, in the form*

$$S_{\lambda,\mu}(n;j,k) = S(n;j,k)\lambda^{k-j}\mu^{n-k} \tag{18}$$

for some coefficients $S(n;j,k)$*. These coefficients satisfy the recurrence relation*

$$S(n+1;m,k) = S(n;m-1,k-1) + \sum_{j=m}^{n} \binom{j}{m-1} \{S(n;j,k-1) + S(n;j,k)\}. \tag{19}$$

Proof. Both claims follow by induction using the recurrence (16). If the factorization (18) holds true for $S_{\lambda,\mu}(n;j,k)$, then inserting this into the right-hand side of (16) yields an expression of the form $\lambda^{k-j}\mu^{n+1-k}(\cdots)$ (inside the brackets there is no dependence on λ,μ), showing that $S_{\lambda,\mu}(n+1;j,k)$ has also the factorized form claimed. Cancelling the factors $\lambda^{k-j}\mu^{n+1-k}$ on both sides, the recurrence relation (19) results. □

The Ore-algebra $\mathcal{O}$ corresponds to the case $(\lambda,\mu) = (1,1)$. In this case, the Ore-Stirling numbers $S_{1,1}(n;j,k)$ are given by the numerical coefficients $S(n;j,k)$ due to (18) and satisfy the recurrence (19). Using this interpretation, the Ore-Stirling numbers for arbitrary parameters (λ,μ) are just scaled versions of the ones with parameters $(1,1)$, i.e.,

$$S_{\lambda,\mu}(n;j,k) = S_{1,1}(n;j,k)\lambda^{k-j}\mu^{n-k}. \tag{20}$$

The values of the Ore-Stirling numbers $S_{1,1}(n;j,k)$ for $n \leq 4$ were given above (see Table 1 and Table 2). Let us consider, for $1 \leq n \leq 4$, the sum of the rows, i.e., consider

$$\beta(n,j) = \sum_{k=1}^{n} S_{1,1}(n;j,k).$$

If we sum also over j, we obtain the Ore-Bell numbers, $B_{1,1}(n) = \sum_{j=1}^{n} \beta(n,j)$. The result is displayed in Table 3.

$n\backslash j$	1	2	3	4	$B_{1,1}(n)$
1	1				1
2	2	1			3
3	6	6	1		13
4	26	36	12	1	75

Table 3: The row sums $\beta(n,j)$ of the $S_{1,1}(n;j,k)$ and the Ore-Bell numbers $B_{1,1}(n)$.

These four lines are equal to A079641 in [37]. This sequence is the matrix product of the Stirling numbers of the second kind and the unsigned Stirling numbers of the first kind, see [21]. In particular, $B_{1,1}(n)$ counts preferential arrangements of n objects and is equal to the *Fubini number* F_n (or *ordered Bell number*), A000670 in [37], see the discussion by Knuth [21] for an interpretation of $\beta(n,j)$ and a recent different combinatorial interpretation in [11]. Algebraically, one thus has $\beta(n,j) = \sum_{k=j}^{n} S(n,k)|s(k,j)|$ for $n \leq 4$. In general, we have the following result.

Theorem 3.5. *Let $n \in \mathbb{N}$. The Ore-Stirling numbers are given, for $1 \le j \le k \le n$, by*

$$S_{1,1}(n; j, k) = S(n, k)|s(k, j)|. \tag{21}$$

Combining this with (20)*, one has for arbitrary parameters $\lambda, \mu \in \mathbb{C}$ the result*

$$S_{\lambda,\mu}(n; j, k) = S(n, k)|s(k, j)|\lambda^{k-j}\mu^{n-k}. \tag{22}$$

Proof. It only remains to show (21). As discussed above, the relation holds true for $n \le 4$. To show it holds true in general, we use the recurrence relation (19). Using the induction hypothesis on the right-hand side for n, the recurrence relation becomes

$$\begin{aligned} S_{1,1}(n+1; m, k) =& S(n, k-1)|s(k-1, m-1)| \\ &+ \sum_{j=m}^{n} \binom{j}{m-1} \{S(n, k-1)|s(k-1, j)| + S(n, k)|s(k, j)|\} \\ =& S(n, k-1)|s(k-1, m-1)| \\ &+ S(n, k-1)\left(\sum_{j=m-1}^{n} \binom{j}{m-1}|s(k-1, j)| - |s(k-1, m-1)|\right) \\ &+ S(n, k)\left(\sum_{j=m-1}^{n} \binom{j}{m-1}|s(k, j)| - |s(k, m-1)|\right). \end{aligned}$$

Using identity (17), one obtains

$$\begin{aligned} S_{1,1}(n+1; m, k) =& S(n, k-1)\left(|s(k-1, m-1)| + |s(k, m)| - |s(k-1, m-1)|\right) \\ &+ S(n, k)\left(|s(k+1, m)| - |s(k, m-1)|\right) \\ =& S(n, k-1)|s(k, m)| + S(n, k)k|s(k, m)|, \end{aligned}$$

where we used in the last step the recurrence relation for $|s(k+1, m)|$. Using now the recurrence relation $S(n+1, k) = S(n, k-1) + kS(n, k)$, the last equation becomes $S_{1,1}(n+1; m, k) = S(n+1, k)|s(k, m)|$, showing the assertion for $n + 1$. □

Remark 3.6. *Above we observed that $S(n, k)$ resp. $|s(n, j)|$ appear as coefficients on the diagonal resp. last column in Table 1 ($n = 3$) and Table 2 ($n = 4$). Note first that* (22) *implies $S_{\lambda,\mu}(n; j, j) = S(n, j)\mu^{n-j}$, explaining the first observation. Similarly, $S_{\lambda,\mu}(n; j, n) = |s(n, j)|\lambda^{n-j}$, explaining the second observation.*

Note that choosing $\mu = 1 - \lambda$ in (22) gives

$$S_{\lambda,1-\lambda}(n; j, k) = S(n, k)|s(k, j)|\lambda^{k-j}(1-\lambda)^{n-k}. \tag{23}$$

This is the one-parameter interpolation sought-after in (8). Indeed, one easily checks that for $\lambda \to 0$ one has $S_{\lambda,1-\lambda}(n; j, k) \to S(n, k)\delta_{j,k}$, and that $S_{\lambda,1-\lambda}(n; j, k) \to |s(n, j)|\delta_{k,n}$ for $\lambda \to 1$.

Let us turn to the Ore-Bell numbers. Using the above expression for $S_{1,1}(n; j, k)$, we can determine the Ore-Bell numbers $B_{1,1}(n)$ in the Ore algebra $\mathcal{O}$. Patrias and Pylyavskyy [30] already found the connection to the Fubini numbers F_n in the context of dual filtered graphs.

Corollary 3.7 ([30]). *Let $n \in \mathbb{N}$. The Ore-Bell numbers $B_{1,1}(n)$ are given by the Fubini numbers F_n.*

Proof. Inserting (21) into the definition of $B_{1,1}(n)$, one obtains

$$B_{1,1}(n) = \sum_{j=1}^{n}\sum_{k=j}^{n} S(n, k)|s(k, j)| = \sum_{k=1}^{n} S(n, k)\sum_{j=1}^{k}|s(k, j)| = \sum_{k=1}^{n} S(n, k)k! = F_n,$$

where we used that $\sum_{j=1}^{k}|s(k, j)| = k!$ (and the last equation is the defining equation of the Fubini numbers). □

What about the Ore-Bell numbers in general? Using (22), one finds (for $\lambda\mu \neq 0$)

$$B_{\lambda,\mu}(n) = \sum_{j=1}^{n}\sum_{k=j}^{n} S(n,k)|s(k,j)|\lambda^{k-j}\mu^{n-k} = \mu^n \sum_{k=1}^{n} S(n,k)\left(\frac{\lambda}{\mu}\right)^k \sum_{j=1}^{k} |s(k,j)|\lambda^{-j}.$$

For $\lambda = 1$, it is straightforward to continue. Recalling that the Fubini polynomials (or ordered Bell polynomials) are defined by $F_n(x) = \sum_{k=1}^{n} S(n,k)k!x^k$ (with $F_n(1) = F_n$), one finds (for $\mu \neq 0$)

$$B_{1,\mu}(n) = \mu^n \sum_{k=1}^{n} S(n,k)\left(\frac{1}{\mu}\right)^k k! = \mu^n F_n(\frac{1}{\mu}),$$

which reduces for $\mu = 1$ to $B_{1,1}(n) = F_n$ from above. For $\mu \to 0$, only the summand $k = n$ survives, yielding $B_{1,0}(n) = S(n,n)n! = n!$, as mentioned above. In general, we use the expansion of the rising factorial, $x^{\overline{n}} = \sum_{j=0}^{n} |s(n,j)|x^j$ (see, e.g., (6.11) in [12]), to obtain the following result.

Corollary 3.8. *Let $n \in \mathbb{N}$. For $\lambda\mu \neq 0$, the Ore-Bell numbers are given by*

$$B_{\lambda,\mu}(n) = \mu^n \sum_{k=1}^{n} S(n,k)\left(\frac{\lambda}{\mu}\right)^k \left(\frac{1}{\lambda}\right)^{\overline{k}}.$$

Remark 3.9. *From above, one obtains for the one-parameter family of Ore-Bell numbers that*

$$B_{\lambda,1-\lambda}(n) = \sum_{j=1}^{n}\sum_{k=j}^{n} S(n,k)|s(k,j)|\lambda^{k-j}(1-\lambda)^{n-k}.$$

Using again that $\lambda^{k-j}(1-\lambda)^{n-k} \to \delta_{j,k}$ for $\lambda \to 0$ (resp. $\delta_{k,n}$ for $\lambda \to 1$), one easily checks that $B_{\lambda,1-\lambda}(n) \to B(n)$ for $\lambda \to 0$ (resp. $n!$ for $\lambda \to 1$), as expected. However, it would be nice to have a more explicit expression for $B_{\lambda,1-\lambda}(n)$.

As alternative to the above derivation of the explicit value of $S_{\lambda,\mu}(n; j, k)$ we can start from (16) and use generating functions. Define $S_{\lambda,\mu}(n; m; v) = \sum_{k=m}^{n} S_{\lambda,\mu}(n; m, k)v^k$. Then (16) gives

$$S_{\lambda,\mu}(n+1; m; v) = vS_{\lambda,\mu}(n; m-1; v) + (\lambda v + \mu)\sum_{j=m}^{n}\binom{j}{m-1}\lambda^{j-m}S_{\lambda,\mu}(n; j; v).$$

Define $S_{\lambda,\mu}(n; u, v) = \sum_{m=1}^{n} S_{\lambda,\mu}(n; m; v)u^m$. By multiplying with u^m and summing over $m = 1, 2, \ldots, n+1$, we obtain

$$\begin{aligned}
S_{\lambda,\mu}(n+1; u, v) &= uvS_{\lambda,\mu}(n; u, v) + (\lambda v + \mu)\sum_{m=1}^{n}\sum_{j=m}^{n}\binom{j}{m-1}\lambda^{j-m}u^m S_{\lambda,\mu}(n; j; v)\\
&= uvS_{\lambda,\mu}(n; u, v) + (\lambda v + \mu)\sum_{j=1}^{n}\sum_{m=1}^{j}\binom{j}{m-1}\lambda^{j-m}u^m S_{\lambda,\mu}(n; j; v)\\
&= uvS_{\lambda,\mu}(n; u, v) + (uv + \frac{\mu u}{\lambda})\sum_{j=1}^{n}((u+\lambda)^j - u^j)S_{\lambda,\mu}(n; j; v)\\
&= uvS_{\lambda,\mu}(n; u, v) + (uv + \frac{\mu u}{\lambda})(S_{\lambda,\mu}(n; u+\lambda, v) - S_{\lambda,\mu}(n; u, v)).
\end{aligned}$$

Define $S_{\lambda,\mu}(x, u, v) = \sum_{n\geq 1} S_{\lambda,\mu}(n; u, v)x^n$. The above recurrence can then be written as

$$S_{\lambda,\mu}(x, u, v) - uvx = uvxS_{\lambda,\mu}(x, u, v) + (uv + \frac{\mu u}{\lambda})x(S_{\lambda,\mu}(x, u+\lambda, v) - S_{\lambda,\mu}(x, u, v)),$$

which leads to

$$S_{\lambda,\mu}(x,u,v) = \frac{uvx}{1+\frac{\mu ux}{\lambda}} + \frac{(uv+\frac{\mu u}{\lambda})x}{1+\frac{\mu ux}{\lambda}} S_{\lambda,\mu}(x,u+\lambda,v).$$

By iterating this equation an infinite number of times, we obtain the following formula,

$$S_{\lambda,\mu}(x,u,v) = vx\sum_{j\geq 0}(v\lambda+\mu)^j x^j \prod_{i=0}^{j} \frac{u+i\lambda}{\lambda+\mu(u+i\lambda)x}. \tag{24}$$

For example,

$$\begin{aligned}S_{\lambda,\mu}(x,u,v) =& uvx + uv(uv+v\lambda+\mu)x^2 + uv(u^2v^2+3uv^2\lambda+2v^2\lambda^2+3\mu uv+3\mu v\lambda+\mu^2)x^3\\ &+ uv(u^3v^3+6u^2v^3\lambda+11uv^3\lambda^2+6v^3\lambda^3+6\mu u^2v^2+18\mu uv^2\lambda+12\mu v^2\lambda^2+7\mu^2 uv\\ &+7\mu^2 v\lambda+\mu^3)x^4+\cdots.\end{aligned}$$

Recall that the Stirling numbers of the first, respectively second kind are given by

$$\prod_{j=0}^{k-1}(x+j) = \sum_{a=0}^{k}(-1)^{k-a}s(k,a)x^a, \quad \frac{x^k}{\prod_{j=1}^{k}(1-jx)} = \sum_{a\geq k} S(a,k)x^a.$$

Hence,

$$\begin{aligned}S_{\lambda,\mu}(x,u,v) &= \lambda vx\sum_{j\geq 0}\frac{(v\lambda+\mu)^j x^j\lambda^j}{(\lambda+\mu ux)^{j+1}}\prod_{i=0}^{j}\frac{u/\lambda+i}{1+\frac{\lambda\mu x}{\lambda+\mu ux}i}\\ &= v\sum_{j\geq 0}\sum_{a\geq j}\sum_{b=0}^{j+1}(-1)^{a+1-b}s(j+1,b)S(a,j)\frac{u^b\lambda^{j+1-b}\mu^{a-j}(v+\mu/\lambda)^j x^{a+1}}{(1+\mu ux/\lambda)^{a+1}}.\end{aligned}$$

By finding the coefficient of x^n, we have

$$S_{\lambda,\mu}(n;u,v) = v\sum_{j=0}^{n-1}\sum_{a=j}^{n-1}\sum_{b=0}^{j+1}(-1)^{n-a}s(j+1,b)S(a,j)\binom{n-1}{a}u^{b+n-1-a}\lambda^{j-b-n+2+a}(v+\mu/\lambda)^j\mu^{n-1-j}.$$

Thus, by finding the coefficient of u^m, we obtain

$$S_{\lambda,\mu}(n;m;v) = v\sum_{j=0}^{n-1}\sum_{a=j}^{n-1}(-1)^{m+1+a}s(j+1,m+a+1-n)S(a,j)\binom{n-1}{a}\lambda^{j+1-m}\mu^{n-1-j}(v+\mu/\lambda)^j.$$

Finally, by finding the coefficient of v^k, we obtain the following result.

Theorem 3.10. *For all $1\leq m\leq k\leq n$, the Ore-Stirling numbers are given by*

$$S_{\lambda,\mu}(n;m,k) = \sum_{j=0}^{n-1}\sum_{a=j}^{n-1}(-1)^{m+1+a}s(j+1,m+a+1-n)S(a,j)\binom{n-1}{a}\binom{j}{k-1}\lambda^{k-m}\mu^{n-k}.$$

Moreover, their generating function is given by (24).

By comparing the expression given in Theorem 3.10 with the one given in (22), we obtain, using $|s(k,m)| = (-1)^{k-m}s(k,m)$, the following identity,

$$(-1)^k S(n,k)s(k,m) = \sum_{j=0}^{n-1}\sum_{a=j}^{n-1}(-1)^{a+1}s(j+1,m+a+1-n)S(a,j)\binom{n-1}{a}\binom{j}{k-1}.$$

Before turning to a combinatorial interpretation of the Ore-Stirling numbers, we consider briefly the first nontrivial examples of the Ore-Lah numbers $S^{(2,1)}_{\lambda,\mu}(n;j,k)$ introduced in Remark 2.3. For $n = 2$, one finds

$$(U^2D)^2 = U^4D^2 + 2\lambda U^3D^2 + 2\mu U^3D + \lambda^2U^2D^2 + \lambda\mu U^2D,$$

showing that one has more nonvanishing Ore-Lah numbers than Ore-Stirling numbers, see Table 4. In general, one may use an induction and (14) to show that one has an expansion $(U^2D)^n = U^{2n}D^n + \cdots + \lambda\mu U^2D$, yielding many nonvanishing numbers $S^{(2,1)}_{\lambda,\mu}(n;j,k)$.

$j\backslash k$	1	2
2	$\lambda\mu$	λ^2
3	2μ	2λ
4		1

Table 4: The nonvanishing Ore-Lah numbers $S^{(2,1)}_{\lambda,\mu}(2;j,k)$.

4. Concerning a combinatorial interpretation

Due to the simple product structure (22) for the generalized Ore-Stirling numbers $S_{\lambda,\mu}(n;j,k)$ we are led to a simple interpretation for $S_{\lambda,\mu}(n;j,k)$ in terms of rook numbers and file numbers. Let us recall some terminology following [39]. For $n \in \mathbb{N}$, we let $[n] = \{1,2,\ldots,n\}$. A *board* B is a subset of $[n]\times[m] \subset \mathbb{Z}^2$, where n and m are positive integers, see Figure 2. Intuitively, we think of a board as an array of squares arranged in rows and columns. An element $(i,j) \in B$ is then represented by a square in the i-th column and j-th row. We consider the columns numbered from left to right, and rows numbered from bottom to top, so that the square $(1,1)$ appears in the left corner on the bottom (note that this is different from [39]). A board B is a *Ferrers board* if there is a non-increasing sequence of positive integers $h(B) = (h_1,h_2,\ldots,h_n)$ such that $B = \{(i,j)\,|\,i \le n \text{ and } h_1 - h_i + 1 \le j \le h_1\}$. For example, the board in Figure 2 is a Ferrers board with *height vector* $(3,2,2,1)$. Given a Ferrers board B, we call a placement of k rooks in B such that there is at most one rook in each row a *file placement of k rooks*, or also a *k-file placement*, see Figure 2 for an example.

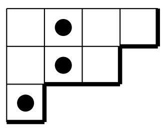

Figure 2: A Ferrers board with a 3-file placement which is not a 3-rook placement.

The set of all k-file placements on B will be denoted by $\mathcal{F}_k(B)$, and $f_k(B) = |\mathcal{F}_k(B)|$ will be called *k-th file number of B*.

Remark 4.1. *Note that due to our convention of drawing the boards we have to consider the rows for a file-placement. In the literature, one finds another convention of drawing boards and considers then the columns for file-placements, see, e.g., [5, 32].*

A *k-rook placement* is a special kind of k-file placement where in addition no two rooks are in the same column. The set of all k-rook placements on B will be denoted by $\mathcal{R}_k(B)$, and $r_k(B) = |\mathcal{R}_k(B)|$ will be called *k-th rook number of B*.

Example 4.2. *The* staircase board J_n *is defined by its height vector* $(n-1, n-2, \ldots, 2, 1)$ *(i.e.,* $h_1(J_n) = n-1$ *and* $h_{k+1}(J_n) = h_k(J_n) - 1$, *for* $k = 1, \ldots, n-1$*). It is well known (see, e.g., [5]) that*

$$r_{n-k}(J_n) = S(n,k), \quad f_{n-k}(J_n) = |s(n,k)|. \tag{25}$$

To any word ω in letters D, U one can associate a Ferrers board B_ω outlined by the path Γ_ω associated to ω. The path Γ_ω results by starting in $(0,0)$, and reading the word ω from left to right, associating to D (resp. U) a step to the right (resp. up); the board B_ω lies above Γ_ω. For example, the board associated to $\omega = DUDDUDU$ is shown in Figure 2 (with Γ_ω drawn in bold). As another example, if $\omega = (UD)^n$, then the associated board is the staircase board $J_n = B_{(UD)^n}$. Thus, inserting (22) into (7) and using (25), we can write

$$\begin{aligned}(UD)^n &= \sum_{j=1}^{n}\sum_{k=j}^{n} S(n,k)|s(k,j)|\lambda^{k-j}\mu^{n-k}U^jD^k \\ &= \sum_{j=1}^{n}\sum_{k=j}^{n} r_{n-k}(J_n)\mu^{n-k}f_{k-j}(J_k)\lambda^{k-j}U^jD^k. \end{aligned}\tag{26}$$

Introducing $l = n-j$ and $\sigma = k-(n-l)$, and switching the order of summation, this shows the following result.

Theorem 4.3. *Let the letters* D, U *satisfy* (5). *Then the normal ordering coefficients of* $\omega = (UD)^n$ *in* $\mathcal{A}_{\lambda,\mu}$ *are given by*

$$(UD)^n = \sum_{\sigma=0}^{n-1}\sum_{l=\sigma}^{n-1} r_\sigma(J_n)\mu^\sigma f_{l-\sigma}(J_{n-\sigma})\lambda^{l-\sigma}\; U^{n-l}D^{n-\sigma}. \tag{27}$$

For $\lambda \to 0$, only coefficients with $l = \sigma$ do not vanish, implying (with $\mu = 1$) in $\mathcal{A}_{0,1} = \mathcal{W}$ that

$$(UD)^n = \sum_{\sigma=0}^{n-1} r_\sigma(J_n)\; U^{n-\sigma}D^{n-\sigma}, \tag{28}$$

which is due to (25) the correct result for the Weyl algebra, see (1). Note that one has for arbitrary words ω in the Weyl algebra containing m letters U and n letters D (with $m \leq n$) that

$$\omega = \sum_{\sigma=0}^{m} r_\sigma(B_\omega)\; U^{m-\sigma}D^{n-\sigma}, \tag{29}$$

where B_ω is the Ferrers board outlined by ω, see [39]. Similarly, for $\mu \to 0$, only coefficients with $\sigma = 0$ do not vanish, implying (with $\lambda = 1$) in $\mathcal{A}_{1,0} = \mathcal{S}$ that

$$(UD)^n = \sum_{l=0}^{n-1} f_l(J_n)\, U^{n-l}D^n, \tag{30}$$

which is due to (25) the correct result for the shift algebra, see (2). For arbitrary words ω in the shift algebra containing m letters U and n letters D (with $m \leq n$) one has that

$$\omega = \sum_{\sigma=0}^{m} f_\sigma(B_\omega)\; U^{m-\sigma}D^n, \tag{31}$$

see [32]. Thus, (27) is the common generalization of (28) and (30). Turning from the particular words $\omega = (UD)^n$ to arbitrary words, one has the normal ordering result (29) in $\mathcal{W}$ and (31) in $\mathcal{S}$, and it would be interesting to find their common generalization, i.e., an interpretation for the normal ordering coefficients of ω in $\mathcal{A}_{\lambda,\mu}$ in terms of rook numbers and file numbers of the board B_ω. However, as (27) shows, already in the case $\omega = (UD)^n$ not only the board $J_n = B_{(UD)^n}$ is involved, but also some other "derived" boards $J_{n-\sigma}$.

Remark 4.4. *Let us consider the word $\omega = D^2U^2$ in $\mathcal{A}_{1,1} = \mathcal{O}$ with associated board $B = B_\omega = (2, 2)$. Using* (5) *(with $\lambda = \mu = 1$) repeatedly, one finds*

$$D^2U^2 = (U^2 + 4U + 4)D^2 + (4U + 6)D + 2I. \tag{32}$$

On the other hand, the rook numbers and file numbers of B are easy to determine,

$$r_0(B) = 1,\ r_1(B) = 4,\ r_2(B) = 2,\ f_0(B) = 1,\ f_1(B) = 4,\ f_2(B) = 4.$$

Let us denote by B^σ the (unknown) "derived" board from B (i.e., the analog of $J_{n-\sigma}$ in (27)*), and let us assume that the sought-after formula has a similar structure like* (27)*. Thus, one expects*

$$D^2U^2 = \sum_{\sigma=0}^{2}\sum_{l=\sigma}^{2} r_\sigma(B) f_{l-\sigma}(B^\sigma) U^{2-l} D^{2-\sigma}.$$

Using $r_\sigma(B)$ from above and noting that $B^0 = B$, this would imply ($f_0(B) = 1$ for any board B)

$$D^2U^2 = (U^2 + 4U + 4)D^2 + (4U + 4f_1(B^1))D + 2I. \tag{33}$$

Comparing (32) *with* (33) *shows that the "derived" board B^1 should satisfy $4f_1(B^1) = 6$ – which is impossible. Thus, the simple product structure for the coefficients of $U^{2-l}D^{2-\sigma}$ does not hold true. Presumably, it has to be replaced for the general case by a sum which reduces for the staircase board to a single summand.*

5. Conclusion and avenues for future research

In the present paper, we introduced Ore-Stirling numbers as normal ordering coefficients in the generalized Ore algebra $\mathcal{A}_{\lambda,\mu}$ (where $\mathcal{A}_{1,1}$ is the Ore algebra of [30]), thereby unifying considerations in the Weyl algebra ($\mathcal{W} = \mathcal{A}_{0,1}$) and the shift algeba ($\mathcal{S} = \mathcal{A}_{1,0}$). Several properties of the Ore-Stirling numbers as well as the associated Ore-Bell numbers were derived. However, many topics were not touched. In the following list, we mention – with different amount of detail – some natural questions which in our opinion merit closer inspection.

1. Derive a recurrence relation for the Ore-Bell numbers $B_{\lambda,\nu}(n)$.
2. As generalization of the last point, derive an analog of Spivey's formula for $B_{\lambda,\nu}(n)$. That is, express $B_{\lambda,\nu}(n+m)$ in terms of $B_{\lambda,\nu}(k)$, $S_{\lambda,\nu}(m; r, s)$ and other combinatorial numbers, like binomial coefficients. The original Spivey identity due to Spivey [38] is the following identity for the Bell numbers, $B_{n+m} = \sum_{k=0}^{n}\sum_{j=0}^{m} j^{n-k}\binom{n}{k}S(m, j)B_k$. Equivalently,

$$B_{0,1}(n+m) = \sum_{k=0}^{n}\sum_{j=0}^{m} j^{n-k}\binom{n}{k} S_{0,1}(m; j, j) B_{0,1}(k).$$

   On the other hand, one has the "dual Spivey identity", derived by Mező [28], where one has $(n+m)! = \sum_{k=0}^{n}\sum_{j=0}^{m} m^{\overline{n-k}}|s(m, j)|\binom{n}{k}k!$ Recalling $n! = B_{1,0}(n)$ and $|s(m, j)| = S_{1,0}(m; j, m)$, this can be written equivalently as

$$B_{1,0}(n+m) = \sum_{k=0}^{n}\sum_{j=0}^{m} m^{\overline{n-k}}\binom{n}{k} S_{1,0}(m; j, m) B_{1,0}(k).$$

For both identities, one has proofs using normal ordering (for the first identity in the Weyl algebra $\mathcal{W} = \mathcal{A}_{0,1}$ [14], for the second identity in the shift algebra $\mathcal{S} = \mathcal{A}_{1,0}$ [34]), so one should try to derive the sought-after identity for $B_{\lambda,\nu}(n+m)$ using normal ordering in the generalized Ore algebra $\mathcal{A}_{\lambda,\mu}$. For some different generalizations, see [36].

3. Derive an analog of the Dobiński formula for $B_{\lambda,\nu}(n)$. Recall that the classical Dobiński formula for the Bell numbers is given by $B_n = \frac{1}{e}\sum_{k=1}^{\infty}\frac{k^n}{k!}$, see [25] for more details and some history of this formula, as well as several generalizations of it.
4. Derive properties of the generalized Ore-Stirling numbers mentioned in Remark 2.3.
5. Generalize Theorem 4.3 to arbitrary words $\omega \in \mathcal{A}_{\lambda,\mu}$, i.e., give an interpretation for the normal ordering coefficients of ω in terms of rook numbers and file numbers of the associated board B_ω. As Theorem 4.3 and the discussion in Remark 4.4 shows, some kind of "derived" boards B_ω^σ are expected to be relevant. Note that B_ω^σ is a subboard of B_ω so it can alternatively be interpreted as B_ω where it is forbidden to place rooks in some cells (depending on σ).
6. For the Weyl algebra $\mathcal{W}$, one has a description of normal ordering in terms of *contractions*, see [24, Section 10] or [1, Note 1]. Given a word ω in letters D, U we call a k-contraction (with $k = 0, 1, \ldots$) of ω the choice of k pairs (D, U) where D precedes U, with subsequent replacement of these letters D, U by the symbol $\emptyset$. For example, the word $\omega = (UD)^3 = UDUDUD$ has exactly one 0-contraction (leaving ω as it is), 3 possible 1-contractions ($U\emptyset\emptyset DUD, U\emptyset UD\emptyset D, UDU\emptyset\emptyset D$), and one 2-contraction ($U\emptyset\emptyset\emptyset\emptyset D$). Let us denote the set of k-contractions of ω by $C_k(\omega)$, and let $C(\omega) = \cup_k C_k(\omega)$. If $\pi \in C(\omega)$, then $:\pi:$ means that all symbols $\emptyset$ are deleted and all letters U are written to the left of all letters D (as if they would commute). For example, $:UDU\emptyset\emptyset D := U^2D^2$. Then one has the following result *(Wick's theorem)*,
$$\omega = \sum_{\pi\in C(\omega)} :\pi:$$
Contractions are closely related to set partitions, see [24]. For example, for $\omega = (UD)^n$ one notes that $:\pi := U^{n-k}D^{n-k}$ if $\pi \in C_k(\omega)$, hence
$$(UD)^n = \sum_{k=0}^{n-1} |C_k((UD)^n)| U^{n-k}D^{n-k} = \sum_{\ell=1}^{n} |C_{n-\ell}((UD)^n)| U^\ell D^\ell,$$
showing by comparison with (1) that $|C_{n-\ell}((UD)^n)| = S(n,\ell)$. In the shift algebra $\mathcal{S}$, the result analogous to (1) is (2) where instead of $S(n,k)$ the $|s(n,k)|$ appear. It would be very nice to have an analog of Wick's theorem for normal ordering in the shift algebra. Thus, one should find an analog to contractions and to the operation $::$ (the sought-after analogous structure $\mathcal{P}_k(\omega)$ should be closely related to permutations with k cycles for ω). Finally, the analog of Wick's theorem for the Ore algebra $\mathcal{O}$ should be considered, too
7. Study q-deformed Ore-Stirling numbers. Recall that the q-deformed Weyl algebra $\mathcal{W}_q$ can be defined by letters D, U satisfying $DU - qUD = I$. The normal ordering coefficients of $(UD)^n$ in $\mathcal{W}_q$ are q-deformed Stirling numbers of the second kind $S^q(n,k)$. Thus, it is natural to introduce the q-deformed generalized Ore algebra $\mathcal{A}_{\lambda,\mu}(q)$ by the q-deformed variant of (5), i.e., by $DU - qUD = \lambda D + \mu I$. Similar to (7), the q-deformed Ore-Stirling numbers $S^q_{\lambda,\mu}(n; j, k)$ are then defined as normal ordering coefficients of $(UD)^n$ in $\mathcal{A}_{\lambda,\mu}(q)$. For $q \to 1$, they should reduce to $S_{\lambda,\mu}(n; j, k)$. The limit $q \to -1$ of $S^q_{\lambda,\mu}(n; j, k)$ might also be interesting.
8. Consider the three-parameter algebra $\mathcal{A}_{\lambda,\mu,\nu}$ generated by D, U with commutation relation
$$DU - UD = \lambda D + \nu U + \mu I.$$
(This corresponds to an algebra of type $(1, \nu, \lambda, \mu)$ in [22], see (4).) For example, $\mathcal{A}_{1,0,1}$ is the *excedance algebra* $\mathcal{E}$ where $DU - UD = D + U$, see [7]. One would then consider the unit cube $[0,1]^3 \subset \mathbb{R}^3$ as space of parameters (λ, μ, ν). Note that the algebra $\mathcal{A}_{0,0,1}$ – where $DU - UD = U$ – is essentially the shift algebra; we denote it by $\mathcal{S}^*$. In a similar fashion, $\mathcal{A}_{0,1,1}$ – where $DU - UD = U + I$ – is essentially

the Ore algebra; we denote it by $\mathcal{O}^*$. As an extension of Figure 1, we draw the unit cube of parameters $(\lambda, \mu, \nu) \in [0,1] \times [0,1] \times [0,1]$ and label the vertices with the corresponding algebras in Figure 3. Clearly, one can ask questions analogous to the ones considered above. For example, by considering

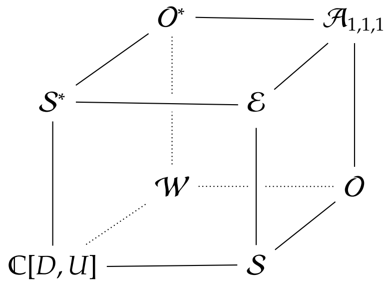

Figure 3: The unit cube of parameters (λ, μ, ν) and resulting algebras $\mathcal{A}_{\lambda,\mu,\nu}$.

normal ordering $(UD)^n$ in $\mathcal{A}_{\lambda,\mu,\nu}$, one can define numbers $S_{\lambda,\mu,\nu}(n; j, k, l)$ and study their properties. Recall that choosing $(\lambda, 1-\lambda, 0)$ corresponds to an interpolation between $\mathcal{W}$ and $\mathcal{S}$, see (6), hence between Stirling numbers of the second kind and unsigned Stirling numbers of the first kind, see (23). In a similar fashion, by letting $\nu = 1 - \mu$, one obtains an interpolation $\mathcal{A}_{\lambda,\mu,1-\mu}$ between $\mathcal{E} = \mathcal{A}_{1,0,1}$ and $\mathcal{O} = \mathcal{A}_{1,1,0}$. Normal ordering in the excedance algebra $\mathcal{E}$ was considered by Clark and Ehrenborg [7], and a combinatorial interpretation for the normal ordering coefficients was given by them. They also established a connection to *Genocchi numbers* (A001469 in [37]).

References

[1] P. Blasiak and P. Flajolet, *Combinatorial models of creation-annihilation*, Sém. Lothar. Combin. **65** (2011), Art. B65c.

[2] G. Benkart, S.A. Lopes and M. Ondrus, *A parametric family of subalgebras of the Weyl algebra. II: Irreducible modules*, Recent developments in algebraic and combinatorial aspects of representation theory. Contemp. Math. 602, Amer. Math. Soc., Providence, RI (2013), 73–98.

[3] G. Benkart, S.A. Lopes and M. Ondrus, *A parametric family of subalgebras of the Weyl algebra. I: Structure and automorphisms*, Trans. Am. Math. Soc. **367** (2015), 1993–2021.

[4] E. Briand, S.A. Lopes and M. Rosas, *Normally ordered forms of powers of differential operators and their combinatorics*, J. Pure Appl. Algebra **224** (2020), Art. 106312.

[5] F. Butler, M. Can, J. Haglund and J.B. Remmel, *Rook Theory Notes*, book project http://www.math.ucsd.edu/ remmel/files/Book.pdf.

[6] F.J. Castro-Jiménez and H. Cobo Pablos, *On certain polynomial systems involving Stirling numbers of second kind*, J. Symb. Comput. **109** (2022), 325–350.

[7] E. Clark and R. Ehrenborg, *The excedance algebra*, Discrete Math. **313** (2013), 1429–1435.

[8] H. Cobo, *On (yet another) family of Stirling polynomials of the second kind*, Integers **21** (2021), Paper #A106.

[9] A. Dzhumadil'daev and D. Yeliussizov, *Path decompositions of digraphs and their applications to Weyl algebra*, Adv. in Appl. Math. **67** (2015), 36–54.

[10] A. Dzhumadil'daev and D. Yeliussizov, *Walks, partitions, and normal ordering*, Electron. J. Combin. **22** (2015), Art. 4.10.

[11] N. Early, *Canonical Bases for Permutohedral Plates*, Preprint, 2017, arXiv:1712.08520.

[12] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete mathematics: a foundation for computer science. 2nd ed*, Amsterdam: Addison-Wesley Publishing Group, 1994.

[13] L.C. Hsu and P.J.-S. Shiue, *A unified approach to generalized Stirling numbers*, Adv. Appl. Math. **20** (1998), 366–384.

[14] J. Katriel, *On a generalized recurrence for Bell numbers*, J. Integer Seq. **11** (2008), Art. 08.3.8.

[15] T. Kim and D.S. Kim, *Some identities involving degenerate Stirling numbers arising from normal ordering*, AIMS Math. **7** (2022), 17357–17368.

[16] T. Kim and D.S. Kim, *Degenerate Whitney numbers of first and second kind of dowling lattices*, Russ. J. Math. Phys. **29** (2022), 358–377.

[17] T. Kim and D.S. Kim, *Some Identities on Degenerate r-Stirling Numbers via Boson Operators*, Russ. J. Math. Phys. **29** (2022), 508–517.

[18] T. Kim and D.S. Kim, *Degenerate r-Whitney numbers and degenerate r-Dowling polynomials via boson operators*, Adv. Appl. Math. **140** (2022), Art. 102394.

[19] T. Kim, D.S. Kim and H.K. Kim, *Degenerate r-Bell Polynomials Arising from Degenerate Normal Ordering*, J. Math. **2022** (2022), Art. 2626249.

[20] T. Kim, D.S. Kim and H.K. Kim, *Normal ordering of degenerate integral powers of number operator and its applications*, Appl. Math. Sci. Eng. **30** (2022), 440–447.

[21] D. E. Knuth, *Convolution polynomials*, Mathematica J. **2** (1992), 67–78.

[22] V. Levandovskyy, C. Koutschan and O. Motsak, *On two-generated non-commutative algebras subject to the affine relation*, in: Computer algebra in scientific computing. 13th international workshop, CASC 2011, Kassel, Germany, September 5–9, 2011. Proceedings, 309–320.

[23] M.M. Mangontarum and J. Katriel, *On q-boson operators and q-analogues of the r-Whitney and r-Dowling numbers*, J. Integer Seq. **18** (2015), Art. 15.9.8.

[24] T. Mansour, *Combinatorics of set partitions*, CRC Press, Boca Raton, FL, 2012.

[25] T. Mansour and M. Schork, *Commutation relations, normal ordering, and Stirling numbers*, CRC Press, Boca Raton, FL, 2016.

[26] T. Mansour, M. Schork and M. Shattuck, *On a new family of generalized Stirling and Bell numbers*, Electron. J. Combin. **18** (2011), Art. 77.

[27] T. Mansour, M. Schork and M. Shattuck, *The generalized Stirling and Bell numbers revisited*, J. Integer Seq. **15** (2012), Art. 12.8.3.

[28] I. Mező, *The dual of Spivey's Bell number formula*, J. Integer Seq. **15** (2012), Art. 12.2.4.

[29] I. Mező, *Combinatorics and number theory of counting sequences*, CRC Press, Boca Raton, FL, 2020.

[30] R. Patrias and P. Pylyavskyy, *Dual filtered graphs*, Algebr. Comb. **1** (2018), 441–500.

[31] H. Scherk, *De evolvenda functione* $(yd \cdot yd \cdot yd \ldots yd\, X)/dx^n$ *disquisitiones nonnullae analyticae*, (Ph.D. Thesis), University of Berlin, 1823.

[32] M.J. Schlosser and M. Yoo, *Weight-dependent commutation relations and combinatorial identities*, Discrete Math. **341** (2018), 2308–2325.

[33] M. Schork, *Recent developments in combinatorial aspects of normal ordering*, Enumer. Combin. Appl. **1** (2021), Article S2S2.

[34] M. Schork, *Normal ordering in the shift algebra and the dual of Spivey's identity*, Quaest. Math. **46** (2023), 173–180.

[35] M. Schork, *File placements, fractional matchings, and normal ordering*, Ann. Comb. **26** (2022), 857–871.

[36] M. Shattuck, *Generalizations of Bell Number Formulas of Spivey and Mező*, Filomat **30** (2016), 2683–2694.

[37] N.J.A. Sloane, *The On-line Encyclopedia of Integer Sequences*, http://oeis.org/ .

[38] M. Z. Spivey, *A generalized recurrence for Bell numbers*, J. Integer Seq. **11** (2008), Art. 08.2.5.

[39] A. Varvak, *Rook numbers and the normal ordering problem*, J. Combin. Theory Ser. A **112** (2005), 292–307.