Filomat 37:18 (2023), 6133–6150
https://doi.org/10.2298/FIL2318133F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Inertial iterative method for a generalized mixed equilibrium, variational inequality and a fixed point problems for a family of quasi-ϕ-nonexpansive mappings

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Abstract. We introduce an inertial type gradient projection hybrid iterative method for finding a common solution of generalized mixed equilibrium, variational inequality and fixed point problems in a two-uniformly convex and uniformly smooth Banach space. Next, we analyze the strong convergence for a common solution of problem. Furthermore, we carry out some consequences and present a numerical example to show and tell the applicability of main theorem. Our result improves, unifies, generalizes and extends ones from several earlier works.

1. Introduction

Let Y^* denotes the dual space of a real Banach space Y. We denote the value of the functional $j \in Y^*$ at $x_1 \in Y$ by $\langle x_1, j\rangle$ and the norm of Y or Y^* by $\|.\|$. Let $P \neq \emptyset$ be a subset of Y. A mapping $J : Y \to 2^{Y^*}$ such that

$$Jx_1 = \{x_2 \in Y^* : \langle x_2, x_1\rangle = \|x_1\|^2 = \|x_2\|^2\}, \quad \forall x_1 \in Y.$$

is called normalized duality mapping.

Let $\mathbb{G} : P \times P \to \mathbb{R}$, $b : P \times P \to \mathbb{R}$ be bifunctions and $D : P \to Y^*$ be a nonlinear mapping, where $\mathbb{R}$ is the set of all real numbers. In this paper, we consider generalized mixed equilibrium problem (in brief, GMEP)as: Find $u_1 \in P$ such that

$$\mathbb{G}(u_1, u_2) + \langle Du_1, u_2 - u_1\rangle + b(u_1, u_2) - b(u_1, u_1) \geq 0, \quad \forall u_2 \in P. \tag{1}$$

The solution set of GMEP(1) is denoted by Sol(GMEP(1)).

If $D \equiv 0$ then GMEP(1) convert to generalized equilibrium problem (in brief, GEP): Find $u_1 \in P$ such that

$$\mathbb{G}(u_1, u_2) + b(u_1, u_2) - b(u_1, u_1) \geq 0, \quad \forall u_2 \in P. \tag{2}$$

2020 *Mathematics Subject Classification*. 47H10, 47J22, 47J25
Keywords. Quasi-ϕ-nonexpansive mapping, fixed point problem, variational inequality problem, generalized mixed equilibrium problem, inertial hybrid iterative method, Banach space
Received: 29 October 2020; Revised: 03 September 2022; Accepted: 19 January 2023
Communicated by Adrian Petrusel
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The solution set of GEP(2) is denoted by Sol(GEP(2)).

If $D \equiv 0$ and $b \equiv 0$ then GMEP(1) becomes equilibrium problem (in brief, EP): Find $u_1 \in P$ such that

$$\mathbb{G}(u_1, u_2) \geq 0, \;\; \forall u_2 \in P, \tag{3}$$

The solution set of EP(3) is denoted by Sol(EP(3)) and (3) introduced by Blum and Oettli [2].

The variational inequality problem (in brief, VIP): Find $u_1 \in P$ such that

$$\langle u_2 - u_1, Bu_1 \rangle \geq 0, \;\; \forall u_2 \in P, \tag{4}$$

where $B : P \to Y^*$ be a nonlinear mapping. VIP(4) is studied by Hartmann and Stampacchia [9] and Sol(VIP(4)) denotes its solution.

Let $T : P \to P$ be a nonlinear mapping. we define fixed point problem (in brief, FPP): Find $u_1 \in P$ such that

$$F(T) = \{u_1 \in P : Tu_1 = u_1\}. \tag{5}$$

Takahashi et al. [18] proposed an algorithm in 2009 as:

$$\left.\begin{array}{l} x_0 \in P, \\ u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ z_n \in P \text{ such that } g(z_n, v) + \dfrac{1}{r_n}\langle v - z_n, Jz_n - Ju_n\rangle \geq 0, \;\forall v \in P, \\ P_n = \{w \in P : \phi(w, z_n) \leq \phi(w, x_n)\}, \\ Q_n = \{w \in P : \langle x_n - w, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{P_n \bigcap Q_n} x_0. \end{array}\right\} \tag{6}$$

Recently, Kazmi and Ali [14], studied an iterative result for finding a common solution of EP (3) and FPP (5) for an asymptotically quasi-ϕ-nonexpansive mapping. For further study of some generalizations of algorithms (6), see[7, 8, 10–12, 20].

In 2008, Mainge [15] development and studied the following inertial method:

$$\left.\begin{array}{ll} z_n & = u_n + \theta_n(u_n - u_{n-1}), \\ u_{n+1} & = (1-\alpha_n)z_n + \alpha_n Tz_n. \end{array}\right\} \tag{7}$$

In a short while ago, Dong et al. [4, 5] studied inertial iterative result in Hilbert space frame.

It is important to highlight that in the framework of Banach space, the inertial iterative algorithm is still unexplored.

Therefore, inspired and motivated by the endeavor of Dong et al. [5], Mainge [15] and Takahashi et al. [18], we proposed an inertial type gradient projection hybrid iterative algorithm for finding a common solution of GMEP(1), VIP(4) for a γ-ism and FPP(5) for a family of quasi-ϕ-nonexpansive mappings in two-uniformly convex and uniformly smooth Banach space. Next, we analyze the strong convergence for a common solution of problem. Furthermore, we carry out some consequences and present a numerical example to show and tell the applicability of main theorem.

2. Preliminaries

We offered some necessary definitions and results which are needed in succession.

Let $S = \{x_1 \in Y : \|x_1\| = 1\}$ be the unit sphere of Y and if $\frac{\|x_1+x_2\|}{2} < 1$, $\forall x_1, x_2 \in S$ with $x_1 \neq x_2$ then Y is said to be strictly convex. If for any $\varepsilon \in (0, 2]$ there exists a $\delta > 0$ such that

$$\|x_1 - x_2\| \geq \varepsilon \text{ implies } \frac{\|x_1 + x_2\|}{2} \leq 1 - \delta, \quad \forall x_1, x_2 \in S,$$

then Y is called uniformly convex. Note that it is strictly convex and reflexive.

The space Y is called smooth if $\lim_{s\to 0} \frac{\|x_1 + sx_2\| - \|x_1\|}{s}$ exists, $\forall x_1, x_2 \in S$ and uniformly smooth if the limit is attained uniformly, $\forall x_1, x_2 \in S$. The space Y enjoys Kadec-Klee property if for any $\{x_n\} \subset Y$ and $x_1 \in Y$ with $x_n \rightharpoonup x_1$ and $\|x_n\| \to \|x_1\|$ then $\|x_n - x_1\| \to 0$ as $n \to \infty$.

A mapping $\phi : Y \times Y \to \mathbb{R}$ such that

$$\phi(x_1, x_2) = \|x_1\|^2 - 2\langle x_1, Jx_2\rangle + \|x_2\|^2, \quad \forall x_1, x_2 \in Y, \tag{8}$$

is called Lyapunov function.

From (8), we have

$$(\|x_1\| - \|x_2\|)^2 \leq \phi(x_1, x_2) \leq (\|x_1\| + \|x_2\|)^2, \quad \forall x_1, x_2 \in Y, \tag{9}$$

$$\phi(x_1, J^{-1}(\lambda Jx_2 + (1 - \lambda)Jx_3)) \leq \lambda\phi(x_1, x_2) + (1 - \lambda)\phi(x_1, x_3), \quad \forall x_1, x_2 \in Y,\ \lambda \in [0, 1], \tag{10}$$

and

$$\phi(x_1, x_2) = \|x_1\|\|Jx_1 - Jx_2\| + \|x_2\|\|x_1 - x_2\|, \quad \forall x_1, x_2 \in Y. \tag{11}$$

Remark 2.1. $\phi(x_1, x_2) = 0 \iff x_1 = x_2$, $\forall x_1, x_2 \in Y$ *provided Y be smooth, reflexive and strictly convex Banach space.*

Definition 2.2. *A function $T : P \to Y^*$ is known as*

(i) *monotone if $\langle x_1 - x_2, Tx_1 - Tx_2\rangle \geq 0, \quad \forall x_1, x_2 \in Y$;*
(ii) *$\gamma-$ inverse strongly monotone (in short, ism) if $\exists\, \gamma > 0$ such that $\langle x_1 - x_2, Tx_1 - Tx_2\rangle \geq \gamma\|Tx_1 - Tx_2\|^2$, $\forall x_1, x_2 \in Y$;*
(iii) *Lipschitz continuous if $\exists\, L > 0$ such that $\|Tx_1 - Tx_2\| \leq L\|x_1 - x_2\|$.*

If T is $\gamma-$ ism then it is Lipschitz continuous with $\frac{1}{\gamma}$ as a constant.

Lemma 2.3. [21] *Let Y be a 2-uniformly convex and smooth Banach space. Then, $\forall x_1, x_2 \in Y$, $\phi(x_1, x_2) \geq c\|x_1 - x_2\|^2$, where $0 < c \leq 1$ and called two-uniformly convex constant.*

Lemma 2.4. [21] *Let Y be a two-uniformly convex Banach space, then*

$$\|x_1 - x_2\| \leq \frac{2}{c}\|Jx_1 - Jx_2\|, \quad \forall x_1, x_2 \in Y,$$

where c be defined in Lemma 2.3.

Lemma 2.5. [13] *Let Y be an uniformly convex and smooth Banach space and let $\{u_n\}, \{v_n\} \subset Y$ with either $\{u_n\}$ or $\{v_n\}$ is bounded. If $\lim_{n\to\infty} \phi(u_n, v_n) = 0$ then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.*

Remark 2.6. *Using* (11), *it is accessible that the converse of Lemma 2.5 is correct provided $\{u_n\}$ and $\{v_n\}$ both are bounded.*

Definition 2.7. [3, 16] *Assume $T : P \to P$ be a function. Then:*

(i) *a point $u_0 \in P$ is called an asymptotic fixed point of T if $\{u_n\} \subset P$ with $u_n \rightharpoonup u_0$ such that $\lim_{n\to\infty} \|Tu_n - u_n\| = 0$. $\widehat{F}(T)$ denotes asymptotic fixed points of T.*

(ii) *T is called relatively nonexpansive if $\widehat{F}(T) = F(T) \neq \emptyset$ and $\phi(u_0, Tu) \leq \phi(u_0, u)$, $\forall u \in P, u_0 \in F(T)$.*

(iii) *T is called quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(u_0, Tu) \leq \phi(u_0, u)$, $\forall u \in P, u_0 \in F(T)$.*

Lemma 2.8. [19] *Let Y be a uniformly convex and smooth Banach space, $P \subset Y$ be closed convex and $T : P \to P$ be closed and quasi-ϕ-nonexpansive function. Then, $F(T)$ is closed and convex.*

Lemma 2.9. [13] *Let Y be an uniformly and smooth convex Banach space. Then, $\exists\ g : [0, 2r] \to \mathbb{R}$ a strictly increasing, continuous and convex function, for $r > 0$ such that $g(0) = 0$ and $g(\|x_1 - x_2\|) \leq \phi(x_1, x_2)$, $\forall x_1, x_2 \in B_r$, where B_r be the closed ball of Y.*

Lemma 2.10. [22] *Let $B_r(0)$ be a closed ball of a uniformly convex Banach space Y, where $r > 0$. For $\{x_1, x_2, ..., x_N\} \subset B_r(0)$ and $\{\lambda_1, \lambda_2, ..., \lambda_N\}$ be positive numbers with $\sum_{i=1}^{N} \lambda_i = 1$, $\exists\ g : [0, 2r) \to [0, \infty)$ a continuous strictly increasing and convex function with $g(0) = 0$ such that*

$$\|\sum_{n=1}^{N} \lambda_n x_n\|^2 \leq \sum_{n=1}^{N} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|),\ \ i, j = 1, 2, ..., N,\ i < j.$$

Lemma 2.11. [17] *Let $P \neq \emptyset$ be closed convex subset of Y and $B : P \to Y^*$ be monotone and hemicontinuous function. Then, VIP(4) is closed and convex.*

Lemma 2.12. [13] *Let $P \neq \emptyset$ be closed convex subset of a strictly convex, reflexive and smooth Banach space Y. Then, $\exists$ a unique element $x_0 \in P$ such that $\phi(x_0, x_1) = \inf_{u\in P} \phi(u, x_1)$, for $x_1 \in Y$.*

Definition 2.13. [1] *A map $\Pi_P : Y \to P$ is said to be generalized projection if $\Pi_P x_1 = u_0$, for any $x_1 \in Y$ and u_0 be the solution of $\phi(u_0, x_1) = \inf_{u\in P} \phi(u, x_1)$.*

Lemma 2.14. [1] *Let $P \neq \emptyset$ be closed convex subset of a strictly convex, reflexive and smooth Banach space Y. Then*

$$\phi(u, \Pi_P x_1) + \phi(\Pi_P x_1, x_1) \leq \phi(u, x_1), \quad \forall u \in P \quad \text{and} \quad x_1 \in Y.$$

Also, for $x_1 \in Y$ and $u \in P$,

$$u = \Pi_P x_1 \iff \langle u - v, Jx_1 - Ju\rangle \geq 0, \quad \forall v \in P.$$

Assumption 2.15. *The bifunction $\mathbb{G} : P \times P \longrightarrow \mathbb{R}$ satisfies as:*

(i) *$\mathbb{G}(u, u) = 0$, $\forall u \in P$;*

(ii) *$\mathbb{G}(u, v) + \mathbb{G}(v, u) \leq 0$, $\forall u \in P$ i.e., $\mathbb{G}$ is monotone;*

(iii) *the mapping $u \mapsto \mathbb{G}(u, v)$ is upper hemicontinuity, $\forall v \in P$;*

(iv) *the mapping $v \mapsto \mathbb{G}(u, v)$, $\forall u \in P$ is lower semicontinuous and convex.*

Assumption 2.16. *The bifunction $b : P \times P \to \mathbb{R}$ satisfies as:*

(i) *$b(u, u) - b(u, v) - b(v, u) + b(v, v) \geq 0$, $\forall u, v \in P$, i.e., skew-symmetric;*

(ii) *convex in second argument and continuous.*

Lemma 2.17. [6] *Let Y be a strictly convex, uniformly smooth and reflexive Banach space and $P \subset Y$ be closed convex. Let $D : P \to Y^*$ be a continuous and monotone mapping, let $\mathbb{G} : P \times P \longrightarrow \mathbb{R}$ and $b : P \times P \to \mathbb{R}$ be bifunctions satisfying Assumption 2.15 and 2.16, respectively. For $x_1 \in Y$ and $r > 0$, define $\mathbb{T}_r : Y \to P$ such that:*

$$\mathbb{T}_r x_1 = \left\{v \in P : \mathbb{G}(v, u) + \langle Dv, u - v\rangle + b(v, u) - b(v, v) + \frac{1}{r}\langle u - v, Jv - Jx_1\rangle \geq 0, \forall u \in P\right\}. \tag{12}$$

Then the following holds:

(a) $\mathbb{T}_r$ *is single valued;*
(b) $\mathbb{T}_r$ *is firmly nonexpansive, i.e.,* $\forall x_1, x_2 \in Y$,

$$\langle \mathbb{T}_r x_1 - \mathbb{T}_r x_2, J\mathbb{T}_r x_1 - J\mathbb{T}_r x_2\rangle \leq \langle \mathbb{T}_r x_1 - \mathbb{T}_r x_2, Jx_1 - Jx_2\rangle;$$

(c) $F(\mathbb{T}_r) = \mathrm{Sol(GMEP(1))}$ *is closed and convex;*
(d) $\mathbb{T}_r$ *is quasi-*ϕ*-nonexpansive;*
(e) $\phi(u_0, \mathbb{T}_r x_1) + \phi(\mathbb{T}_r x_1, x_1) \leq \phi(u_0, x_1), \ \forall\, u_0 \in F(\mathbb{T}_r)$.

In continuation, the mapping $\Phi : Y \times Y^* \to \mathbb{R}$, defined by

$$\Phi(x_1, x_1^*) = \|x_1\|^2 - \langle x_1, x_1^*\rangle + \|x_1^*\|^2.$$

Examine that $\Phi(x_1, x_1^*) = \phi(x_1, J^{-1}x_1^*)$.

Lemma 2.18. [1] *Let Y be a strictly convex, smooth and reflexive Banach space. Then,*

$$\Phi(x_1, x_1^*) + 2\langle J^{-1}x_1^* - x_1, x_2^*\rangle \leq \Phi(x_1, x_1^* + x_2^*), \quad \forall x_1 \in Y,\ x_1^*, x_2^* \in Y^*.$$

3. Main Result

In this section, we provided our main theorem:

Theorem 3.1. *Let Y be a 2-uniformly convex and uniformly smooth real Banach space with dual Y^* and let $P \subset Y$ be nonempty closed and convex. Let $B : P \to Y^*$ be a $\gamma-$ ism mapping with constant $\gamma \in (0,1)$. Let $\mathbb{G} : P \times P \to \mathbb{R}$ and $b : P \times P \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.15 and 2.16, respectively and $D : P \to Y^*$ be a continuous and monotone mapping,. For each $i = 1, 2, ..., N$, let $T_i : P \to P$ be closed quasi-ϕ nonexpansive mappings such that $\Gamma := \mathrm{Sol(GMEP(1))} \bigcap \mathrm{Sol(VIP(4))} \bigcap (\bigcap_{i=1}^{N} \mathrm{F}(T_i)) \neq \emptyset$. Let $\{x_n\}$ generated by schemes:*

$$\left.\begin{aligned}
&x_0, x_1 \in P, \ \ P_1 := P,\\
&w_n = x_n + \theta_n(x_n - x_{n-1}),\\
&y_n = \Pi_C J^{-1}(Jw_n - \mu_n B w_n),\\
&v_n = J^{-1}(\alpha_{n,0} J w_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i w_n),\\
&z_n = J^{-1}(\delta_n J y_n + (1-\delta_n) J v_n),\\
&u_n = \mathbb{T}_{r_n} z_n,\\
&P_n = \{z \in P : \phi(z, u_n) \leq \phi(z, w_n)\},\\
&Q_n = \{z \in P : \langle x_n - z, Jx_n - Jx_0\rangle \leq 0\},\\
&x_{n+1} = \Pi_{P_n \bigcap Q_n} x_0, \ \ \forall n \geq 1.
\end{aligned}\right\} \tag{13}$$

Consider $\{\alpha_{n,i}\}$ and $\{\delta_n\}$ be sequences in $[0,1]$ and $\{\theta_n\} \subset (0,1)$ satisfying:

(i) $\sum_{i=0}^{N} \alpha_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} \geq 0$;
(iii) $\limsup_{n\to\infty} \delta_n < 1$;
(iv) $r_n \in [a, \infty)$, *for some* $a > 0$;
(v) $\{\mu_n\} \subset (0, \infty)$ *satisfying the condition* $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{c^2\gamma}{2}$, *where c be defined in Lemma 2.3.*

Then, $\{x_n\}$ strongly converges to x^*, where $x^* = \Pi_\Gamma x_0$, generalized projection of Y onto Γ.

Proof. We divide the proof into several steps.

Step 1. First, we prove that Γ is closed and convex.

By Lemmas 2.8, 2.11 and 2.17, $\Gamma \neq \emptyset$ be closed and convex and thus $\Pi_\Gamma x_0$ is well defined.

Step 2. Next, prove that $P_n \bigcap Q_n$ is closed and convex. From (13), it is obvious that Q_n is closed and convex. Clearly, $P_1 = P$ is closed and convex. Moreover, P_n be closed. Next, we prove the convexity of P_n. For $q_1, q_2 \in P_n$, we see that $q_1, q_2 \in P$. This adopt $tq_1 + (1-t)q_2 \in P$, where $t \in (0,1)$, and thus

$$\phi(q_1, u_n) \leq \phi(q_1, w_n) \tag{14}$$

and

$$\phi(q_2, u_n) \leq \phi(q_2, w_n). \tag{15}$$

The above two inequalities are equivalent to

$$2\langle q_1, Jw_n\rangle - 2\langle q_1, Ju_n\rangle \quad \leq \quad \|w_n\|^2 - \|u_n\|^2 \tag{16}$$

and

$$2\langle q_2, Jw_n\rangle - 2\langle q_2, Ju_n\rangle \quad \leq \quad \|w_n\|^2 - \|u_n\|^2. \tag{17}$$

By (16) and (17), we have

$$2\langle tq_1 + (1-t)q_2, Jw_n\rangle - 2\langle tq_1 + (1-t)q_2, Ju_n\rangle \leq \|w_n\|^2 - \|u_n\|^2. \tag{18}$$

Hence, we have

$$\phi(tq_1 + (1-t)q_2, u_n) \leq \phi(tq_1 + (1-t)q_2, w_n). \tag{19}$$

This implies that $tq_1 + (1-t)q_2 \in P_n$ and hence P_n is closed and convex. So, $P_n \bigcap Q_n$ is closed and convex, $\forall\, n \geq 1$.

Step 3. We claim that $\Gamma \subset P_n \bigcap Q_n$, $\forall n \geq 1$.

Let $x^* \in \Gamma$ and so

$$\begin{aligned} \phi(x^*, u_n) &= \phi(x^*, \mathbb{T}_{r_n} z_n) \\ &\leq \phi(x^*, z_n) \qquad (20) \\ &= \phi(x^*, J^{-1}(\delta_n Jy_n + (1-\delta_n)Jv_n)) \\ &\leq \delta_n \phi(x^*, y_n) + (1-\delta_n)\phi(x^*, v_n). \qquad (21) \end{aligned}$$

Using Lemma 2.10, we compute

$$\begin{aligned} \phi(x^*, v_n) &= \phi(x^*, J^{-1}(\alpha_{n,0}Jw_n + \sum_{i=1}^N \alpha_{n,i} JT_i w_n)) \\ &= \|x^*\|^2 - 2\langle x^*, \alpha_{n,0}Jw_n + \sum_{i=1}^N \alpha_{n,i} JT_i w_n\rangle + \|\alpha_{n,0}Jw_n + \sum_{i=1}^N \alpha_{n,i} JT_i w_n\|^2 \\ &\leq \|x^*\|^2 - 2\alpha_{n,0}\langle x^*, Jw_n\rangle - 2\sum_{i=1}^N \alpha_{n,i}\langle x^*, JT_i w_n\rangle \\ &\quad + \alpha_{n,0}\|Jw_n\|^2 + \sum_{i=1}^N \alpha_{n,i}\|JT_i w_n\|^2 - \alpha_{n,0}\alpha_{n,i} g\|Jw_n - JT_i w_n\| \\ &= \|x^*\|^2 - 2\alpha_{n,0}\langle x^*, Jw_n\rangle + \alpha_{n,0}\|Jw_n\|^2 \\ &\quad + \sum_{i=1}^N \alpha_{n,i}\|JT_i w_n\|^2 - \alpha_{n,0}\alpha_{n,j} g\|Jw_n - JT_i w_n\| \end{aligned}$$

$$
\begin{aligned}
&= \alpha_{n,0}\phi(x^*, w_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(x^*, T_i w_n) - \alpha_{n,0}\alpha_{n,j} g\|Jw_n - JT_i w_n\| \\
&\leq \alpha_{n,0}\phi(p, w_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(p, w_n) - \alpha_{n,0}\alpha_{n,i} g\|Jw_n - JT_i w_n\| \\
&\leq \sum_{i=0}^{N} \alpha_{n,i}\phi(x^*, w_n) - \alpha_{n,0}\alpha_{n,i} g\|Jw_n - JT_i w_n\| \\
&\leq \phi(x^*, w_n) - \alpha_{n,0}\alpha_{n,i} g\|Jw_n - JT_i w_n\|. \qquad (22) \\
&\leq \phi(x^*, w_n) \qquad (23)
\end{aligned}
$$

Using Lemmas 2.4 and 2.18, we compute

$$
\begin{aligned}
\phi(x^*, y_n) &= \phi(x^*, \Pi_C J^{-1}(Jw_n - \mu_n B w_n)) \\
&\leq \phi(x^*, J^{-1}(Jw_n - \mu_n B w_n)) \\
&= \Phi(x^*, Jw_n - \mu_n B w_n) \\
&\leq \Phi(x^*, (Jw_n - \mu_n B w_n) + \mu_n B w_n) - 2\langle J^{-1}(Jw_n - \mu_n B w_n) - x^*, \mu_n B w_n\rangle \\
&= \Phi(x^*, Jw_n) - 2\mu_n\langle J^{-1}(Jw_n - \mu_n D w_n) - x^*, B w_n\rangle \\
&= \phi(x^*, w_n) - 2\langle w_n - x^*, B w_n\rangle - 2\mu_n\langle J^{-1}(Jw_n - \mu_n B w_n) - w_n, B w_n\rangle \\
&= \phi(x^*, w_n) - 2\langle w_n - x^*, B w_n - B x^*\rangle - 2\mu_n\langle J^{-1}(Jw_n - \mu_n B w_n) - w_n, B w_n\rangle \\
&\leq \phi(x^*, w_n) - 2\mu_n\gamma\|Bw_n\|^2 + 2\mu_n\|J^{-1}(Jw_n - Bw_n) - J^{-1}Jw_n\|\|Bw_n\|^2 \\
&\leq \phi(x^*, w_n) - 2\mu_n\gamma\|Bw_n\|^2 + \frac{4\mu_n^2}{c^2}\|Bw_n\|^2 \\
&= \phi(x^*, w_n) - 2\mu_n(\gamma - \frac{2\mu_n}{c^2})\|Bw_n\|^2 \qquad (24)
\end{aligned}
$$

which combined with $\mu_n < \frac{c^2\gamma}{2}$, we have that

$$\phi(x^*, y_n) \leq \phi(x^*, w_n). \qquad (25)$$

By (21) (23) and (25) we observe that

$$\phi(x^*, u_n) \leq \phi(x^*, w_n). \qquad (26)$$

This implies that $x^* \in P_n$. Therefore, $\Gamma \subset P_n$, $\forall n \geq 1$.

After a while, by using induction we prove that $\Gamma \subset P_n \cap Q_n$, $\forall n \geq 1$. From $Q_1 = P$, we get $\Gamma \subset P_1 \cap Q_1$. Let $\Gamma \subset P_j \cap Q_j$, for arbitrary $j \in N$. So, $\exists\, x_{j+1} \in P_j \cap Q_j$ such that $x_{j+1} = \prod_{P_j \cap Q_j} x$. From the concept of x_{j+1}, we get, for all $x^* \in P_j \cap Q_j$,

$$\langle x_{j+1} - x^*, Jx_0 - Jx_{j+1}\rangle \geq 0.$$

Since $\Gamma \subset P_j \bigcap Q_j$, we have

$$\langle x_{j+1} - x^*, Jx_0 - Jx_{j+1}\rangle \geq 0, \quad \forall x^* \in \Gamma \qquad (27)$$

and hence $x^* \in Q_{j+1}$. So, we have $\Gamma \subset Q_{j+1}$. Therefore, we obtain $\Gamma \subset P_{j+1} \cap Q_{j+1}$.
Thus, $\Gamma \subset P_n \cap Q_n$, $\forall n \geq 1$. This means that $\{x_n\}$ is well-defined.

Step 4. Next, claim that $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, , $\{v_n\}$, $\{z_n\}$, $\{u_n\}$ are bounded, $\lim_{n\to\infty} \phi(x_n, x_0)$ exists and $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$.

By (13), we get $x_n = \Pi_{Q_n} x_0$. From $x_n = \Pi_{Q_n} x_0$ and Lemma 2.14, we get

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{Q_n} x_0, x_0) \\ &\leq \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \leq \phi(u, x_0), \quad \forall u \in \Gamma \subset Q_n.\end{aligned}$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded and hence, $\{x_n\}$ is bounded because of (9). Further,

$$\begin{aligned}\phi(x^*, x_n) &= \phi(x^*, \Pi_{P_{n-1} \bigcap Q_{n-1}} x_0) \\ &= \phi(x^*, x_0) - \phi(x_n, x_0),\end{aligned}$$

implies that $\{\phi(x^*, x_n)\}$ is bounded. Hence, $\{T_i x_n\}$ is also bounded because of the fact $\phi(x^*, T_i x_n) \leq \phi(x^*, x_n)$, $\forall p \in \Gamma$. Thus, $\{w_n\}$ is also bounded. From (23), it follows that $\{v_n\}$ is also bounded. By (25) and (26), $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are also bounded.

From $x_{n+1} = \Pi_{P_n \bigcap Q_n} x_0 \in Q_n$ and $x_n \in \Pi_{Q_n} x_0$, we get

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \ \ \forall n \geq 1.$$

This prove that $\{\phi(x_n, x_0)\}$ is nondecreasing. Thus, $\lim_{n\to\infty} \phi(x_n, x_0)$ exists because of the boundedness of $\{\phi(x_n, x_0)\}$. Further, we get

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0), \ \ \forall n \geq 1,\end{aligned}$$

which intends

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{28}$$

Using Lemma 2.5, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{29}$$

Step 5. We prove that $x_n \to x^*$, $z_n \to x^*$ and $u_n \to x^*$ as $n \to \infty$, where x^* be an arbitrary point in P.

As Y is reflexive and $\{x_n\}$ is bounded, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. On account of, $P_n \cap Q_n$ is closed and convex therefore $x^* \in P_n \cap Q_n$. Using weakly lower semicontinuity of $\|\cdot\|^2$, we get

$$\begin{aligned}\phi(x^*, x_0) &= \|x^*\|^2 - 2\langle x^*, Jx_0\rangle + \|x_0\|^2 \\ &\leq \liminf_{k\to\infty}(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2) \\ &= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\ &\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \\ &\leq \phi(x^*, x_0),\end{aligned}$$

which implies that $\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(x^*, x_0)$. Hence, $\lim_{k\to\infty} \|x_{n_k}\| = \|x^*\|$. Further, $x_{n_k} \to x^*$ as $k \to \infty$ because of Kadec-Klee property of Y. Since $\lim_{n\to\infty} \phi(x_n, x_0)$ exists therefore it yield that $\lim_{n\to\infty} \phi(x_n, x_0) = \phi(x^*, x_0)$. If $\exists$ subsequence $\{x_{n_j}\}$ of $\{x_n\}$ with $x_{n_j} \to \tilde{x}$ as $j \to \infty$, then

$$\begin{aligned}\phi(x^*, \tilde{x}) &= \lim_{k,j\to\infty} \phi(x_{n_k}, x_{n_j}) \\ &= \lim_{k,j\to\infty} \phi(x_{n_k}, \Pi_{Q_{n_j}} x_0) \\ &\leq \lim_{k,j\to\infty} \{\phi(x_{n_k}, x_0) - \phi(x_{n_j}, x_0)\} = 0,\end{aligned}$$

that is, $x^* = \tilde{x}$ and thus $x_n \to x^*$ as $n \to \infty$.

Since $\|w_n - x_n\| = \|\theta_n(x_n - x_{n-1})\| \le \|x_n - x_{n-1}\|$ and using (29), we get

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{30}$$

By Remark 2.6 and using bundedness of $\{w_n\}$, we have

$$\lim_{n\to\infty} \phi(x_n, w_n) = 0. \tag{31}$$

By (29) and (30), we have

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0, \tag{32}$$

it follows from Remark 2.6

$$\lim_{n\to\infty} \phi(x_{n+1}, w_n) = 0. \tag{33}$$

As $x_{n+1} = \prod_{P_n \bigcap Q_n} x_0 \in P_n$, we have

$$\phi(x_{n+1}, u_n) \quad \le \quad \phi(x_{n+1}, w_n).$$

Using (33), we get

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{34}$$

By (9), we have

$$\lim_{n\to\infty} (\|x_{n+1}\| - \|u_n\|) = 0,$$

which intend

$$\lim_{n\to\infty} \|u_n\| = \|x^*\|, \text{ provided } \lim_{n\to\infty} \|x_n\| = \|x^*\|. \tag{35}$$

Hence, we have

$$\lim_{n\to\infty} \|Ju_n\| = \lim_{n\to\infty} \|u_n\| = \|x^*\| = \|Jx^*\|, \tag{36}$$

which suggest that $\{\|Ju_n\|\}$ is bounded. Since Y and Y^* are reflexive, we may consider $Ju_n \rightharpoonup y^* \in Y^*$. Thanks to the reflexivity of Y, $J(Y) = Y^*$, i.e., $\exists\, y \in Y$ such that $Jy = y^*$, which intend

$$\phi(x_{n+1}, u_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2$$

$$\phi(x_{n+1}, u_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2.$$

Further, in above equation taking limit infimum as $n \to \infty$, we have

$$\begin{aligned} 0 \quad &\ge \quad \|x^*\|^2 - 2\langle x^* x, y^*\rangle + \|y^*\|^2 \\ &= \quad \|x^*\|^2 - 2\langle x^*, Jy\rangle + \|Jy\|^2 \\ &= \quad \|x^*\|^2 - 2\langle x^*, Jy\rangle + \|y\|^2 \\ &= \quad \phi(x^*, y), \end{aligned}$$

i.e., $x^* = y$ and hence, $y^* = Jx^*$. Thus, $Ju_n \rightharpoonup Jx^* \in Y^*$. Thanks to Kadec-Klee property of Y^* and (36), we get

$$\lim_{n\to\infty} \|Ju_n - Jx^*\| = 0.$$

By the demicontinuity of J^{-1}, we have $u_n \rightharpoonup x^*$. Thanks to Kadec-Klee property of Y and (35), we get

$$\lim_{n\to\infty} u_n = x^*. \tag{37}$$

By the weakly lower semicontinuity of $\|\cdot\|^2$ and for any $\hat{x} \in \Gamma$, we calculate

$$\begin{aligned}
\phi(\hat{x}, x^*) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx^*\rangle + \|x^*\|^2 \\
&\leq \liminf_{n\to\infty}(\|\hat{x}\|^2 - 2\langle \hat{x}, Ju_n\rangle + \|u_n\|^2) \\
&= \liminf_{n\to\infty} \phi(\hat{x}, u_n) \\
&\leq \limsup_{n\to\infty} \phi(\hat{x}, u_n) \\
&= \limsup_{n\to\infty}(\|\hat{x}\|^2 - 2\langle \hat{x}, Ju_n\rangle + \|u_n\|^2) \\
&\leq \phi(\hat{x}, x^*),
\end{aligned}$$

which intend

$$\lim_{n\to\infty} \phi(\hat{x}, u_n) = \phi(\hat{x}, x^*). \tag{38}$$

As $x_n \to x^*$, $n \to \infty$ and (37), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{39}$$

By the uniform continuity of J, we get

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{40}$$

By the concept of ϕ and for any $\hat{x} \in \Gamma$, we calculate

$$\begin{aligned}
\phi(\hat{x}, x_n) - \phi(\hat{x}, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle \hat{x}, Jx_n - Ju_n\rangle \\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|\hat{x}\|\|Jx_n - Ju_n\|.
\end{aligned}$$

By (39) and (40), we get

$$\lim_{n\to\infty} \{\phi(\hat{x}, x_n) - \phi(\hat{x}, u_n)\} = 0. \tag{41}$$

By (38) and (41), we get

$$\lim_{n\to\infty} \phi(\hat{x}, x_n) = \phi(\hat{x}, x^*). \tag{42}$$

Again, by using weakly lower semicontinuity of $\|\cdot\|^2$ and for any $\hat{x} \in \Gamma$, we get

$$\begin{aligned}
\phi(\hat{x}, x^*) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx^*\rangle + \|x^*\|^2 \\
&\leq \liminf_{n\to\infty}(\|\hat{x}\|^2 - 2\langle \hat{x}, Jw_n\rangle + \|w_n\|^2) \\
&= \liminf_{n\to\infty} \phi(\hat{x}, w_n) \\
&\leq \limsup_{n\to\infty} \phi(\hat{x}, w_n) \\
&= \limsup_{n\to\infty}(\|\hat{x}\|^2 - 2\langle \hat{x}, Jw_n\rangle + \|w_n\|^2) \\
&\leq \phi(\hat{x}, x^*),
\end{aligned}$$

which yield

$$\lim_{n\to\infty} \phi(\hat{x}, w_n) = \phi(\hat{x}, x^*). \tag{43}$$

Hence, for any $\hat{x} \in \Gamma \subset P_n$ and by (23), we have

$$\phi(\hat{x}, v_n) \quad \leq \quad \phi(\hat{x}, w_n). \tag{44}$$

Using (43) and (44), we get

$$\lim_{n\to\infty} \phi(\hat{x}, v_n) = \phi(\hat{x}, x^*). \tag{45}$$

By (20), (26), Lemma 2.17(e) and $u_n = \mathbb{T}_{r_n} z_n$, we have for any $\hat{x} \in \Gamma$

$$\begin{aligned}\phi(u_n, z_n) &= \phi(\mathbb{T}_{r_n} z_n, z_n)\\ &\leq \phi(\hat{x}, z_n) - \phi(\hat{x}, \mathbb{T}_{r_n} z_n)\\ &= \phi(\hat{x}, w_n) - \phi(\hat{x}, u_n).\end{aligned}$$

By (38), (43) and taking $n \to \infty$, we get

$$\lim_{n\to\infty} \phi(u_n, z_n) = 0, \tag{46}$$

and hence from (9), we have

$$\lim_{n\to\infty} (\|u_n\| - \|z_n\|) = 0.$$

By relation (35), we have

$$\lim_{n\to\infty} \|z_n\| = \|x^*\|, \tag{47}$$

and hence

$$\lim_{n\to\infty} \|Jz_n\| = \|Jx^*\|, \tag{48}$$

i.e., $\{\|Jz_n\|\}$ is bounded in Y^*. By reflexivity of Y^*, we consider $Jz_n \rightharpoonup y^* \in Y^*$ as $n \to \infty$. As $J(Y) = Y^*$ $\exists\, y \in Y$ such that $Jy = y^*$. Thus,

$$\begin{aligned}\phi(u_n, z_n) &= \|u_n\|^2 - 2\langle u_n, Jz_n\rangle + \|z_n\|^2\\ &= \|u_n\|^2 - 2\langle u_n, Jz_n\rangle + \|Jz_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ in above equation, we have

$$\begin{aligned}0 &\geq \|x^*\|^2 - 2\langle x^*, y^*\rangle + \|y^*\|^2\\ &= \|x^*\|^2 - 2\langle x^*, Jy\rangle + \|Jy\|^2\\ &= \|x^*\|^2 - 2\langle x^*, Jy\rangle + \|y\|^2\\ &= \phi(x^*, y).\end{aligned}$$

From Remark 2.1, we have $x^* = y$, i.e., $y^* = Jx^*$. Thus, $Jz_n \rightharpoonup Jx^* \in Y^*$. Thanks to Kadec-Klee property of Y^* and (48)

$$\lim_{n\to\infty} \|Jz_n - Jx^*\| = 0.$$

Using demicontinuity of J^{-1} in above yield $z_n \rightharpoonup x^*$. Thanks to Kadec-Klee property of Y and (47), we get

$$\lim_{n\to\infty} z_n = x^*.$$

Step 6. Next, claim that $x^* \in \Gamma$.

By Lemma 2.5 and (46), we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{49}$$

By the uniform continuity of J, we get

$$\lim_{n\to\infty} \|Ju_n - Jz_n\| = 0. \tag{50}$$

Further, by (30), (39) and (49), we get

$$\begin{aligned} \|w_n - z_n\| &\le \|w_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{51}$$

Again, by uniform continuity of J, we have

$$\lim_{n\to\infty} \|Jw_n - Jz_n\| = 0. \tag{52}$$

By (20), (21), (23) and (24), we obtain for any $\hat{x} \in \Gamma$

$$\begin{aligned} \phi(\hat{x}, z_n) &\le \delta_n \phi(\hat{x}, y_n) + (1-\delta_n)\phi(\hat{x}, v_n) \\ &\le \phi(\hat{x}, w_n) - 2\mu_n\delta_n(\gamma - \frac{2\mu_n}{c^2})\|Bw_n\|^2, \qquad (53) \\ & \qquad (54) \end{aligned}$$

this implies that

$$\begin{aligned} 2\mu_n\delta_n(\gamma - \frac{2\mu_n}{c^2})]\|Bw_n\|^2 &\le \phi(\hat{x}, w_n) - \phi(\hat{x}, z_n) \\ &= \|w_n\|^2 - \|z_n\|^2 - 2\langle \hat{x}, Jw_n - Jz_n\rangle \\ &\le \|w_n - z_n\|(\|w_n\| + \|z_n\|) + 2\|\hat{x}\|\|Jw_n - Jz_n\|, \end{aligned} \tag{55}$$

it follows from (51),(52), (55) and $\mu_n\delta_n(\gamma - \frac{2\mu_n}{c^2}) > 0$ that

$$\lim_{n\to\infty} \|Bw_n\| = 0. \tag{56}$$

Since B is γ-ism and so $\frac{1}{\gamma}$-Lipschitz continuous. It immediately follows from $\lim_{n\to\infty} w_n = x^*$ and (56) that $x^* \in B^{-1}(0)$. Thus, $x^* \in \mathrm{Sol(VIP(4))}$.

Furthermore, combining (13) with (56) yields that

$$\begin{aligned} \lim_{n\to\infty} \|y_n - x^*\| &= \lim_{n\to\infty} \|\Pi_C J^{-1}(Jw_n - \mu_n Bw_n) - \Pi_C x^*\| \\ &\le \lim_{n\to\infty} \|J^{-1}(Jw_n - \mu_n Bw_n) - x^*\| \\ &= 0. \end{aligned} \tag{57}$$

Using Lemma 2.4 and 2.18, we estimate

$$\begin{aligned} \phi(w_n, y_n) &= \phi(w_n, \Pi_C J^{-1}(Jw_n - \mu_n Bw_n)) \\ &\le \phi(w_n, J^{-1}(Jw_n - \mu_n Bw_n)) \\ &\le \Phi(w_n, (Jw_n - \mu_n Bw_n)) \\ &\le \Phi(w_n, (Jw_n - \mu_n Bw_n) + \mu_n Bw_n) - 2\langle J^{-1}(Jw_n - \mu_n Bw_n) - w_n, \mu_n Bw_n\rangle \\ &= \phi(w_n, w_n) + 2\langle J^{-1}(Jw_n - \mu_n Bw_n) - w_n, -\mu_n Bw_n\rangle \\ &= 2\mu_n\langle J^{-1}(Jw_n - \mu_n Bw_n) - w_n, -Bw_n\rangle \\ &\le \|J^{-1}(Jw_n - \mu_n Bw_n) - J^{-1}Jw_n\| \\ &\le \frac{4}{c^2}\mu_n^2\|Bw_n\|^2, \end{aligned} \tag{58}$$

then using (56) we obtain that

$$\lim_{n\to\infty} \phi(w_n, y_n) = 0. \tag{59}$$

By Lemma 2.5, we get

$$\lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{60}$$

Further, by (37), (57) and (60), we get

$$\begin{aligned} \|u_n - w_n\| &= \|u_n - y_n + y_n - w_n\| \\ &\leq \|u_n - y_n\| + \|w_n - y_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{61}$$

From $r_n \geq a$ and (50), we have

$$\lim_{n\to\infty} \frac{\|Ju_n - Jz_n\|}{r_n} = 0. \tag{62}$$

By $u_n = \mathbb{T}_{r_n} z_n$, we obtain

$$\mathbb{G}(u_n, v) + \langle Du_n, v - u_n\rangle + b(v, u_n) - b(u_n, u_n) + \frac{1}{r_n}\langle v - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall v \in P.$$

Using Assumption 2.15(ii), we have

$$\begin{aligned} \frac{1}{r_n}\langle v - u_n, Ju_n - Jz_n\rangle &\geq -\mathbb{G}(u_n, v) + \langle Du_n, u_n - v\rangle - b(v, u_n) + b(u_n, u_n) \\ &\geq \mathbb{G}(v, u_n) + \langle Du_n, u_n - v\rangle - b(v, u_n) + b(u_n, u_n). \end{aligned}$$

Letting $n \to \infty$, from (62) and by Assumption 2.15 (iv), we obtain

$$\mathbb{G}(v, x^*) + \langle Dx^*, x^* - v\rangle - b(v, x^*) + b(x^*, x^*) \leq 0, \quad \forall v \in P.$$

For all $t \in (0, 1]$ and $v \in P$, setting $v_t := tv + (1 - t)x^*$. Hence, $v_t \in P$ and thus

$$\mathbb{G}(v_t, x^*) + \langle Dx^*, x^* - v_t\rangle - b(v_t, x^*) + b(x^*, x^*) \leq 0.$$

By Assumption 2.15(i)-(iv), we get

$$\begin{aligned} 0 &= \mathbb{G}(v_t, v_t) \\ &\leq t\mathbb{G}(v_t, v) + (1 - t)\mathbb{G}(v_t, x^*) \\ &\leq t\mathbb{G}(v_t, v) + (1 - t)[b(v_t, x^*) - b(x^*, x^*) + \langle Dx^*, v_t - x^*\rangle]. \\ &\leq t\mathbb{G}(v_t, v) + t(1 - t)[b(v, x^*) - b(x^*, x^*) + \langle Dx^*, v - x^*\rangle], \end{aligned}$$

which yields

$$\mathbb{G}(x^*, v) + \langle Dx^*, v - x^*\rangle + b(v, x^*) - b(x^*, x^*) \geq 0, \quad \forall v \in P.$$

Thus, $x^* \in \mathrm{Sol(GMEP(1))}$.

Further, claim that $x^* \in \bigcap_{i=1}^{N} \mathrm{F}(T_i)$.

Using (21), (22) into (25), we have for any $\hat{x} \in \Gamma$

$$\begin{aligned} \phi(\hat{x}, u_n) &\leq \delta_n\phi(\hat{x}, y_n) + (1 - \delta_n)\phi(\hat{x}, v_n) \\ &\leq \delta_n\phi(\hat{x}, w_n) + (1 - \delta_n)[\phi(\hat{x}, w_n) - \alpha_{n,0}\alpha_{n,j}g\|Jw_n - JT_iw_n\|] \\ &\leq \phi(\hat{x}, w_n) - (1 - \delta_n)\alpha_{n,0}\alpha_{n,j}g\|Jw_n - JT_iw_n\|. \end{aligned}$$

This implies that

$$(1-\delta_n)\alpha_{n,0}\alpha_{n,j}g\|Jw_n - JT_iw_n\| \le \phi(\hat{x}, w_n) - \phi(\hat{x}, u_n). \tag{63}$$

Now,

$$\begin{aligned}\phi(\hat{x}, w_n) - \phi(\hat{x}, u_n) &= \|w_n\|^2 - \|u_n\|^2 - 2\langle \hat{x}, Jw_n - Ju_n\rangle \\ &\le \|w_n - u_n\|(\|w_n\| + \|u_n\|) + 2\|\hat{x}\|\|Jw_n - Ju_n\|.\end{aligned}$$

Using (61) and the property of J in above inequality, we have

$$\lim_{n\to\infty}(\phi(\hat{x}, w_n) - \phi(\hat{x}, u_n)) = 0. \tag{64}$$

By Lemma 2.9 and given conditions in (63), we have

$$\lim_{n\to\infty} g(\|JT_iw_n - Jw_n\|) = 0.$$

Using the concept of g

$$\lim_{n\to\infty} \|JT_iw_n - Jw_n\| = 0,$$

which yield

$$\lim_{n\to\infty} \|T_iw_n - w_n\| = 0. \tag{65}$$

By (32), (60), (49) and (61), we observe that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{w_n\}$ and $\{z_n\}$ all have the same asymptotic behaviour, hence from (65), we have that

$$\lim_{n\to\infty} \|T_ix_n - x_n\| = 0. \tag{66}$$

This means that $x^* = T_ix^*$, i.e., $x^* \in \bigcap_{i=1}^{N} \mathrm{F}(T_i)$. Then, $x^* \in \mathrm{Sol(GMEP(1))} \cap \mathrm{Sol(VIP(4))} \bigcap(\bigcap_{i=1}^{N} \mathrm{F}(T_i))$.

Step 7. Finally, we show $x^* = \Pi_\Gamma x_0$. Taking $k \to \infty$ in (27), we obtain

$$\langle x^* - \hat{x}, Jx_0 - Jx^*\rangle \ge 0, \ \ \forall \hat{x} \in \Gamma.$$

Using Lemma 2.14, we get $x^* = \Pi_\Gamma x_0$. □

We provided some consequences from our main Theorem 3.1:

Corollary 3.2. *Let Y be a uniformly convex and uniformly smooth real Banach space with dual Y^* and let $P \subset Y$ be nonempty closed and convex. Let $\mathbb{G} : P \times P \to \mathbb{R}$ and $b : P \times P \to \mathbb{R}$ be bifunctions satisfying Assumptions* 2.15 *and* 2.16, *respectively and $D : P \to Y^*$ be a continuous and monotone mapping. For each $i = 1, 2, ..., N$, let $T_i : P \to P$ be closed quasi-ϕ nonexpansive mappings such that $\Gamma := \mathrm{Sol(GMEP(1))} \bigcap(\bigcap_{i=1}^{N} \mathrm{F}(T_i)) \ne \emptyset$. Let $\{x_n\}$ generated by schemes:*

$$\left.\begin{aligned}
&x_0, x_1 \in P, \ \ P_1 := P, \\
&w_n = x_n + \theta_n(x_n - x_{n-1}), \\
&v_n = J^{-1}(\alpha_{n,0}Jw_n + \sum_{i=1}^{N} \alpha_{n,i}JT_iw_n), \\
&z_n = J^{-1}(\delta_nJw_n + (1-\delta_n)Jv_n), \\
&u_n = \mathbb{T}_{r_n}z_n, \\
&P_n = \{z \in P : \phi(z, u_n) \le \phi(z, w_n)\}, \\
&Q_n = \{z \in P : \langle x_n - z, Jx_n - Jx_0\rangle \le 0\}, \\
&x_{n+1} = \Pi_{P_n \bigcap Q_n}x_0, \ \ \forall n \ge 1.
\end{aligned}\right\} \tag{67}$$

Consider $\{\alpha_{n,i}\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ and $\{\theta_n\} \subset (0, 1)$ satisfying:

(i) $\sum\limits_{i=0}^{N} \alpha_{n,i} = 1$;

(ii) $\liminf\limits_{n\to\infty} \alpha_{n,0}\alpha_{n,i} \geq 0$;

(iii) $\limsup_{n\to\infty} \delta_n < 1$;

(iv) $r_n \in [a, \infty)$, *for some* $a > 0$.

Then, $\{x_n\}$ *strongly converges to* x^*, *where* $x^* = \Pi_\Gamma x_0$, *generalized projection of* Y *onto* Γ.

Corollary 3.3. *Let* Y *be a uniformly convex and uniformly smooth real Banach space with dual* Y^* *and let* $P \subset Y$ *be nonempty closed and convex. Let* $\mathbb{G} : P \times P \to \mathbb{R}$ *be bifunction satisfying Assumption* 2.15 *and* $D : P \to Y^*$ *be a continuous and monotone mapping. For each* $i = 1, 2, ..., N$, *let* $T_i : P \to P$ *be closed quasi-*ϕ *nonexpansive mappings such that* $\Gamma := \mathrm{Sol(GEP(2))} \bigcap (\bigcap\limits_{i=1}^{N} \mathrm{F}(T_i)) \neq \emptyset$. *Let* $\{x_n\}$ *generated by schemes:*

$$\left.\begin{array}{l} x_0, x_1 \in P, \ \ P_1 := P, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = J^{-1}(\alpha_{n,0}Jw_n + \sum\limits_{i=1}^{N} \alpha_{n,i}JT_iw_n), \\ z_n = J^{-1}(\delta_nJw_n + (1-\delta_n)Jv_n), \\ u_n = \mathbb{T}_{r_n}z_n, \\ P_n = \{z \in P : \phi(z, u_n) \leq \phi(z, w_n)\}, \\ Q_n = \{z \in P : \langle x_n - z, Jx_n - Jx_0\rangle \leq 0\}, \\ x_{n+1} = \Pi_{P_n \bigcap Q_n}x_0, \ \ \forall n \geq 1. \end{array}\right\} \tag{68}$$

Consider $\{\alpha_{n,i}\}$ *and* $\{\delta_n\}$ *be sequences in* $[0, 1]$ *and* $\{\theta_n\} \subset (0, 1)$ *satisfying:*

(i) $\sum\limits_{i=0}^{N} \alpha_{n,i} = 1$;

(ii) $\liminf\limits_{n\to\infty} \alpha_{n,0}\alpha_{n,i} \geq 0$;

(iii) $\limsup_{n\to\infty} \delta_n < 1$;

(iv) $r_n \in [a, \infty)$, *for some* $a > 0$.

Then, $\{x_n\}$ *strongly converges to* x^*, *where* $x^* = \Pi_\Gamma x_0$, *generalized projection of* Y *onto* Γ.

Remark 3.4. *If* Y *is a Hilbert space, then we have* $Y^* = Y$, $J = J^{-1} = I$, *an identity mapping,* $\phi(x_1, x_2) = \|x_1 - x_2\|^2$, for all $x_1, x_2 \in Y$, $c = 1$, *the two uniformly convex constant,* $\Pi_P = \mathbb{P}_P$, *projection mapping onto* P *and nonexpansive mappings* T_i, *for each* $i = 1, 2, ..., N$ *with* $\bigcap\limits_{i=1}^{N} \mathrm{F}(T_i) \neq \emptyset$ *are quasi-*ϕ *nonexpansive mappings. Thus, if one replaces quasi-*ϕ *nonexpansive mappings into nonexpansive mappings with* $\bigcap\limits_{i=1}^{N} \mathrm{F}(T_i) \neq \emptyset$ *in a Hilbert space then the assertions of Theorem* 3.1 *remain valid.*

4. Numerical Example

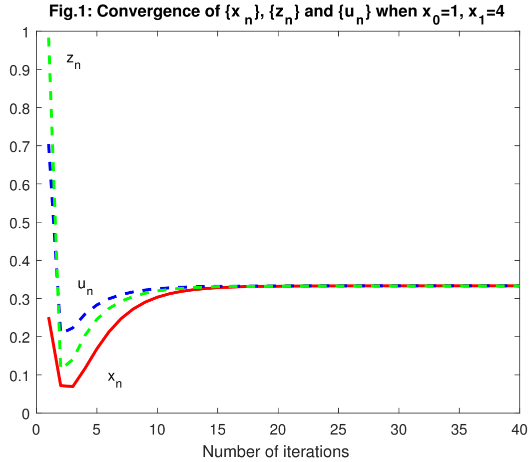
Example 4.1. *Let $Y = \mathbb{R}$, $P = [a, b]$, where $a, b \in \mathbb{R}$ but fixed, and let $\mathbb{G} : P \times P \to \mathbb{R}$ be defined by $\mathbb{G}(u, v) = (u-1)(v-u), \forall u, v \in P$ and $b : P \times P \to \mathbb{R}$ be defined by $b(u, v) = uv$, $\forall u, v \in P$; let $D : P \to \mathbb{R}$ be defined by $D(u) = u$, $\forall u \in P$. Let $B : P \to \mathbb{R}$ be defined by $Bu = (3u-1)$; let $T_i : P \to P$ be defined by $T_i u = \frac{u+i}{1+3i}u$. Setting $\{\mu_n\} = \{\frac{0.9}{n}\}, r_n = \frac{1}{4}$, $\theta_n = 0.9$, $\alpha_{n,0} = \frac{1}{2}$, $\sum_{i=1}^{N} \alpha_{n,i} = \frac{1}{2}$ such that $\sum_{i=0}^{N} \alpha_{n,i} = 1$ and $\{\delta_n\} = \{\frac{1}{n^3}\}$, $\forall n \geq 1$. Let $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ be generated by the hybrid iterative algorithm (13) converges to $x^* = \{\frac{1}{3}\} \in \Gamma$:*

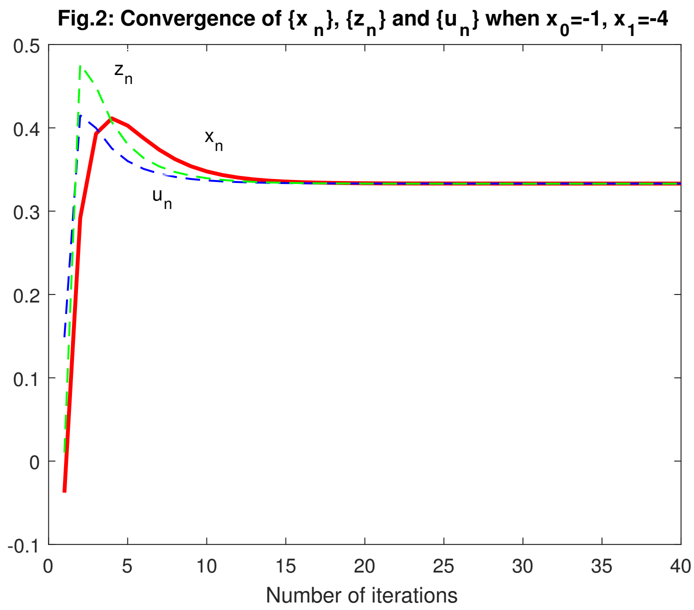
Proof. Obviously $\mathbb{G}$ and b satisfy Assumptions 2.15 and 2.16, respectively and D is continuous and monotone and hence $\mathrm{Sol(GMEP(1))} = \{\frac{1}{3}\} \neq \emptyset$. Also, B is $\frac{1}{3}$-ism and $\mathrm{Sol(VIP(4))} = \{\frac{1}{3}\} \neq \emptyset$. And T is quasi-ϕ-nonexpansive with $\mathrm{Fix}(T_i) = \{\frac{1}{3}\}$. Thus, $\Gamma := \mathrm{Sol(GMEP(1))} \cap \mathrm{Sol(VIP(4))} \cap \mathrm{F}(T_i) = \{\frac{1}{3}\} \neq \emptyset$. The iterative scheme (13) becomes following scheme after simplification: Initial values given x_0,x_1,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = \mathbb{P}_P(w_n - \mu_n B w_n) = \begin{cases} 0, & \text{if} x < 0, \\ 1, & \text{if} x > 1, \\ w_n - \mu_n \frac{w_n}{2}, & \text{otherwise}. \end{cases}, \\ v_n = \alpha_{n,0} w_n + \sum_{i=1}^{N} \alpha_{n,i} w_n; \quad z_n = \delta_n y_n + (1-\delta_n) v_n; \quad u_n = \dfrac{1 + 4z_n}{7}; \\ C_n = [e_n, \infty), \quad \text{where } e_n = \dfrac{u_n + w_n}{2}; \\ Q_n = [x_n, \infty); \\ x_{n+1} = \mathbb{P}_{P_n \cap Q_n} x_0, \quad \forall n \geq 1, \ \mathbb{P} \text{ denotes the metric projection.} \end{cases} \tag{69}$$

Finally, using the software Matlab 7.8.0, we have following figures which show that $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge to $\hat{x} = \{\frac{1}{3}\}$ as $n \to +\infty$. □

Fig.1: Convergence of {x n}, {z n} and {u n} when x 0=1, x 1=4





5. Conclusions

We proposed an inertial type gradient projection hybrid iterative algorithm for finding a common solution of GMEP(1), VIP(4) for a γ-ism and FPP(5) for a family of quasi-ϕ-nonexpansive mappings in two-uniformly convex and uniformly smooth Banach space. Theorem 3.1 is an upgrade of the result of [16] and [5] in the following sense:

(i) In [16], the authors studied and analyzed a convergence theorem for a relatively nonexpansive mapping whereas in our Theorem 3.1, a convergence theorem is showed for a family of quasi-ϕ nonexpansive mappings.

(ii) In [5], the authors studied convergence analysis theorem in a real Hilbert space for one nonexpansive mapping where as in our Theorem, we studied in the much more general 2-uniformly convex and uniformly smooth Banach space and for a family of quasi-ϕ nonexpansive mappings.

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