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Geometric characterizations of canal hypersurfaces in Euclidean spaces

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Abstract. In the present paper, firstly we obtain the general expression of canal hypersurfaces in Euclidean n-space and deal with canal hypersurfaces in Euclidean 4-space E^4. We compute Gauss map, Gaussian curvature and mean curvature of canal hypersurfaces in E^4 and obtain an important relation between the mean and Gaussian curvatures as $3H\rho = K\rho^3 - 2$. We prove that, the flat canal hypersurfaces in Euclidean 4-space are only circular hypercylinders or circular hypercones and minimal canal hypersurfaces are only generalized catenoids. Also, we state the expression of tubular hypersurfaces in Euclidean spaces and give some results about Weingarten tubular hypersurfaces in E^4.

1. Introduction

A canal surface is formed by the envelope of the spheres whose centers lie on a curve and radius vary depending on this curve [5]. In this sense, let $\Lambda := \alpha(u) = (a(u), b(u), c(u))$ be a regular space curve and $\rho(u)$ be a C^1-function with $\rho > 0$ and $|\dot{\rho}| < \|\dot{\alpha}\|$. The envelope of the one parameter family of spheres

$$(x - \alpha(u))^2 - \rho(u)^2 = 0 \tag{1}$$

is called a *canal surface* and Λ its *directrix* in Euclidean 3-space. Also, the parametric representation of canal surfaces can be given by

$$\mathbf{x} = \mathbf{x}(u, v) := \alpha(u) - \frac{\rho(u)\dot{\rho}(u)}{\|\dot{\alpha}(u)\|^2}\dot{\alpha}(u) + \frac{\rho(u)\sqrt{\|\dot{\alpha}(u)\|^2 - \dot{\rho}(u)^2}}{\|\dot{\alpha}(u)\|}(e_1(u)\cos(v) + e_2(u)\sin(v)), \tag{2}$$

where $\{e_1, e_2\}$ is an orthonormal base orthogonal to tangent vector $\dot{\alpha}$. In case of a constant radius function $\rho(u)$, the envelope is called *tubular* or *pipe surface* (see [7]). Canal surfaces (especially *tubular* surfaces) have been applied to many fields, such as the solid and the surface modeling for CAD/CAM, construction of blending surfaces, shape re-construction and so on.

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In this context, canal and tubular (hyper)surfaces have been studied by many mathematicians in different spaces. For instance, the notion of special conformally flat spaces which generalizes that of subprojective spaces has been introduced in [2] and the authors have proved that every canal hypersurface of a Euclidean space is a special conformally flat space and it is a subprojective space if and only if it is a surface of revolution. In [8], a relationship between the caustics of a submanifold of general dimension and of a canal hypersurface of the submanifold in Euclidean space has been investigated and as a consequence, it has been seen that these caustics are same. Analytic and algebraic properties of canal surfaces have been studied in [20]. In [1], the authors have shown that canal surfaces and tube surfaces can be obtained by the quaternion product and by the matrix representation and also in [16], it is shown that any canal surface to a rational spine curve and a rational radius function possesses rational parametrizations. The principal curvatures and principal curvature lines on canal surfaces have been determined in [4] and by means of a connection of the differential equations for these curvature lines and real Riccati equations, it has been established that canal surfaces have at most two isolated periodic principal lines. Some interesting and important relations about the Gaussian curvature, the mean curvature and the second Gaussian curvature have been found and based on these relations, some canal surfaces have been characterized in [12]. Classification of cyclic surfaces which is formed by movement of a circle of variable or constant radius under any law in a three dimensional space and geometrical research of canal surfaces have been given in [13].

Furthermore, for different studies of canal and tubular surfaces in different spaces such as Minkowskian, Galilean and pseudo-Galilean, we refer to [9], [10], [11], [14], [15], [18], [19], etc.

In the second section of this paper, we obtain the general expression of canal hypersurfaces in Euclidean n-space. In the third section, after recalling some basic notions about hypersurfaces and stating the expression of canal hypersurfaces in 4-dimensional Euclidean space, we obtain the Gaussian curvature and the mean curvature of canal hypersurfaces in E^4 and give an important relation between these curvatures. Moreover, we study on tubular hypersurfaces in this section.

2. Expression of Canal Hypersurfaces in Euclidean *n*-Space

Let a center curve $\alpha : I \subseteq \mathbb{R} \to E^n$ be a curve with non-zero curvature and arc-length parametrization. Then, the parametrization of the envelope of hypersphere defining the canal hypersurface X in E^n can be given by

$$X(v_1, v_2, v_3, ..., v_{n-1}) - \alpha(v_1) = \sum_{i=1}^{n} a_i(v_1, v_2, v_3, ..., v_{n-1}) F_i(v_1), \ i \in \{1, 2, 3, ..., n\}, \tag{3}$$

where $F_i(v_1)$ are Frenet vectors of $\alpha(v_1)$ and a_i are differentiable functions of $v_1, v_2, v_3, ..., v_{n-1}$ on the interval I. Furthermore, since $X(v_1, v_2, v_3, ..., v_{n-1})$ lies on the hypersphere, we have

$$\langle X(v_1, v_2, v_3, ..., v_{n-1}) - \alpha(v_1), X(v_1, v_2, v_3, ..., v_{n-1}) - \alpha(v_1)\rangle = \rho^2(v_1) \tag{4}$$

which leads to from (3) that

$$\sum_{i=1}^{n} (a_i(v_1, v_2, v_3, ..., v_{n-1}))^2 = \rho^2(v_1) \tag{5}$$

and

$$\sum_{i=1}^{n} a_i(v_1, v_2, v_3, ..., v_{n-1})(a_i(v_1, v_2, v_3, ..., v_{n-1}))_{v_1} = \rho(v_1)\rho'(v_1). \tag{6}$$

Here, $\rho(v_1)$ is the radius function of hypersurface X and we note that, throughout this study, we state $\rho'(v_1) = \frac{d\rho(v_1)}{dv_1}$, $(a_i(v_1, v_2, v_3, ..., v_{n-1}))_{v_i} = \frac{\partial a_i(v_1,v_2,v_3,...,v_{n-1})}{\partial v_i}$, $(X_i(v_1, v_2, v_3, ..., v_{n-1}))_{v_i} = \frac{\partial X_i(v_1,v_2,v_3,...,v_{n-1})}{\partial v_i}$, $i \in \{1, 2, ..., n-1\}$.

We know that [3], the Frenet n-frame $F_i(v_1)$ of the curve $\alpha(v_1)$ satisfy the relations

$$\left.\begin{aligned} &F_1'(v_1) = k_1(v_1)F_2(v_1),\\ &F_i'(v_1) = -k_{i-1}(v_1)F_{i-1}(v_1) + k_i(v_1)F_{i+1}(v_1), \ \ i \in \{2,3,...,n-1\},\\ &F_n'(v_1) = -k_{n-1}(v_1)F_{n-1}(v_1), \end{aligned}\right\} \tag{7}$$

where k_i are the i-th curvatures of the curve $\alpha(v_1)$.

So, differentiating (3) with respect to v_1 and using the Frenet formula (7), we get

$$\begin{aligned} (X(v_1, v_2, v_3, ..., v_{n-1}))_{v_1} &= F_1(v_1) + \sum_{i=1}^{n} (a_i(v_1, v_2, v_3, ..., v_{n-1}))_{v_1} F_i(v_1)\\ &+ a_1(v_1, v_2, v_3, ..., v_{n-1})k_1(v_1)F_2(v_1)\\ &+ \sum_{i=2}^{n-1} a_i(v_1, v_2, v_3, ..., v_{n-1})(-k_{i-1}(v_1)F_{i-1}(v_1) + k_i(v_1)F_{i+1}(v_1))\\ &+ a_n(v_1, v_2, v_3, ..., v_{n-1})(-k_{n-1}(v_1)F_{n-1}(v_1)). \end{aligned} \tag{8}$$

Furthermore, $X(v_1, v_2, v_3, ..., v_{n-1}) - \alpha(v_1)$ is a normal vector to the canal hypersurfaces, which implies that

$$\langle X(v_1, v_2, v_3, ..., v_{n-1}) - \alpha(v_1), (X(v_1, v_2, v_3, ..., v_{n-1}))_{v_i}\rangle = 0, \ i \in \{1, 2, 3, ..., n-1\}. \tag{9}$$

Then, taking $i = 1$ in (9), from (3), (5), (6) and (8) we obtain

$$\left.\begin{aligned} &a_1(v_1, v_2, v_3, ..., v_{n-1}) = -\rho(v_1)\rho'(v_1),\\ &\sum_{i=2}^{n} (a_i(v_1, v_2, v_3, ..., v_{n-1}))^2 = \rho^2(v_1)(1 - (\rho'(v_1))^2). \end{aligned}\right\} \tag{10}$$

From (10), let us take

$$\left.\begin{aligned} &a_2(v_1, v_2, v_3, ..., v_{n-1}) = \pm\rho(v_1)\sqrt{1 - (\rho'(v_1))^2} \prod_{k=2}^{n-1} \cos(x_k),\\ &a_i(v_1, v_2, v_3, ..., v_{n-1}) = \pm\rho(v_1)\sqrt{1 - (\rho'(v_1))^2} \sin(v_{n+1-i}) \prod_{k=n+2-i}^{n-1} \cos(x_k),\\ &a_n(v_1, v_2, v_3, ..., v_{n-1}) = \pm\rho(v_1)\sqrt{1 - (\rho'(v_1))^2} \sin(v_{n-1}), \end{aligned}\right\} \tag{11}$$

where $i \in \{3, 4..., n-1\}$.

So, from (3) and (11), we have

Theorem 2.1. *The canal hypersurface in Euclidean n-space is expressed as*

$$\begin{aligned} X(v_1, v_2, v_3, ..., v_{n-1}) &= \alpha(v_1) - \rho(v_1)\rho'(v_1)F_1(v_1)\\ &\pm \rho(v_1)\sqrt{1 - (\rho'(v_1))^2} \begin{bmatrix} \left(\prod_{k=2}^{n-1} \cos(v_k)\right) F_2(v_1)\\ + \sum_{i=3}^{n-1} \left(\sin(v_{n+1-i}) \prod_{k=n+2-i}^{n-1} \cos(v_k)\right) F_i(v_1)\\ + \sin(v_{n-1})F_n(v_1) \end{bmatrix}. \end{aligned} \tag{12}$$

3. Canal Hypersurfaces in E^4

In this section, we study canal hypersurfaces in Euclidean 4-space E^4 by giving their expressions with the aid of (12).

Since we will deal with canal hypersurface in E^4 and give some important characterizations about them, let we recall some fundamental notions for hypersurfaces in E^4.

If $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ are three vectors in E^4, then the inner product and vector product are defined by

$$\left\langle \overrightarrow{u}, \overrightarrow{v} \right\rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \tag{13}$$

and

$$\overrightarrow{u} \times \overrightarrow{v} \times \overrightarrow{w} = \det \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \tag{14}$$

respectively. Also, the norm of the vector $\overrightarrow{u}$ is $\left\| \overrightarrow{u} \right\| = \sqrt{\left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle}$.

If

$$\begin{aligned} &\Psi : U \subset E^3 \longrightarrow E^4 \\ &(v_1, v_2, v_3) \longrightarrow \Psi(v_1, v_2, v_3) = (\Psi_1(v_1, v_2, v_3), \Psi_2(v_1, v_2, v_3), \Psi_3(v_1, v_2, v_3), \Psi_4(v_1, v_2, v_3)) \end{aligned} \tag{15}$$

is a hypersurface in E^4, then the unit normal vector field, the matrix forms of the first and second fundamental forms are

$$N_\Psi = \frac{\Psi_{v_1} \times \Psi_{v_2} \times \Psi_{v_3}}{\left\| \Psi_{v_1} \times \Psi_{v_2} \times \Psi_{v_3} \right\|}, \tag{16}$$

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{17}$$

and

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{18}$$

respectively. Here $g_{ij} = \left\langle \Psi_{v_i}, \Psi_{v_j} \right\rangle$, $h_{ij} = \left\langle \Psi_{v_i v_j}, N_\Psi \right\rangle$, $\Psi_{v_i} = \frac{\partial \Psi(v_1, v_2, v_3)}{\partial v_i}$, $\Psi_{v_i v_j} = \frac{\partial^2 \Psi(v_1, v_2, v_3)}{\partial v_i v_j}$, $i, j \in \{1, 2, 3\}$. Also, the shape operator of the hypersurface (15) is

$$S = [g_{ij}]^{-1} [h_{ij}], \tag{19}$$

where $[g_{ij}]^{-1}$ is the inverse matrix of the matrix $[g_{ij}]$.

With the aid of (16)-(19), the Gaussian and mean curvatures of a hypersurface in E^4 are given by

$$K = \det(S) = \frac{\det[h_{ij}]}{\det[g_{ij}]} \tag{20}$$

and

$$3H = tr(S), \tag{21}$$

respectively [6].

From (12), the canal hypersurface C in E^4 can be written as

$$C(v_1, v_2, v_3) = \alpha(v_1) - \rho\rho' F_1(v_1) \pm \rho\sqrt{1-\rho'^2}\left[\cos v_2 \cos v_3 F_2(v_1) + \sin v_2 \cos v_3 F_3(v_1) + \sin v_3 F_4(v_1)\right], \quad (22)$$

where $v_1 \in [0, l]$ and $v_2, v_3 \in [0, 2\pi)$. Also, from now on we state $\alpha = \alpha(v_1)$, $\rho = \rho(v_1)$, $F_i = F_i(v_1)$, $i \in \{1,2,3,4\}$, $\rho' = \frac{d\rho(v_1)}{dv_1}$ and we will consider the "±" as "+".

Firstly, from (7) and (22) the derivatives of the canal hypersurface (22) are obtained as

$$C_{v_1} = C^1_{v_1}F_1 + C^2_{v_1}F_2 + C^3_{v_1}F_3 + C^4_{v_1}F_4, \quad (23)$$

$$C_{v_2} = -\rho\sqrt{1-\rho'^2}\sin v_2 \cos v_3 F_2 + \rho\sqrt{1-\rho'^2}\cos v_2 \cos v_3 F_3, \quad (24)$$

$$C_{v_3} = -\rho\sqrt{1-\rho'^2}\cos v_2 \sin v_3 F_2 - \rho\sqrt{1-\rho'^2}\sin v_2 \sin v_3 F_3 + \rho\sqrt{1-\rho'^2}\cos v_3 F_4, \quad (25)$$

where

$$\left.\begin{aligned}
C^1_{v_1} &= 1 - \rho'^2 - k_1\rho\sqrt{1-\rho'^2}\cos v_2 \cos v_3 - \rho\rho'',\\
C^2_{v_1} &= -k_1\rho\rho' - k_2\rho\sqrt{1-\rho'^2}\sin v_2 \cos v_3 + \left(\rho'\sqrt{1-\rho'^2} - \frac{\rho\rho'\rho''}{\sqrt{1-\rho'^2}}\right)\cos v_2 \cos v_3,\\
C^3_{v_1} &= \rho\sqrt{1-\rho'^2}\,(k_2 \cos v_2 \cos v_3 - k_3 \sin v_3) + \left(\rho'\sqrt{1-\rho'^2} - \frac{\rho\rho'\rho''}{\sqrt{1-\rho'^2}}\right)\sin v_2 \cos v_3,\\
C^4_{v_1} &= k_3\rho\sqrt{1-\rho'^2}\sin v_2 \cos v_3 + \left(\rho'\sqrt{1-\rho'^2} - \frac{\rho\rho'\rho''}{\sqrt{1-\rho'^2}}\right)\sin v_3.
\end{aligned}\right\}$$

From (16) and (23)-(25), the unit normal vector field of C in E^4 is

$$N = -\rho' F_1 + \sqrt{1-\rho'^2}\cos v_2 \cos v_3 F_2 + \sqrt{1-\rho'^2}\sin v_2 \cos v_3 F_3 + \sqrt{1-\rho'^2}\sin v_3 F_4. \quad (26)$$

Also, the coefficients of the first fundamental form of C are given by

$$\left.\begin{aligned}
g_{11} &= \frac{1}{1-\rho'^2}\left(\begin{array}{l}
\left(1-\rho'^2\right)Q^2 + \left(\begin{array}{l} k_2\rho\left(1-\rho'^2\right)\sin v_2 \cos v_3 + k_1\rho\rho'\sqrt{1-\rho'^2} \\ +\rho'\left(\rho'^2 + \rho\rho'' - 1\right)\cos v_2 \cos v_3 \end{array}\right)^2 \\
+\left(\begin{array}{l} -k_2\rho\left(1-\rho'^2\right)\cos v_2 \cos v_3 + k_3\rho\left(1-\rho'^2\right)\sin v_3 \\ +\rho'\left(\rho'^2 + \rho\rho'' - 1\right)\sin v_2 \cos v_3 \end{array}\right)^2 \\
+\left(\rho'\left(\rho'^2 + \rho\rho'' - 1\right)\sin v_3 - k_3\rho\left(1-\rho'^2\right)\sin v_2 \cos v_3\right)^2
\end{array}\right),\\
g_{12} &= g_{21} = \rho^2\left(k_1\rho'\sqrt{1-\rho'^2}\sin v_2 + k_2(1-\rho'^2)\cos v_3 - k_3(1-\rho'^2)\cos v_2 \sin v_3\right)\cos v_3,\\
g_{13} &= g_{31} = \rho^2\left(k_1\rho'\sqrt{1-\rho'^2}\cos v_2 \sin v_3 + k_3(1-\rho'^2)\sin v_2\right),\\
g_{22} &= \rho^2(1-\rho'^2)\cos^2 v_3,\\
g_{23} &= g_{32} = 0,\\
g_{33} &= \rho^2(1-\rho'^2),
\end{aligned}\right\} \quad (27)$$

where $Q = \rho\left(k_1\sqrt{1-\rho'^2}\cos v_2 \cos v_3 + \rho''\right) - 1 + \rho'^2$ and it follows that

$$\det[g_{ij}] = \rho^4(1-\rho'^2)Q^2\cos^2 v_3. \quad (28)$$

Now, for obtaining the coefficients of the second fundamental form, let we give the second derivatives $C_{v_iv_j} = \frac{\partial^2 C}{\partial v_i v_j}$ of the canal hypersurface (22):

$$C_{v_1v_1} = C^1_{v_1v_1}F_1 + C^2_{v_1v_1}F_2 + C^3_{v_1v_1}F_3 + C^4_{v_1v_1}F_4, \tag{29}$$

$$\begin{aligned} C_{v_1v_2} &= C_{v_2v_1} \\ &= k_1\rho\sqrt{1-\rho'^2}\sin v_2\cos v_3 F_1 \\ &+ \left(\frac{\cos v_3}{\sqrt{1-\rho'^2}}\left(-\rho'(1-\rho'^2)\sin v_2 + \rho(-k_2(1-\rho'^2)\cos v_2 + \rho'\rho''\sin v_2)\right)\right)F_2 \\ &+ \left(\frac{\cos v_3}{\sqrt{1-\rho'^2}}\left(\rho'(1-\rho'^2)\cos v_2 + \rho(-k_2(1-\rho'^2)\sin v_2 - \rho'\rho''\cos v_2)\right)\right)F_3 \\ &+ k_3\rho\sqrt{1-\rho'^2}\cos v_2\cos v_3 F_4, \end{aligned} \tag{30}$$

$$\begin{aligned} C_{v_1v_3} &= C_{v_3v_1} \\ &= k_1\rho\sqrt{1-\rho'^2}\cos v_2\sin v_3 F_1 \\ &+ \left(\frac{\sin v_3}{\sqrt{1-\rho'^2}}\left(-\rho'(1-\rho'^2)\cos v_2 + \rho(k_2(1-\rho'^2)\sin v_2 + \rho'\rho''\cos v_2)\right)\right)F_2 \\ &+ \left(\frac{1}{\sqrt{1-\rho'^2}}\left(\begin{array}{l} -\rho'(1-\rho'^2)\sin v_2\sin v_3 \\ +\rho\left(\begin{array}{l} -k_2(1-\rho'^2)\cos v_2\sin v_3 \\ -k_3(1-\rho'^2)\cos v_3 + \rho'\rho''\sin v_2\sin v_3 \end{array}\right) \end{array}\right)\right)F_3 \\ &+ \left(\frac{1}{\sqrt{1-\rho'^2}}\left(\begin{array}{l} \rho'(1-\rho'^2)\cos v_3 \\ +\rho(-k_3(1-\rho'^2)\sin v_2\sin v_3 - \rho'\rho''\cos v_3) \end{array}\right)\right)F_4, \end{aligned} \tag{31}$$

$$C_{v_2v_2} = -\rho\sqrt{1-\rho'^2}\cos v_2\cos v_3 F_2 - \rho\sqrt{1-\rho'^2}\sin v_2\cos v_3 F_3, \tag{32}$$

$$C_{v_2v_3} = C_{v_3v_2} = \rho\sqrt{1-\rho'^2}\sin v_2\sin v_3 F_2 - \rho\sqrt{1-\rho'^2}\cos v_2\sin v_3 F_3 \tag{33}$$

and

$$C_{v_3v_3} = -\rho\sqrt{1-\rho'^2}\cos v_2\cos v_3 F_2 - \rho\sqrt{1-\rho'^2}\sin v_2\cos v_3 F_3 - \rho\sqrt{1-\rho'^2}\sin v_3 F_4, \tag{34}$$

where

$$C^1_{v_1v_1} = \frac{1}{\sqrt{1-\rho'^2}}\left(\begin{array}{l} \rho'(-2k_1(1-\rho'^2)\cos v_2\cos v_3 - 3\rho''\sqrt{1-\rho'^2}) \\ +\rho\left(\begin{array}{l} (k_1)^2\rho'\sqrt{1-\rho'^2} - k_1'(1-\rho'^2)\cos v_2\cos v_3 \\ +k_1(k_2(1-\rho'^2)\sin v_2 + 2\rho'\rho''\cos v_2)\cos v_3 - \rho'''\sqrt{1-\rho'^2} \end{array}\right) \end{array}\right),$$

$$C^2_{v_1v_1} = \frac{-1}{(1-\rho'^2)^{\frac{3}{2}}}\left(\begin{array}{l} -(1-\rho'^2)\left(\begin{array}{l} k_1(1-2\rho'^2)\sqrt{1-\rho'^2} - 2k_2\rho'(1-\rho'^2)\sin v_2\cos v_3 \\ +\rho''(1-3\rho'^2)\cos v_2\cos v_3 \end{array}\right) \\ +\rho\left(\begin{array}{l} (1-\rho'^2)^2((k_1)^2 + (k_2)^2)\cos v_2\cos v_3 \\ +k_1'\rho'(1-\rho'^2)^{\frac{3}{2}} + k_2'(1-\rho'^2)^2\sin v_2\cos v_3 \\ +2k_1\rho''(1-\rho'^2)^{\frac{3}{2}} + (\rho''^2 + \rho'\rho'''(1-\rho'^2))\cos v_2\cos v_3 \\ +k_2(1-\rho'^2)(-k_3(1-\rho'^2)\sin v_3 - 2\rho'\rho''\sin v_2\cos v_3) \end{array}\right) \end{array}\right),$$

$$C^3_{v_1v_1} = \frac{1}{(1-\rho'^2)^{\frac{3}{2}}}\left(\begin{array}{l} (1-\rho'^2)\left(\begin{array}{l} 2\rho'(1-\rho'^2)(k_2\cos v_2\cos v_3 - k_3\sin v_3) \\ +\rho''(1-3\rho'^2)\sin v_2\cos v_3 \end{array}\right) \\ -\rho\left(\begin{array}{l} (1-\rho'^2)^2((k_2^2+k_3^2)\sin v_2 - k_2'\cos v_2)\cos v_3 \\ +k_3'(1-\rho'^2)^2\sin v_3 + 2k_3\rho'\rho''(-1+\rho'^2)\sin v_3 \\ -k_2\rho'(-1+\rho'^2)\left(k_1\sqrt{1-\rho'^2}+2\rho''\cos v_2\cos v_3\right) \\ +\sin v_2\cos v_3\left(\rho'\rho'''(1-\rho'^2)+\rho''^2\right) \end{array}\right) \end{array}\right),$$

$$C^4_{v_1v_1} = \frac{1}{(1-\rho'^2)^{\frac{3}{2}}}\left(\begin{array}{l} (1-\rho'^2)(2\rho' k_3(1-\rho'^2)\sin v_2\cos v_3 + \rho''(1-3\rho'^2)\sin v_3) \\ -\rho'\left(\begin{array}{l} k_3(1-\rho'^2)^2(-k_2\cos v_2\cos v_3 + k_3\sin v_3) \\ -(1-\rho'^2)((1-\rho'^2)k_3'\sin v_2\cos v_3 + 2k_3\rho'\rho'') \\ +(\rho''^2-\rho'\rho'''(1-\rho'^2))\sin v_3 \end{array}\right) \end{array}\right).$$

Thus, from (18), (26) and (29)-(34), the coefficients of the second fundamental form are given by

$$\left.\begin{array}{l} h_{11} = \frac{\rho}{\rho'^2-1}\left(\begin{array}{l} \left((k_2)^2\cos^2 v_3 - k_2k_3\cos v_2\sin(2v_3) + (k_3)^2\left(\cos^2 v_3\sin^2 v_2 + \sin^2 v_3\right)\right)(1-\rho'^2)^2 \\ +(k_1)^2(1-\rho'^2)\left((1-\rho'^2)\cos^2 v_2\cos^2 v_3 + \rho'^2\right) + \rho''^2 \\ +2k_1\sqrt{1-\rho'^2}(k_2\rho'(1-\rho'^2)\sin v_2 + \rho''\cos v_2)\cos v_3 \end{array}\right) \\ \quad + k_1\sqrt{1-\rho'^2}\cos v_2\cos v_3 + \rho'', \\ h_{12} = h_{21} = \rho\left(-k_1\rho'\sqrt{1-\rho'^2}\sin v_2 + (1-\rho'^2)(k_3\cos v_2\sin v_3 - k_2\cos v_3)\right)\cos v_3, \\ h_{13} = h_{31} = \rho\left(-k_1\rho'\sqrt{1-\rho'^2}\cos v_2\sin v_3 - k_3(1-\rho'^2)\sin v_2\right), \\ h_{22} = -\rho(1-\rho'^2)\cos^2 v_3, \\ h_{23} = h_{32} = 0, \\ h_{33} = -\rho(1-\rho'^2) \end{array}\right\} \tag{35}$$

and it implies

$$\det[h_{ij}] = \rho^2(1-\rho'^2)\left(\begin{array}{l} (1-\rho'^2)(k_1\sqrt{1-\rho'^2}\cos v_2\cos v_3 + \rho'') \\ -\rho\left(\begin{array}{l} (k_1)^2(1-\rho'^2)\cos^2 v_2\cos^2 v_3 \\ +2k_1\rho''\sqrt{1-\rho'^2}\cos v_2\cos v_3 + \rho''^2 \end{array}\right) \end{array}\right)\cos^2 v_3. \tag{36}$$

So, from (20), (28) and (36), we have

Theorem 3.1. *The Gaussian curvature of the canal hypersurface (22) in Euclidean 4-space is*

$$K = \frac{\left((1-\rho'^2)(k_1\sqrt{1-\rho'^2}\cos v_2\cos v_3 + \rho'') - \rho\left(\begin{array}{l} (k_1)^2(1-\rho'^2)\cos^2 v_2\cos^2 v_3 + \rho''^2 \\ +2k_1\rho''\sqrt{1-\rho'^2}\cos v_2\cos v_3 \end{array}\right)\right)}{\rho^2\left(\rho\left(k_1\sqrt{1-\rho'^2}\cos v_2\cos v_3 + \rho''\right) - 1 + \rho'^2\right)^2}. \tag{37}$$

Theorem 3.2. *The canal hypersurface (22) in Euclidean 4-space is flat if and only if it is a circular hypercylinder or circular hypercone.*

Proof. If the canal hypersurface (22) in Euclidean 4-space is flat, then from (37) it must be

$$k_1\sqrt{1-\rho'^2}(1-\rho'^2-2\rho\rho'')\cos v_2\cos v_3 - \rho(k_1)^2(1-\rho'^2)\cos^2 v_2\cos^2 v_3 + (1-\rho'^2-\rho\rho'')\rho'' = 0. \tag{38}$$

Since the set $\{1, \cos v_2 \cos v_3, \cos^2 v_2 \cos^2 v_3\}$ is linearly independent, we have

$$\left.\begin{array}{l} k_1\sqrt{1-\rho'^2}(1-\rho'^2-2\rho\rho'')=0, \\ \rho(k_1)^2(1-\rho'^2)=0, \\ (1-\rho'^2-\rho\rho'')\rho''=0. \end{array}\right\} \tag{39}$$

From the second equation of (39), since $\rho \neq 0$ and $1-\rho'^2 \neq 0$, we have $k_1 = 0$ and so, (22) is a hypersurface of revolution. Also, in this case, the first equation of (39) holds, too. If we use $k_1 = 0$ in (37), we have

$$K = \frac{\rho''}{\rho^2(1-\rho'^2-\rho\rho'')}. \tag{40}$$

Since the hypersurface is flat, (40) implies $\rho'' = 0$, that is, $\rho(v_1) = av_1 + b$, $a, b \in \mathbb{R}$, $a \neq \pm 1$. From this, the third equation of (39) holds, too. Therefore, (22) is a circular hypercylinder when $a = 0$, or a circular hypercone when $a \neq 0$, $a \neq \pm 1$.

Conversely, if $k_1 = 0$ and $\rho(v_1) = av_1 + b$ (i.e., if (22) is a circular hypercylinder or a circular hypercone), then we have $K = 0$ and this completes the proof. □

Also, after finding the inverse of the matrix of the first fundamental form and using this and (35) in (19), the shape operator of the canal hypersurface (22) is obtained by

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}, \tag{41}$$

where

$$\left.\begin{array}{l} S_{11} = \frac{1}{Q^2}\left(\begin{array}{l} (1-\rho'^2)\left(k_1\sqrt{1-\rho'^2}\cos v_2\cos v_3+\rho''\right) \\ -\rho\left((k_1)^2(1-\rho'^2)\cos^2 v_2\cos^2 v_3+2k_1\rho''\sqrt{1-\rho'^2}\cos v_2\cos v_3+\rho''^2\right) \end{array}\right), \\ S_{21} = \frac{1}{\rho Q}\left(k_1\rho'\sqrt{1-\rho'^2}\sin v_2\sec v_3+k_2(1-\rho'^2)-k_3(1-\rho'^2)\cos v_2\tan v_3\right), \\ S_{31} = \frac{1}{\rho Q^2}\left(\begin{array}{l} Qk_3(1-\rho'^2)\sin v_2 \\ +k_1\rho'\cos v_2\sin v_3\left(-(1-\rho'^2)^{\frac{3}{2}}+\rho\left(\begin{array}{l} k_1(1-\rho'^2)\cos v_2\cos v_3 \\ +\sqrt{1-\rho'^2}\rho'' \end{array}\right)\right) \end{array}\right), \\ S_{22} = S_{33} = -\frac{1}{\rho}, \\ S_{12} = S_{13} = S_{23} = S_{32} = 0. \end{array}\right\}$$

Hence from (21) and (41), we get

Theorem 3.3. *The mean curvature of the canal hypersurface (22) in Euclidean 4-space is*

$$H = \frac{\left(\begin{array}{l} -3\rho^2\left((k_1)^2(1-\rho'^2)\cos^2 v_2\cos^2 v_3+2k_1\rho''\sqrt{1-\rho'^2}\cos v_2\cos v_3+\rho''^2\right) \\ -2(1-\rho'^2)^2+5\rho(1-\rho'^2)(k_1\sqrt{1-\rho'^2}\cos v_2\cos v_3+\rho'') \end{array}\right)}{3\rho\left(\rho\left(k_1\sqrt{1-\rho'^2}\cos v_2\cos v_3+\rho''\right)-1+\rho'^2\right)^2}. \tag{42}$$

Theorem 3.4. *The canal hypersurface (22) in Euclidean 4-space is minimal if and only if it is a hypersurface of revolution parametrized by*

$$C(v_1,v_2,v_3) = (v_1-\rho\rho', \pm\rho\sqrt{1-\rho'^2}\cos v_2\cos v_3, \pm\rho\sqrt{1-\rho'^2}\sin v_2\cos v_3, \pm\rho\sqrt{1-\rho'^2}\sin v_3), \tag{43}$$

where $\rho(v_1)$ *is given by* $\int \frac{d\rho}{\sqrt{1-\left(\frac{a}{\rho}\right)^{\frac{4}{3}}}} = \pm v_1 + b$, $a, b \in \mathbb{R}$.

Proof. If the canal hypersurface (22) in Euclidean 4-space is minimal, then from (42) it must be

$$k_1\sqrt{1-\rho'^2}(5\rho(1-\rho'^2)-6\rho^2\rho'')\cos v_2\cos v_3-3\rho^2(k_1)^2(1-\rho'^2)\cos^2 v_2\cos^2 v_3 \\ +(2-2\rho'^2-3\rho\rho'')(-1+\rho'^2+\rho\rho'')=0. \tag{44}$$

Since the set $\{1,\cos v_2\cos v_3,\cos^2 v_2\cos^2 v_3\}$ is linearly independent, we have

$$\left.\begin{array}{l} k_1\sqrt{1-\rho'^2}(5\rho(1-\rho'^2)-6\rho^2\rho'')=0, \\ 3\rho^2(k_1)^2(1-\rho'^2)=0, \\ (2-2\rho'^2-3\rho\rho'')(-1+\rho'^2+\rho\rho'')=0. \end{array}\right\} \tag{45}$$

From the second equation of (45), we have $k_1=0$ and then the first equation of (45) holds, too. If we use $k_1=0$ in (42), we have

$$H=\frac{2-2\rho'^2-3\rho\rho''}{3\rho(-1+\rho'^2+\rho\rho'')}. \tag{46}$$

So, if the canal hypersurface (22) in Euclidean 4-space is minimal, then from (46) $\rho(v_1)$ must satisfy the differential equation

$$2-2\rho'(v_1)^2-3\rho(v_1)\rho''(v_1)=0. \tag{47}$$

Now, let us solve (47).

If we take $\rho'(v_1)=f(v_1)$, we get

$$\rho''=f'=\frac{df}{d\rho}\frac{d\rho}{dv_1}=\frac{df}{d\rho}f. \tag{48}$$

Using (48) in (47), we have

$$3\rho\frac{df}{d\rho}f+2f^2-2=0. \tag{49}$$

From (47), $\rho'(v_1)=f(v_1)\neq 0$ and so we reach that

$$\frac{3f}{2(1-f^2)}df=\frac{d\rho}{\rho}. \tag{50}$$

By integrating (50), we have

$$f=\pm\sqrt{1-\left(\frac{a}{\rho}\right)^{\frac{4}{3}}}, \tag{51}$$

where a is constant. Since $\rho'=\frac{d\rho}{dv_1}=f$, from (51) we get

$$\int\frac{d\rho}{\sqrt{1-\left(\frac{a}{\rho}\right)^{\frac{4}{3}}}}=\pm\int dv_1. \tag{52}$$

Since $k_1=0$, without loss of generality, we can suppose the curve $\alpha(v_1)$ as $\alpha(v_1)=(v_1,0,0,0)$ and $F_1=(1,0,0,0)$, $F_2=(0,1,0,0)$, $F_3=(0,0,1,0)$, $F_4=(0,0,0,1)$. Then, (22) can be parametrized by (43) and from (52), $\rho(v_1)$ satisfies $\int\frac{d\rho}{\sqrt{1-\left(\frac{a}{\rho}\right)^{\frac{4}{3}}}}=\pm v_1+b,\ a,b\in\mathbb{R}$.

Conversely, if (22) is parametrized by (43), where $\rho(v_1)$ satisfies $\int\frac{d\rho}{\sqrt{1-\left(\frac{a}{\rho}\right)^{\frac{4}{3}}}}=\pm v_1+b,\ a,b\in\mathbb{R}$, then we have $H=0$ and this completes the proof. □

Also, we know that [17], the only minimal hypersurface of revolution (except the hyperplane) in Euclidean space is the generalized catenoid. Thus, from the last Theorem, we have

Corollary 3.5. *The canal hypersurface (22) is minimal if and only if it is a generalized catenoid.*

Here, from (37) and (42), we can state the following theorem which gives an important relation between Gaussian and mean curvatures:

Theorem 3.6. *The Gaussian curvature K and the mean curvature H of canal hypersurfaces (22) in Euclidean 4-space satisfy*

$$H = \frac{1}{3}(K\rho^2 - \frac{2}{\rho}). \tag{53}$$

Now, if

$$H_{v_i}K_{v_j} - H_{v_j}K_{v_i} = 0, \; i \neq j, \; i, j = 1, 2, 3, \tag{54}$$

holds on a hypersurface, then we call the hypersurface as $(H, K)_{\{ij\}}$-Weingarten hypersurface, where $X_{v_i} = \frac{\partial X}{\partial v_i}$. So, from (37) and (42) we have

Theorem 3.7. *The canal hypersurface (22) in Euclidean 4-space is $(H, K)_{\{23\}}$-Weingarten hypersurface.*

Also, from (41) we have

$$\det(S - \kappa I_3) = -\frac{(\kappa\rho + 1)^2}{\rho^2 Q^2} \left(\begin{array}{l} \kappa\rho^2 \left(\begin{array}{l} 2k_1\rho'' \sqrt{1-\rho'^2} \cos v_2 \cos v_3 \\ +(k_1)^2 \left(1-\rho'^2\right) \cos^2 v_2 \cos^2 v_3 + \rho''^2 \end{array} \right) \\ +\rho \left(\begin{array}{l} 2k_1 \sqrt{1-\rho'^2} \left(\kappa\rho'^2 - \kappa + \rho''\right) \cos v_2 \cos v_3 \\ +\rho'' \left(2\kappa\rho'^2 - 2\kappa + \rho''\right) \\ +(k_1)^2 \left(1-\rho'^2\right) \cos^2 v_2 \cos^2 v_3 \end{array} \right) \\ -\left(1-\rho'^2\right)\left(\kappa\rho'^2 - \kappa + k_1 \sqrt{1-\rho'^2} \cos v_2 \cos v_3 + \rho''\right) \end{array} \right). \tag{55}$$

By solving the equation $\det(S - \kappa I_3) = 0$ from (55), we obtain the principal curvatures of the canal hypersurfaces (22) in E^4 as follows:

Theorem 3.8. *The principal curvatures of the canal hypersurfaces (22) in E^4 are*

$$\kappa_1 = \kappa_2 = -\frac{1}{\rho} \quad \textit{and} \quad \kappa_3 = K\rho^2. \tag{56}$$

Here, we know that if $\rho(v_1) = \lambda$ is a constant, then the canal hypersurface is called tubular or pipe hypersurface and from (12) the tubular hypersurface in E^n can be given by

$$X(v_1, v_2, v_3, ..., v_{n-1}) = \alpha(v_1) \pm \lambda \left[\begin{array}{l} \left(\prod_{k=2}^{n-1} \cos(x_k)\right) F_2(v_1) \\ +\sum_{i=3}^{n-1} \left(\sin(v_{n+1-i}) \prod_{k=n+2-i}^{n-1} \cos(x_k)\right) F_i(v_1) \\ +\sin(v_{n-1}) F_n(v_1) \end{array} \right]. \tag{57}$$

So, from (57) the tubular hypersurface in E^4 is

$$T(v_1, v_2, v_3) = \alpha(v_1) \pm \lambda \left[\cos v_2 \cos v_3 F_2(v_1) + \sin v_2 \cos v_3 F_3(v_1) + \sin v_3 F_4(v_1)\right]. \tag{58}$$

Here, by taking "±" as "+" in (58), we get

Theorem 3.9. *The Gaussian and mean curvatures of the tubular hypersurface (58) in Euclidean 4-space are*

$$K = \frac{k_1 \cos v_2 \cos v_3}{\lambda^2(1 - k_1\lambda \cos v_2 \cos v_3)} \tag{59}$$

and

$$H = \frac{2 - 3k_1\lambda \cos v_2 \cos v_3}{3\lambda(-1 + k_1\lambda \cos v_2 \cos v_3)} \tag{60}$$

respectively.

So, from (59) and (60) we get

Theorem 3.10. *The tubular hypersurface (58) in Euclidean 4-space is* $(H, K)_{\{12\}}$ *and* $(H, K)_{\{13\}}$*- Weingarten hypersurface.*

Also, we know that, a hypersurface is called a linear Weingarten hypersurface, if it satisfies

$$aH + bK = c, \tag{61}$$

where a, b, c are not all zero constants. Thus, from (53), we have

Theorem 3.11. *The tubular hypersurface (58) in Euclidean 4-space is a linear Weingarten hypersurface.*

Proof. The equation (53) on the tubular hypersurface (58) implies

$$-3\lambda H + \lambda^3 K = 2. \tag{62}$$

So, from the definition of a linear Weingarten hypersurface, the proof completes. □

4. Conclusion and Future Work

In this study, firstly we obtain the general expression of canal hypersurfaces in Euclidean n-space and we deal with canal hypersurfaces in Euclidean 4-space with the aid of this expression. In this sense, we obtain the Gaussian curvature and the mean curvature of canal hypersurfaces in E^4 and give an important relation between the Gaussian and mean curvatures. Also by taking the radius function as a constant, we state the tubular hypersurfaces in Euclidean spaces and give some results about tubular hypersurfaces in E^4. In this context, we prove that the tubular hypersurfaces are linear Weingarten hypersurfaces in E^4.

We hope that, this study will give a new perspective to readers who deal with the canal hypersurfaces in E^4 and E^n. As open problems, this study can be handled in Minkowskian, Galilean and pseudo-Galilean 4-spaces in the near future. Also, some important characterizations for the Laplace-Beltrami operators on the canal hypersurfaces or different classifications for canal hypersurfaces in different 4-dimensional spaces can be investigated.

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