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Some weighted Hadamard and Ostrowski-type fractional inequalities for quasi-geometrically convex functions

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Abstract. It is well knowledge that the purpose of inequality is to develop various approaches to mathematical problem solving. In order to prove the originality and existence of mathematical techniques, it is now necessary to seek exact inequalities. In the present research, we propose some novel weighted fractional Ostrowski-type inequalities for functions which are differentiable and satisfy quasi-geometrically convex using a new identity. Moreover, outcomes for functions with a bounded first derivative are proved. Finally, some examples are given to illustrate the investigated results. The obtained results generalize and refine previously known results.

1. Introduction

Integral inequalities are regarded as a terrific tool for developing both qualitative and quantitative aspects in mathematics. This is because integral inequalities may be used to analyse functions. Because of the increasing demand for productive uses of these inequalities, interest has been continually growing, and this is done in order to satisfy those demands. Many researchers had looked at these inequalities, and those researchers had employed a variety of methods in order to investigate and give their findings regarding these inequalities [1]- [30].

In the past twenty years, the discipline of fractional calculus theory has seen a rise in both its popularity and its significance. This may be attributed to the fact that the theory has various appealing applications and a wide appeal in the fields of physics, engineering, chemistry, and biology, such as viscosity, electromagnetism, rheology, selective transport, electrical networks, and fluid flow. Numerous research have recently begun to concentrate their attention on discrete versions of this fractional calculus, making use of the benefits offered by the time scale theory as well as the references that are included within it. The fundamental idea that forms the basis for the context of this application of fractional calculus can be characterized in two different ways. The first method is known as the Riemann-Liouville method. This method involves changing the integral operator with a single integral by repeating it $n!$ times and then utilizing the well-known Cauchy formula, which transforms n! into the Gamma function. This method was

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developed by Riemann and Liouville. The Grünwald-Letnikov method is the second one, and it requires first iterating the derivative n times and then fractionalizing the binomial coefficients using the Gamma function. This method was developed by Grünwald and Letnikov. The fractional derivatives that were derived through the use of this calculus appeared to be difficult to understand and lost several of the most important qualities that are normally associated with derivatives, such as the product rule and the chain rule. On the other hand, the behavior of these fractional operators' semigroup characteristics is appropriate in several other contexts. We refer the interested reader to [10, 11, 16, 33, 35, 36].

Assumed that a real function called χ has been defined on a nonempty interval of a real line called $\mathbb{R}$. It is claimed that the function χ is convex on I if and only if inequality

$$\chi(\varrho\eta + (1-\varrho)\mu) \leq \varrho\chi(\eta) + (1-\varrho)\chi(\mu)$$

holds for all $\eta, \mu \in I$ and $\varrho \in [0,1]$. The concept of a quasi-convex function is a generalization of the concept of a convex function. To be more specific, we say that a function $\chi : [x,y] \subset \mathbb{R} \to \mathbb{R}$ has been called quasi-convex on $[x,y]$ if the following conditions are met:

$$\chi(\varrho\eta + (1-\varrho)\mu) \leq \max\{\chi(\eta), \chi(\mu)\}$$

for all $\eta, \mu \in [x,y]$ and $\varrho \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [12]).

It is important to keep in mind that Niculescu first described and thought about a class of mappings that he called a GA-convex mappings, as follows:

Definition 1.1. *[31, 32] Suppose that function $\chi : I \subseteq (0,\infty) \to \mathbb{R}$ is called GA-convex, if*

$$\chi\left(\eta^{\varrho}\mu^{1-\varrho}\right) \leq \varrho\chi(\eta) + (1-\varrho)\chi(\mu)$$

for all $\eta, \mu \in I$ and $\varrho \in [0,1]$.

In 2013, İmdat İşcan introduced the concept of the quasi-geometrically convex functions:

Definition 1.2. *[11] Suppose that function $\chi : I \subseteq (0,\infty) \to \mathbb{R}$ is called quasi-geometrically convex on I if*

$$\chi\left(\eta^{\varrho}\mu^{1-\varrho}\right) \leq \max\left[\chi(\eta), \chi(\mu)\right]$$

for all $\eta, \mu \in I$ and $\varrho \in [0,1]$.

In a significant number of real-world issues, it is essential to constrain one quantity using another quantity. Inequalities from classical mathematics, such as Ostrowski's inequality, are extremely helpful for achieving this goal. The Ostrowski's inequality was introduced by Alexander Ostrowski in [34], and with the passing of the years, generalizations on the same, involving derivatives of the function under study, have taken place.

Theorem 1.3. *Let $\chi : [\eta, \chi_2] \subseteq \mathbb{R} \to \mathbb{R}$ be continuous on $[\eta,\mu]$ and differentiable function on (η,μ), whose derivative $\chi' : (\eta,\mu) \to \mathbb{R}$ is bounded on (η,μ), i.e., $|\chi'(\tau)| \leq \mathcal{M}$. Then the following inequality holds:*

$$\left|\chi(x) - \frac{1}{\mu-\eta}\int_{\eta}^{\mu}\chi(\tau)\,d\tau\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{\eta+\mu}{2}\right)^2}{(\mu-\eta)^2}\right](\mu-\eta)\,\mathcal{M}, \tag{1}$$

for all $x \in [\eta,\mu]$. The constant $\frac{1}{4}$ is the best possible constant.

The inequality (1) can be written in equivalent form as follows:

$$\left|\chi(x) - \frac{1}{\mu-\eta}\int_{\eta}^{\mu}\chi(\tau)\,d\tau\right| \leq \frac{\mathcal{M}}{\mu-\eta}\left[\frac{(x-\eta)^2 + (\mu-x)^2}{2}\right].$$

For the most recent outcomes, improvements, parallels, generalizations, and novelties pertaining to Ostrowski's type inequalities, see [3–5, 13, 17, 21–23].

We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 1.4. *Let $\chi \in L[\eta, \mu]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{\eta^+}^{\varsigma} f$ and $J_{\mu^-}^{\varsigma} f$ of order $\varsigma > 0$ with $\mu > \eta \geq 0$ are defined by*

$$J_{\eta^+}^{\varsigma} \chi(x) = \frac{1}{\Gamma(\varsigma)} \int_{\eta}^{x} \left(\ln \frac{x}{\varrho}\right)^{\varsigma-1} \chi(\varrho) \frac{d\varrho}{\varrho}, \quad x > \eta$$

and

$$J_{\mu^-}^{\varsigma} \chi(x) = \frac{1}{\Gamma(\varsigma)} \int_{x}^{\mu} \left(\ln \frac{\varrho}{x}\right)^{\varsigma-1} \chi(\varrho) \frac{d\varrho}{\varrho}, \quad x < \mu$$

respectively, and Gamma function $\Gamma(\varsigma)$ is represented as $\Gamma(\varsigma) = \int_0^{\infty} e^{-\varphi} \varphi^{\varsigma-1} d\varphi$ (see [16]).

In this article, we establish a new identity and then utilize it to derive new weighted inequalities of the Ostrowski type for quasi-geometrically convex functions. We do this by using the novel identity that we just constructed. Next, we will discuss more findings for functions that have a first derivative that is constrained inside a certain range. Furthermore, a few examples are provided to highlight the findings of the investigation. The results are a generalization and improvement of earlier findings.

2. Main results

Lemma 2.1. *Let $\chi : J \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on J° and $\eta, \mu \in J^\circ$ with $\eta < \mu$. If $\omega : [\eta, \mu] \to (0, \infty)$ is a continuous, positive, and geometrically symmetric to $\sqrt{\eta\mu}$ and $\chi' \in L[\eta, \mu]$. Then we have $\varrho \in [0, 1]$ and $\varsigma > 0$,*

$$\begin{aligned} &J_{x^-}^{\varsigma} \omega\chi(\eta) + J_{x^+}^{\varsigma} \omega\chi(\mu) - \left[J_{x^-}^{\varsigma} \omega(\eta) + J_{x^+}^{\varsigma} \omega(\mu)\right] \chi(x) \\ &= \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)} \int_0^1 p_1(\varrho) \chi'(\mu^{\varrho} x^{1-\varrho}) d\varrho - \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)} \int_0^1 p_2(\varrho) \chi'(\eta^{\varrho} x^{1-\varrho}) d\varrho, \end{aligned} \tag{2}$$

where

$$p_1(\varrho) := \int_{\varrho}^{1} (1-\rho)^{\varsigma-1} \omega(\mu^{\rho} x^{1-\rho}) d\rho \tag{3}$$

and

$$p_2(\varrho) := \int_{\varrho}^{1} (1-\rho)^{\varsigma-1} \omega(\eta^{\rho} x^{1-\rho}) d\rho. \tag{4}$$

Proof. We apply part-by-part integration and change of variable ($u = \mu^{\rho}x^{1-\rho}$) to obtain

$$\int_0^1 p_1(\varrho)\chi'(\mu^{\varrho}x^{1-\varrho})d\varrho = \int_0^1 \left(\int_{\varrho}^1 (1-\rho)^{\varsigma-1}\omega(\mu^{\rho}x^{1-\rho})d\rho\right)\chi'(\mu^{\varrho}x^{1-\varrho})d\varrho$$
$$= \frac{1}{\ln\left(\frac{\mu}{x}\right)}\left(\int_{\varrho}^1 (1-\rho)^{\varsigma-1}\omega(\mu^{\rho}x^{1-\rho})d\rho\right)\chi(\mu^{\varrho}x^{1-\varrho})\Bigg|_{\varrho=0}^{\varrho=1} + \frac{1}{\ln\left(\frac{\mu}{x}\right)}\int_0^1 (1-\varrho)^{\varsigma-1}\omega(\mu^{\varrho}x^{1-\varrho})\chi(\mu^{\varrho}x^{1-\varrho})d\varrho$$
$$= -\frac{1}{\ln\left(\frac{\mu}{x}\right)}\left(\int_0^1 (1-\rho)^{\varsigma-1}\omega(\mu^{\rho}x^{1-\rho})d\rho\right)\chi(x) + \frac{1}{\ln\left(\frac{\mu}{x}\right)}\int_0^1 (1-\varrho)^{\varsigma-1}\omega(\mu^{\varrho}x^{1-\varrho})\chi(\mu^{\varrho}x^{1-\varrho})d\varrho$$
$$= -\frac{1}{(\ln\left(\frac{\mu}{x}\right))^{\varsigma+1}}\left(\int_x^{\mu}\left(\ln\frac{\mu}{u}\right)^{\varsigma-1}\omega(u)\frac{du}{u}\right)\chi(x) + \frac{1}{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}\int_x^{\mu}\left(\ln\frac{\mu}{u}\right)^{\varsigma-1}\omega(u)\chi(u)\frac{du}{u}$$
$$= -\frac{\Gamma(\varsigma)}{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}\left(J_{x^+}^{\varsigma}\omega(\mu)\right)\chi(x) + \frac{\Gamma(\varsigma)}{(\ln\left(\frac{\mu}{x}\right))^{\varsigma+1}}\left(J_{x^+}^{\varsigma}\omega\chi(\mu)\right). \tag{5}$$

Now, multiplying (5) by $\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}$, we get

$$\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\chi'(\mu^{\varrho}x^{1-\varrho})d\varrho = J_{x^+}^{\varsigma}\omega\chi(\mu) - \left(J_{x^+}^{\varsigma}\omega(\mu)\right)\chi(x). \tag{6}$$

Similar work give

$$\int_0^1 p_2(\varrho)\chi'(\eta^{\varrho}x^{1-\varrho})d\varrho = \frac{\Gamma(\varsigma)}{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}\left(J_{x^-}^{\varsigma}\omega(\eta)\right)\chi(x) - \frac{\Gamma(\varsigma)}{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}J_{x^-}^{\varsigma}\omega\chi(\eta). \tag{7}$$

Multiplying (7) by $\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}$, we obtain

$$\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\chi'(\eta^{\varrho}x^{1-\varrho})d\varrho = \left(J_{x^-}^{\varsigma}\omega(\eta)\right)\chi(x) - J_{x^-}^{\varsigma}\omega\chi(\eta). \tag{8}$$

By taking the difference between (6) and (8), we get

$$\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\chi'(\mu^{\varrho}x^{1-\varrho})d\varrho - \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\chi'(\eta^{\varrho}x^{1-\varrho})d\varrho$$
$$= J_{x^-}^{\varsigma}\omega\chi(\eta) + J_{x^+}^{\varsigma}\omega\chi(\mu) - \left[J_{x^-}^{\varsigma}\omega(\eta) + J_{x^+}^{\varsigma}\omega(\mu)\right]\chi(x), \tag{9}$$

which is the desired result. □

Theorem 2.2. *Let the conditions of Lemma 2.1 hold and let $|\chi'|$ is quasi-geometrically convex then the following inequality for fractional integrals holds:*

$$|J_{x^-}^{\varsigma}\omega\chi(\eta) + J_{x^+}^{\varsigma}\omega\chi(\mu) - \left[J_{x^-}^{\varsigma}\omega(\eta) + J_{x^+}^{\varsigma}\omega(\mu)\right]\chi(x)| \le \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\max\left\{|\chi'(x)|, \left|\chi'(\mu)\right|\right\}\|\omega\|_{[x,\mu],\infty}$$
$$+ \frac{\left(\ln\left(\frac{\eta}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\max\left\{|\chi'(x)|, \left|\chi'(\eta)\right|\right\}\|\omega\|_{[\eta,x],\infty}. \tag{10}$$

Proof. We have the following by deduction from the Lemma 2.1, the property of the modulus, and the fact that $|\chi'|$ is quasi-geometrically convex:

$$\begin{aligned}
&\left|J^{\varsigma}_{x^-}\omega\chi(\eta) + J^{\varsigma}_{x^+}\omega\chi(\mu) - \left[J^{\varsigma}_{x^-}\omega(\eta) + J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x)\right| \\
&\le \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 \left|p_1(\varrho)\right|\left|\chi'(\mu^{\varrho}x^{1-\varrho})\right|d\varrho + \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 \left|p_2(\varrho)\right|\left|\chi'(\eta^{\varrho}x^{1-\varrho})\right|d\varrho \\
&\le \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\max\left\{|\chi'(x)|,\left|\chi'(\mu)\right|\right\}\int_0^1\left|\int_{\varrho}^1(1-\rho)^{\varsigma-1}\omega(\mu^{\rho}x^{1-\rho})d\rho\right|d\varrho \\
&+ \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\max\left\{|\chi'(x)|,\left|\chi'(\eta)\right|\right\}\int_0^1\left|\int_{\varrho}^1(1-\rho)^{\varsigma-1}\omega(\eta^{\rho}x^{1-\rho})d\rho\right|d\varrho \\
&\le \left(\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\max\left\{|\chi'(x)|,\left|\chi'(\mu)\right|\right\}\|\omega\|_{[x,\mu],\infty}\right. \\
&\left.+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\max\left\{|\chi'(x)|,\left|\chi'(\eta)\right|\right\}\|\omega\|_{[\eta,x],\infty}\right)\int_0^1\left|\int_{\varrho}^1(1-\rho)^{\varsigma-1}d\rho\right|d\varrho \\
&\qquad = \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\max\left\{|\chi'(x)|,\left|\chi'(\mu)\right|\right\}\|\omega\|_{[x,\mu],\infty} + \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\max\left\{|\chi'(x)|,\left|\chi'(\eta)\right|\right\}\|\omega\|_{[\eta,x],\infty}. \quad (11)
\end{aligned}$$

The proof has been completed. □

Example 2.3. *Let us consider $\chi:(0,\infty)\to\mathbb{R}$ defined by $\chi(x) = x+\frac{1}{x}$. It is clear that the function $\left|\chi'(x)\right| = \left|1-\frac{1}{x^2}\right|$ is quasi-geometrically convex on about $\left[\frac{1}{2},2\right]$. The function $\omega:\left[\frac{1}{2},2\right]\to[0,\infty)$ defined by*

$$\omega(x) = \begin{cases} \frac{1}{x}, & x\in\left[\frac{1}{2},1\right), \\ x, & x\in[1,2], \end{cases}$$

is geometrically symmetric about 1. *We observed that for $\varsigma=\frac{1}{2}$, we get that*

$$\begin{aligned}
&J^{\varsigma}_{x^-}\omega\chi(\eta) + J^{\varsigma}_{x^+}\omega\chi(\mu) \\
&= \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^2(\ln 2\varrho)^{-\frac{1}{2}}\,\omega\chi(\varrho)\frac{d\varrho}{\varrho} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^2\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\omega\chi(\varrho)\frac{d\varrho}{\varrho} \\
&= \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^1(\ln(2\varrho))^{-\frac{1}{2}}\frac{1}{\varrho}\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^1\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\frac{1}{\varrho}\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho} \\
&+ \frac{1}{\Gamma(\frac{1}{2})}\int_1^2(\ln(2\varrho))^{-\frac{1}{2}}\varrho\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho} + \frac{1}{\Gamma(\frac{1}{2})}\int_1^2\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\varrho\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho} = \frac{4\left(\sqrt{2\pi}erf\left(\sqrt{\ln(4)}\right)+\sqrt{\ln(2)}\right)}{\sqrt{\pi}} \\
&\qquad + \frac{2\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\left(erfi\left(\sqrt{\ln(16)}\right)-erfi\left(\sqrt{\ln(4)}\right)\right)-2\sqrt{\ln(2)}+2\sqrt{\ln(4)}\right)}{\sqrt{\pi}}.
\end{aligned}$$

We also observe that

$$\begin{aligned}
&\left[J^{\varsigma}_{x^-}\omega(\eta)+J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x)\\
&=\left[\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{2}\left(\ln\left(2\varrho\right)\right)^{-\frac{1}{2}}\omega(\varrho)\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{2}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\omega(\varrho)\frac{d\varrho}{\varrho}\right]\left(x+\frac{1}{x}\right)\\
&=\left(\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{1}\left(\ln\left(2\varrho\right)\right)^{-\frac{1}{2}}\frac{d\varrho}{\varrho^2}+\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{1}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\frac{d\varrho}{\varrho^2}\right)\left(x+\frac{1}{x}\right)\\
&+\left(\frac{1}{\Gamma(\frac{1}{2})}\int_{1}^{2}\left(\ln\left(2\varrho\right)\right)^{-\frac{1}{2}}d\varrho+\frac{1}{\Gamma(\frac{1}{2})}\int_{1}^{2}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}d\varrho\right)\left(x+\frac{1}{x}\right)\\
&\qquad=\left[4erf\left(\sqrt{\ln(2)}\right)-erfi\left(\sqrt{\ln(2)}\right)+erfi\left(\sqrt{\ln(4)}\right)\right]\left(x+\frac{1}{x}\right).
\end{aligned}$$

Hence for $x\in\left[\frac{1}{2},2\right]$, *we get*

$$\begin{aligned}
&\psi\left(x\right):=J^{\varsigma}_{x^-}\omega\chi(\eta)+J^{\varsigma}_{x^+}\omega\chi(\mu)-\left[J^{\varsigma}_{x^-}\omega(\eta)+J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x)\\
&=\frac{4\left(\sqrt{2\pi}erf\left(\sqrt{\ln(4)}\right)+\sqrt{\ln(2)}\right)}{\sqrt{\pi}}\\
&+\frac{2\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\left(erfi\left(\sqrt{\ln(16)}\right)-erfi\left(\sqrt{\ln(4)}\right)\right)-2\sqrt{\ln(2)}+2\sqrt{\ln(4)}\right)}{\sqrt{\pi}}\\
&\qquad-\left[4erf\left(\sqrt{\ln(2)}\right)-erfi\left(\sqrt{\ln(2)}\right)+erfi\left(\sqrt{\ln(4)}\right)\right]\left(x+\frac{1}{x}\right).
\end{aligned}$$

Now

$$\begin{aligned}
&\varphi\left(x\right):=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\max\left\{\left|\chi'(x)\right|,\left|\chi'(\mu)\right|\right\}\|\omega\|_{[x,\mu],\infty}+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\\
&\times\max\left\{\left|\chi'(x)\right|,\left|\chi'(\eta)\right|\right\}\|\omega\|_{[\eta,x],\infty}=\frac{\left(\ln\frac{2}{x}\right)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\max\left\{\left|1-\frac{1}{x^2}\right|,\frac{3}{4}\right\}\|\omega\|_{[x,2],\infty}\\
&+\frac{\left(\ln 2x\right)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\max\left\{\left|1-\frac{1}{x^2}\right|,3\right\}\|\omega\|_{[\frac{1}{2},x],\infty}\le\|\omega\|_{[\frac{1}{2},2],\infty}\\
&\qquad\times\left[\frac{\left(\ln\frac{2}{x}\right)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\max\left\{\left|1-\frac{1}{x^2}\right|,\frac{3}{4}\right\}+\frac{\left(\ln 2x\right)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\max\left\{\left|1-\frac{1}{x^2}\right|,3\right\}\right].
\end{aligned}\tag{12}$$

Corollary 2.4. *Letting* $x=\sqrt{\eta\mu}$ *in Theorem 2.2, we get the following:*

$$\begin{aligned}
&\Big|J^{\varsigma}_{\sqrt{\eta\mu}^-}\omega\chi(\eta)+J^{\varsigma}_{\sqrt{\eta\mu}^+}\omega\chi(\mu)\\
&-\left[J^{\varsigma}_{\sqrt{\eta\mu}^-}\omega(\eta)+J^{\varsigma}_{\sqrt{\eta\mu}^+}\omega(\mu)\right]\chi(\sqrt{\eta\mu})\Big|\le\frac{\|\omega\|_{[\eta,\mu],\infty}\left(\ln\frac{\mu}{\eta}\right)^{\varsigma+1}}{2^{\varsigma+1}\Gamma(\varsigma+2)}\\
&\qquad\times\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|,\left|\chi'(\mu)\right|\right\}+\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|,\left|\chi'(\eta)\right|\right\}\right).
\end{aligned}\tag{13}$$

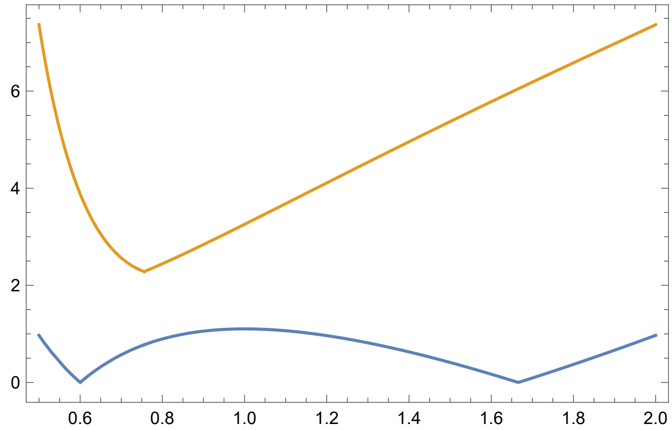

Figure 1: The graph of validates the inequality proved in Theorem (2.2) over the interval $[\frac{1}{2}, 2]$

Corollary 2.5. *Choosing $\omega(u) = \frac{1}{\ln \frac{\mu}{\eta}}$ in Theorem 2.2 , we get*

$$\left| J_{x^-}^{\varsigma} \chi(\eta) + J_{x^+}^{\varsigma} \chi(\mu) - \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma} + \left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma}}{\Gamma(\varsigma+1)} \chi(x) \right| \leq \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)} \times \max\left\{|\chi'(x)|, |\chi'(\mu)|\right\} + \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)} \max\left\{|\chi'(x)|, |\chi'(\eta)|\right\}. \quad (14)$$

Moreover, if we take $x = \sqrt{\eta\mu}$, we obtain

$$\left| J_{x^-}^{\varsigma} \chi(\eta) + J_{x^+}^{\varsigma} \chi(\mu) - \frac{\left(\ln \frac{\mu}{\eta}\right)^{\varsigma}}{2^{\varsigma-1}\Gamma(\varsigma+1)} \chi(\sqrt{\eta\mu}) \right| \leq \frac{\left(\ln \frac{\mu}{\eta}\right)^{\varsigma}}{2^{\varsigma}\Gamma(\varsigma+2)} \max\left\{|\chi'(x)|, |\chi'(\mu)|\right\} + \max\left\{|\chi'(x)|, |\chi'(\eta)|\right\}.$$

Corollary 2.6. *Let $\varsigma = 1$ in Theorem 2.2, then*

$$\left| \int_{\eta}^{\mu} \omega(u)\chi(u)du - \left(\int_{\eta}^{\mu} \omega(u)du \right) \chi(x) \right| \leq \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^2}{2} \max\left\{|\chi'(x)|, |\chi'(\mu)|\right\} \|\omega\|_{[x,\mu],\infty} + \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^2}{2} \max\left\{|\chi'(x)|, |\chi'(\eta)|\right\} \|\omega\|_{[\eta,x],\infty}. \quad (15)$$

Moreover, for $x = \sqrt{\eta\mu}$, we get

$$\left| \int_{\eta}^{\mu} \omega(u)\chi(u)du - \left(\int_{\eta}^{\mu} \omega(u)du \right) \chi\left(\sqrt{\eta\mu}\right) \right| \leq \frac{\left(\ln \frac{\mu}{\eta}\right)^2}{8} \|\omega\|_{[\eta,\mu],\infty} \times \left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|, |\chi'(\mu)|\right\} + \max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|, |\chi'(\eta)|\right\} \right). \quad (16)$$

Theorem 2.7. *Let the conditions of Lemma 2.1 hold and let $|\chi'|^q$ is quasi-geometrically convex, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,* *then*

$$\left|J_{x^-}^{\varsigma}\omega\chi(\eta)+J_{x^+}^{\varsigma}\omega\chi(\mu)-\left[J_{x^-}^{\varsigma}\omega(\eta)+J_{x^+}^{\varsigma}\omega(\mu)\right]\chi(x)\right|$$
$$\leq\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{(\varsigma p+1)^{\frac{1}{p}}\Gamma(\varsigma+1)}\|\omega\|_{[x,\mu],\infty}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{(\varsigma p+1)^{\frac{1}{p}}\Gamma(\varsigma+1)}\|\omega\|_{[\eta,x],\infty}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}. \quad (17)$$

Proof. By using the modulus properties, Lemma 2.1, Hölder's inequality, and the fact that $|\chi'|^q$ is geometrically convex, we get

$$\left|J_{x^-}^{\varsigma}\omega\chi(\eta)+J_{x^+}^{\varsigma}\omega\chi(\mu)-\left[J_{x^-}^{\varsigma}\omega(\eta)+J_{x^+}^{\varsigma}\omega(\mu)\right]\chi(x)\right|$$
$$\leq\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|p_1(\varrho)\right|^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\chi'(\mu^{\varrho}x^{1-\varrho})\right|^q d\varrho\right)^{\frac{1}{q}}$$
$$+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|p_2(\varrho)\right|^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\chi'(\eta^{\varrho}x^{1-\varrho})\right|^q d\varrho\right)^{\frac{1}{q}}\leq\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}$$
$$\times\left(\int_0^1\left|\int_0^1(1-\rho)^{\varsigma-1}\omega(\mu^{\rho}x^{1-\rho})d\rho\right|d\varrho\right)^{\frac{1}{p}}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|\int_{\varrho}^1(1-\rho)^{\varsigma-1}\omega(\eta^{\rho}x^{1-\rho})d\rho\right|d\varrho\right)^{\frac{1}{p}}$$
$$\times\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}\leq\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\|\omega\|_{[x,\mu],\infty}$$
$$\times\left(\int_0^1\left|\int_{\varrho}^1(1-\rho)^{\varsigma-1}d\rho\right|^p d\varrho\right)^{\frac{1}{p}}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}$$
$$\times\|\omega\|_{[\eta,x],\infty}\left(\int_0^1\left|\int_{\varrho}^1(1-\rho)^{\varsigma-1}d\rho\right|^p d\varrho\right)^{\frac{1}{p}}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{(\varsigma p+1)^{\frac{1}{p}}\Gamma(\varsigma+1)}\|\omega\|_{[x,\mu],\infty}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{(\varsigma p+1)^{\frac{1}{p}}\Gamma(\varsigma+1)}\|\omega\|_{[\eta,x],\infty}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}. \quad (18)$$

The proof is completed. □

Example 2.8. *Let us consider $\chi:(0,\infty)\to\mathbb{R}$ defined by $\chi(x)=x+\frac{1}{x}$. It is clear that the function $\left|\chi'(x)\right|^{\frac{3}{2}}=\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}}$ is quasi-geometrically convex on about $\left[\frac{1}{2},2\right]$. The function $\omega:\left[\frac{1}{2},2\right]\to[0,\infty)$ defined by*

$$\omega(x)=\begin{cases}\frac{1}{x}, & x\in\left[\frac{1}{2},1\right),\\ x, & x\in[1,2],\end{cases}$$

is geometrically symmetric about 1. *We observed that for* $\varsigma = \frac{1}{2}$, *we get that*

$$\begin{aligned}
&J^{\varsigma}_{x^-}\omega\chi(\eta) + J^{\varsigma}_{x^+}\omega\chi(\mu) \\
&= \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{2}(\ln(2\varrho))^{-\frac{1}{2}}\,\omega\chi(\varrho)\frac{d\varrho}{\varrho} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{2}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\omega\chi(\varrho)\frac{d\varrho}{\varrho} \\
&= \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{1}(\ln(2\varrho))^{-\frac{1}{2}}\,\frac{1}{\varrho}\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{1}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\frac{1}{\varrho}\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho} \\
&+ \frac{1}{\Gamma(\frac{1}{2})}\int_{1}^{2}(\ln(2\varrho))^{-\frac{1}{2}}\,\varrho\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho} + \frac{1}{\Gamma(\frac{1}{2})}\int_{1}^{2}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\varrho\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho} = \frac{4\left(\sqrt{2\pi}erf\left(\sqrt{\ln(4)}\right)+\sqrt{\ln(2)}\right)}{\sqrt{\pi}} \\
&\qquad + \frac{2\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\left(erfi\left(\sqrt{\ln(16)}\right)-erfi\left(\sqrt{\ln(4)}\right)\right)-2\sqrt{\ln(2)}+2\sqrt{\ln(4)}\right)}{\sqrt{\pi}}.
\end{aligned}$$

We also observe that

$$\begin{aligned}
&\left[J^{\varsigma}_{x^-}\omega(\eta) + J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x) \\
&= \left[\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{2}(\ln(2\varrho))^{-\frac{1}{2}}\,\omega(\varrho)\frac{d\varrho}{\varrho} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{2}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\omega(\varrho)\frac{d\varrho}{\varrho}\right]\left(x+\frac{1}{x}\right) \\
&= \left(\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{1}(\ln(2\varrho))^{-\frac{1}{2}}\,\frac{d\varrho}{\varrho^2} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{1}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\frac{d\varrho}{\varrho^2}\right)\left(x+\frac{1}{x}\right) \\
&+ \left(\frac{1}{\Gamma(\frac{1}{2})}\int_{1}^{2}(\ln(2\varrho))^{-\frac{1}{2}}\,d\varrho + \frac{1}{\Gamma(\frac{1}{2})}\int_{1}^{2}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}d\varrho\right)\left(x+\frac{1}{x}\right) \\
&\qquad = \left[4erf\left(\sqrt{\ln(2)}\right) - erfi\left(\sqrt{\ln(2)}\right) + erfi\left(\sqrt{\ln(4)}\right)\right]\left(x+\frac{1}{x}\right).
\end{aligned}$$

Hence for $x \in \left[\frac{1}{2}, 2\right]$, *we get*

$$\begin{aligned}
&\psi(x) := J^{\varsigma}_{x^-}\omega\chi(\eta) + J^{\varsigma}_{x^+}\omega\chi(\mu) - \left[J^{\varsigma}_{x^-}\omega(\eta) + J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x) \\
&= \frac{4\left(\sqrt{2\pi}erf\left(\sqrt{\ln(4)}\right)+\sqrt{\ln(2)}\right)}{\sqrt{\pi}} \\
&+ \frac{2\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\left(erfi\left(\sqrt{\ln(16)}\right)-erfi\left(\sqrt{\ln(4)}\right)\right)-2\sqrt{\ln(2)}+2\sqrt{\ln(4)}\right)}{\sqrt{\pi}} \\
&\qquad - \left[4erf\left(\sqrt{\ln(2)}\right) - erfi\left(\sqrt{\ln(2)}\right) + erfi\left(\sqrt{\ln(4)}\right)\right]\left(x+\frac{1}{x}\right).
\end{aligned}$$

Now for $x \in \left[\frac{1}{2}, 2\right]$, $p = 3$, $q = \frac{3}{2}$

$$\varphi(x) := \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{(1+\varsigma p)^{\frac{1}{p}}\Gamma(\varsigma+1)}\left(\max\left\{|\chi'(x)|^q, \left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}} \|\omega\|_{[x,\mu],\infty}$$
$$+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{(1+\varsigma p)^{\frac{1}{p}}\Gamma(\varsigma+2)}\left(\max\left\{|\chi'(x)|^q, \left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}} \|\omega\|_{[\eta,x],\infty}$$
$$=\frac{\left(\ln\frac{2}{x}\right)^{\frac{3}{2}}}{\left(\frac{5}{2}\right)^{\frac{1}{3}}\Gamma(\frac{5}{2})}\left(\max\left\{\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}}, \left(\frac{3}{4}\right)^{\frac{3}{2}}\right\}\right)^{\frac{2}{3}} \|\omega\|_{[x,2],\infty}$$
$$+\frac{(\ln 2x)^{\frac{3}{2}}}{\left(\frac{5}{2}\right)^{\frac{1}{3}}\Gamma(\frac{5}{2})}\left(\max\left\{\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}}, 3^{\frac{3}{2}}\right\}\right)^{\frac{2}{3}} \|\omega\|_{[\frac{1}{2},x],\infty}$$
$$\leq\left[\frac{\left(\ln\frac{2}{x}\right)^{\frac{3}{2}}}{\left(\frac{5}{2}\right)^{\frac{1}{3}}\Gamma(\frac{5}{2})}\left(\max\left\{\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}}, \left(\frac{3}{4}\right)^{\frac{3}{2}}\right\}\right)^{\frac{2}{3}}+\frac{(\ln 2x)^{\frac{3}{2}}}{\left(\frac{5}{2}\right)^{\frac{1}{3}}\Gamma(\frac{5}{2})}\left(\max\left\{\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}}, 3^{\frac{3}{2}}\right\}\right)^{\frac{2}{3}}\right]\|\omega\|_{[\frac{1}{2},2],\infty} \quad (19)$$

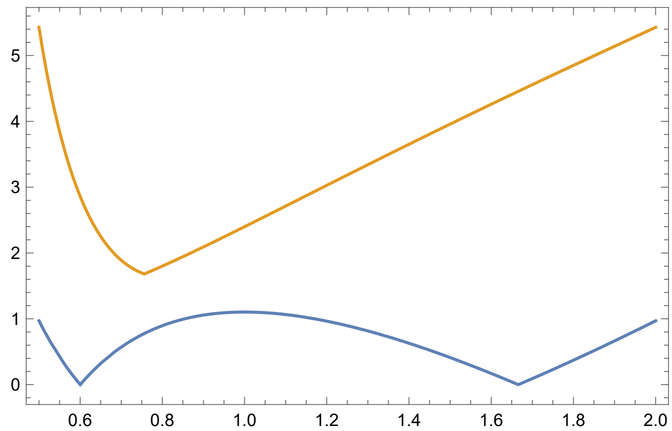

Figure 2: The graph of validates the inequality proved in Theorem (2.7) over the interval $[\frac{1}{2}, 2]$

Corollary 2.9. *If we take* $x = \sqrt{\eta\mu}$ *in Theorem 2.7, we get the following:*

$$\left|J^{\varsigma}_{\sqrt{\eta\mu}^-}\omega\chi(\eta) + J^{\varsigma}_{\sqrt{\eta\mu}^+}\omega\chi(\mu) - \left[J^{\varsigma}_{\sqrt{\eta\mu}^-}\omega(\eta) + J^{\varsigma}_{\sqrt{\eta\mu}^+}\omega(\mu)\right]\chi\left(\sqrt{\eta\mu}\right)\right|$$
$$\leq \frac{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma+1}}{2^{\varsigma+1}(\varsigma p+1)^{\frac{1}{p}}\Gamma(\varsigma+1)}\|\omega\|_{[\eta,\mu],\infty}\left(\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^q, \left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^q, \left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}\right). \quad (20)$$

Corollary 2.10. *Suppose that $\omega(u)=\frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ in Theorem 2.7, we get the following:*

$$\left|J^{\varsigma}_{x^-}\chi(\eta)+J^{\varsigma}_{x^+}\chi(\mu)-\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma}+\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma}}{\Gamma(\varsigma+1)}\chi(x)\right|\leq\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{(\varsigma p+1)^{\frac{1}{p}}\Gamma(\varsigma+1)}$$
$$\times\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{(\varsigma p+1)^{\frac{1}{p}}\Gamma(\varsigma+1)}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{q}}. \quad (21)$$

Moreover, if we take $x=\sqrt{\eta\mu}$, we obtain

$$\left|\frac{2^{\varsigma-1}\Gamma(\varsigma+1)}{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma}}\left(J^{\varsigma}_{\sqrt{\eta\mu}}-\chi(\eta)+J^{\varsigma}_{\sqrt{\eta\mu}}+\chi(\mu)\right)-\chi\left(\sqrt{\eta\mu}\right)\right|\leq\frac{\left(\ln\frac{\mu}{\eta}\right)}{4(\varsigma p+1)^{\frac{1}{p}}}\left(\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{q}}\right). \quad (22)$$

Corollary 2.11. *Let $\varsigma=1$ be in Theorem 2.7, we get the following:*

$$\left|\int_{\eta}^{\mu}\omega(u)\chi(u)du-\left(\int_{\eta}^{\mu}\omega(u)du\right)\chi(x)\right|\leq\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{2}}{(p+1)^{\frac{1}{p}}}\|\omega\|_{[x,\mu],\infty}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}$$
$$+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{2}}{(p+1)^{\frac{1}{p}}}\|\omega\|_{[\eta,x],\infty}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{q}}. \quad (23)$$

Moreover, considering $x=\sqrt{\eta\mu}$, we get

$$\left|\int_{\eta}^{\mu}\omega(u)\chi(u)du-\left(\int_{\eta}^{\mu}\omega(u)du\right)\chi\left(\sqrt{\eta\mu}\right)\right|\leq\frac{\left(\ln\frac{\mu}{\eta}\right)^{2}}{4(p+1)^{\frac{1}{p}}}\|\omega\|_{[\eta,\mu],\infty}\left(\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{2}}\right). \quad (24)$$

Corollary 2.12. *Let $\omega(u)=\frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ be in Corollary 2.11, we get the following inequality:*

$$\left|\frac{1}{\ln\frac{\mu}{\eta}}\int_{\eta}^{\mu}\chi(u)du-\chi(x)\right|\leq\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{2}}{\left(\ln\frac{\mu}{\eta}\right)(p+1)^{\frac{1}{p}}}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}$$
$$+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{2}}{\left(\ln\frac{\mu}{\eta}\right)(p+1)^{\frac{1}{p}}}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{q}}. \quad (25)$$

Theorem 2.13. *Using assumptions of Lemma 2.1 and let $|\chi'|^{q}$ is quasi-geometrically convex, where $q\geq1$, then*

$$\left|J^{\varsigma}_{x^-}\omega\chi(\eta)+J^{\varsigma}_{x^+}\omega\chi(\mu)-\left[J^{\varsigma}_{x^-}\omega(\eta)+J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x)\right|\leq\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\|\omega\|_{[x,\mu],\infty}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}$$
$$+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\|\omega\|_{[\eta,x],\infty}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{q}}. \quad (26)$$

Proof. By using the modulus properties, Lemma 2.1, power mean inequality, and the fact that $|\chi'|^q$ is geometrically convex, we get

$$
\begin{aligned}
&\left|J_{x^-}^{\varsigma}\omega\chi(\eta)+J_{x^+}^{\varsigma}\omega\chi(\mu)-\left[J_{x^-}^{\varsigma}\omega(\eta)+J_{x^+}^{\varsigma}\omega(\mu)\right]\chi(x)\right|\\
&\le\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|p_1(\varrho)\right|d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1\left|p_1(\varrho)\right|\left|\chi'(\mu^{\varrho}x^{1-\varrho})\right|^q d\varrho\right)^{\frac{1}{q}}\\
&+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|p_2(\varrho)\right|d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1\left|p_2(\varrho)\right|\left|\chi'(\eta^{\varrho}x^{1-\varrho})\right|^q d\varrho\right)^{\frac{1}{q}}\\
&\le\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|p_1(\varrho)\right|d\varrho\right)^{1-\frac{1}{q}}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\int_0^1\left|p_1(\varrho)\right|d\varrho\right)^{\frac{1}{q}}\\
&+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|p_2(\varrho)\right|d\varrho\right)^{1-\frac{1}{q}}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\int_0^1\left|p_2(\varrho)\right|d\varrho\right)^{\frac{1}{q}}\\
&=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|p_1(\varrho)\right|d\varrho\right)\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}\\
&+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\left(\int_0^1\left|p_2(\varrho)\right|d\varrho\right)\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\le\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\|\omega\|_{[x,\mu],\infty}\left(\int_0^1\left(\int_{\varrho}^1(1-\rho)^{\varsigma-1}d\rho\right)d\varrho\right)\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}\\
&+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\|\omega\|_{[\eta,x],\infty}\left(\int_0^1\left(\int_{\varrho}^1(1-\rho)^{\varsigma-1}d\rho\right)d\varrho\right)\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\quad=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\|\omega\|_{[x,\mu],\infty}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\|\omega\|_{[\eta,x],\infty}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}
$$

The proof is completed. □

Example 2.14. *Let us consider $\chi:(0,\infty)\to\mathbb{R}$ defined by $\chi(x)=x+\frac{1}{x}$. It is clear that the function $\left|\chi'(x)\right|=\left|1-\frac{1}{x^2}\right|$ is quasi-geometrically convex on about $\left[\frac{1}{2},2\right]$. The function $\omega:\left[\frac{1}{2},2\right]\to[0,\infty)$ defined by*

$$
\omega(x)=\begin{cases}\frac{1}{x}, & x\in\left[\frac{1}{2},1\right),\\ x, & x\in[1,2],\end{cases}
$$

is geometrically symmetric about 1. *We observed that for $\varsigma=\frac{1}{2}$, we get that*

$$
\begin{aligned}
&J_{x^-}^{\varsigma}\omega\chi(\eta)+J_{x^+}^{\varsigma}\omega\chi(\mu)\\
&=\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{2}(\ln(2\varrho))^{-\frac{1}{2}}\,\omega\chi(\varrho)\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{2}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\omega\chi(\varrho)\frac{d\varrho}{\varrho}\\
&\qquad=\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{1}(\ln(2\varrho))^{-\frac{1}{2}}\frac{1}{\varrho}\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^{1}\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\frac{1}{\varrho}\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho}
\end{aligned}
$$

$$+\frac{1}{\Gamma(\frac{1}{2})}\int_1^2 (\ln(2\varrho))^{-\frac{1}{2}}\varrho\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\frac{1}{2})}\int_1^2\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\varrho\left(\varrho+\frac{1}{\varrho}\right)\frac{d\varrho}{\varrho}=\frac{4\left(\sqrt{2\pi}erf\left(\sqrt{\ln(4)}\right)+\sqrt{\ln(2)}\right)}{\sqrt{\pi}}$$
$$+\frac{2\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\left(erfi\left(\sqrt{\ln(16)}\right)-erfi\left(\sqrt{\ln(4)}\right)\right)-2\sqrt{\ln(2)}+2\sqrt{\ln(4)}\right)}{\sqrt{\pi}}.$$

We also observe that

$$\begin{aligned}
&\left[J_{x^-}^{\varsigma}\omega(\eta)+J_{x^+}^{\varsigma}\omega(\mu)\right]\chi(x)\\
&=\left[\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^2(\ln(2\varrho))^{-\frac{1}{2}}\omega(\varrho)\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^2\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\omega(\varrho)\frac{d\varrho}{\varrho}\right]\left(x+\frac{1}{x}\right)\\
&=\left(\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^1(\ln(2\varrho))^{-\frac{1}{2}}\frac{d\varrho}{\varrho^2}+\frac{1}{\Gamma(\frac{1}{2})}\int_{\frac{1}{2}}^1\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}\frac{d\varrho}{\varrho^2}\right)\left(x+\frac{1}{x}\right)\\
&+\left(\frac{1}{\Gamma(\frac{1}{2})}\int_1^2(\ln(2\varrho))^{-\frac{1}{2}}d\varrho+\frac{1}{\Gamma(\frac{1}{2})}\int_1^2\left(\ln\frac{2}{\varrho}\right)^{-\frac{1}{2}}d\varrho\right)\left(x+\frac{1}{x}\right)\\
&\qquad=\left[4erf\left(\sqrt{\ln(2)}\right)-erfi\left(\sqrt{\ln(2)}\right)+erfi\left(\sqrt{\ln(4)}\right)\right]\left(x+\frac{1}{x}\right).
\end{aligned}$$

Hence for $x\in\left[\frac{1}{2},2\right]$, we get

$$\begin{aligned}
&\psi(x):=J_{x^-}^{\varsigma}\omega\chi(\eta)+J_{x^+}^{\varsigma}\omega\chi(\mu)-\left[J_{x^-}^{\varsigma}\omega(\eta)+J_{x^+}^{\varsigma}\omega(\mu)\right]\chi(x)\\
&=\frac{4\left(\sqrt{2\pi}erf\left(\sqrt{\ln(4)}\right)+\sqrt{\ln(2)}\right)}{\sqrt{\pi}}\\
&+\frac{2\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\left(erfi\left(\sqrt{\ln(16)}\right)-erfi\left(\sqrt{\ln(4)}\right)\right)-2\sqrt{\ln(2)}+2\sqrt{\ln(4)}\right)}{\sqrt{\pi}}\\
&\qquad-\left[4erf\left(\sqrt{\ln(2)}\right)-erfi\left(\sqrt{\ln(2)}\right)+erfi\left(\sqrt{\ln(4)}\right)\right]\left(x+\frac{1}{x}\right).
\end{aligned}$$

Now for $x\in\left[\frac{1}{2},2\right]$ and $q=\frac{3}{2}$

$$\begin{aligned}
&\varphi(x):=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+1)}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}\|\omega\|_{[x,\mu],\infty}\\
&+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\left(\max\left\{|\chi'(x)|^q,\left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}\|\omega\|_{[\eta,x],\infty}\\
&=\frac{\left(\ln\frac{2}{x}\right)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\left(\max\left\{\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}},\left(\frac{3}{4}\right)^{\frac{3}{2}}\right\}\right)^{\frac{2}{3}}\|\omega\|_{[x,2],\infty}\\
&+\frac{(\ln 2x)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\left(\max\left\{\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}},3^{\frac{3}{2}}\right\}\right)^{\frac{2}{3}}\|\omega\|_{[\frac{1}{2},x],\infty}\\
&\qquad\le\left[\frac{\left(\ln\frac{2}{x}\right)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\left(\max\left\{\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}},\left(\frac{3}{4}\right)^{\frac{3}{2}}\right\}\right)^{\frac{2}{3}}+\frac{(\ln 2x)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\left(\max\left\{\left|1-\frac{1}{x^2}\right|^{\frac{3}{2}},3^{\frac{3}{2}}\right\}\right)^{\frac{2}{3}}\right]\|\omega\|_{[\frac{1}{2},2],\infty}
\end{aligned}\tag{27}$$

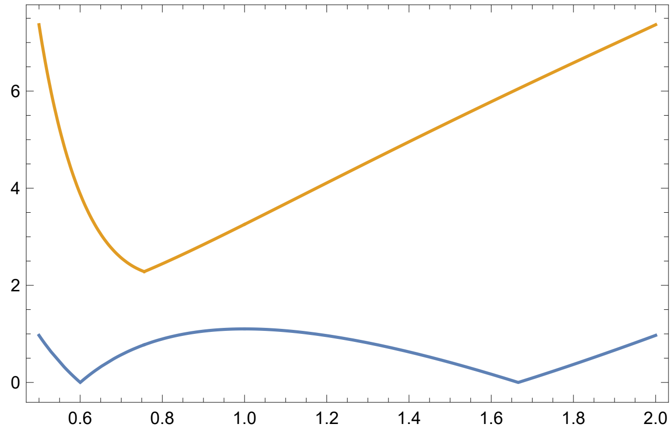

Figure 3: The graph of validates the inequality proved in Theorem (2.13) over the interval $[\frac{1}{2}, 2]$

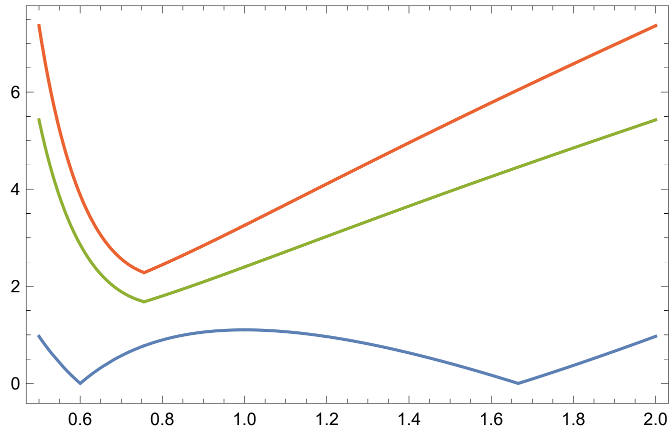

Figure 4: The graphs of error bounds given in Theorem (2.2) , Theorem (2.7) and Theorem (2.13) over the interval $[\frac{1}{2}, 2]$

Remark 2.15. *The comparison of the error bounds is shown in the figure below*

Corollary 2.16. *Taking* $x = \sqrt{\eta\mu}$ *in Theorem 2.13, we get the following:*

$$\left| J^{\varsigma}_{\sqrt{\eta\mu}^-}\omega\chi(\eta) + J^{\varsigma}_{\sqrt{\eta\mu}^+}\omega\chi(\mu) - \left[J^{\varsigma}_{\sqrt{\eta\mu}^-}\omega(\eta) + J^{\varsigma}_{\sqrt{\eta\mu}^+}\omega(\mu)\right]\chi\left(\sqrt{\eta\mu}\right)\right|$$
$$\leq \frac{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma+1}}{2^{\varsigma+1}\Gamma(\varsigma+2)}\|\omega\|_{[\eta,\mu],\infty}\left(\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^q, \left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^q, |\chi'(\eta)|^q\right\}\right)^{\frac{1}{q}}\right). \quad (28)$$

Corollary 2.17. *Letting* $\omega(u) = \frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ *in Theorem 2.13, we get the following inequality:*

$$\left| J^{\varsigma}_{x^-}\chi(\eta) + J^{\varsigma}_{x^+}\chi(\mu) - \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma} + \left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma}}{\Gamma(\varsigma+1)}\chi(x)\right|$$
$$\leq \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\left(\max\left\{|\chi'(x)|^q, \left|\chi'(\mu)\right|^q\right\}\right)^{\frac{1}{q}} + \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma+2)}\left(\max\left\{|\chi'(x)|^q, \left|\chi'(\eta)\right|^q\right\}\right)^{\frac{1}{q}}. \quad (29)$$

Moreover, if we take $x = \sqrt{\eta\mu}$, we obtain

$$\left|\frac{2^{\varsigma-1}\Gamma(\varsigma+1)}{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma}}\left(J^{\varsigma}_{\sqrt{\eta\mu}^-}\chi(\eta)+J^{\varsigma}_{\sqrt{\eta\mu}^+}\chi(\mu)\right)-\chi\left(\sqrt{\eta\mu}\right)\right| \le \frac{\ln\frac{\mu}{\eta}}{4(\varsigma+1)} \times\left(\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}+\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{q}}\right). \quad (30)$$

Corollary 2.18. *When we choose $\varsigma = 1$ in Theorem 2.13, we get the following result:*

$$\left|\int_{\eta}^{\mu}\omega(u)\chi(u)du-\left(\int_{\eta}^{\mu}\omega(u)du\right)\chi(x)\right| \le \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{2}}{2}\|\omega\|_{[x,\mu],\infty}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{2}}{2}\|\omega\|_{[\eta,x],\infty}\left(\max\left\{|\chi'(x)|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{q}}. \quad (31)$$

Moreover, considering $x = \sqrt{\eta\mu}$, we get

$$\left|\int_{\eta}^{\mu}\omega(u)\chi(u)du-\left(\int_{\eta}^{\mu}\omega(u)du\right)\chi\left(\sqrt{\eta\mu}\right)\right| \le \frac{\left(\ln\frac{\mu}{\eta}\right)^{2}}{8(p+1)^{\frac{1}{p}}}\|\omega\|_{[\eta,\mu],\infty}\left(\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^{q},\left|\chi'(\mu)\right|^{q}\right\}\right)^{\frac{1}{q}}\right. \left.+\left(\max\left\{\left|\chi'\left(\sqrt{\eta\mu}\right)\right|^{q},\left|\chi'(\eta)\right|^{q}\right\}\right)^{\frac{1}{2}}\right). \quad (32)$$

3. Further Results

This section begins with the following results.

Theorem 3.1. *Using all conditions of Lemma 2.1 and let there exist constants $m < M$ such that $-\infty < m \le \chi'(x) \le M < +\infty$ for all $x \in J$, then*

$$\left|\Lambda^{\varsigma}(\eta,x,\mu,\omega,\chi)\right| \le \frac{(M-m)\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma+2)}\|\omega\|_{[x,\mu],\infty}+\frac{(M-m)\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma+2)}\|\omega\|_{[\eta,x],\infty}, \quad (33)$$

where

$$\Lambda^{\varsigma}(\eta,x,\mu,\omega,\chi) := J^{\varsigma}_{x^-}\omega\chi(\eta)+J^{\varsigma}_{x^+}\omega\chi(\mu)-\left[J^{\varsigma}_{x^-}\omega(\eta)+J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x) -\frac{(M+m)\left[\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}-\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}\right]}{2\Gamma(\varsigma)}\int_0^1\left(p_1(\varrho)+p_2(\varrho)\right)d\varrho, \quad (34)$$

where p_1, p_2 are defined as in (3) and (4), respectively.

Proof. We get the following result by using the Lemma 2.1,

$$
\begin{aligned}
&J_{x^-}^{\varsigma}\omega\chi(\eta)+J_{x^+}^{\varsigma}\omega\chi(\mu)-\left[J_{x^-}^{\varsigma}\omega(\eta)+J_{x^+}^{\varsigma}\omega(\mu)\right]\chi(x)\\
&=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\chi'(\mu^{\varrho}x^{1-\varrho})d\varrho-\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\chi'(\eta^{\varrho}x^{1-\varrho})d\varrho\\
&=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\left[\chi'(\mu^{\varrho}x^{1-\varrho})-\frac{M+m}{2}+\frac{M+m}{2}\right]d\varrho\\
&-\frac{\left(\ln\left(\frac{\eta}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\left[\chi'(\eta^{\varrho}x^{1-\varrho})-\frac{M+m}{2}+\frac{M+m}{2}\right]d\varrho\\
&=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\left[\chi'(\mu^{\varrho}x^{1-\varrho})-\frac{M+m}{2}\right]d\varrho+\frac{(M+m)\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma)}\int_0^1 p_1(\varrho)d\varrho\\
&\qquad-\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\left[\chi'(\eta^{\varrho}x^{1-\varrho})-\frac{M+m}{2}\right]d\varrho+\frac{(M+m)\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma)}\int_0^1 p_2(\varrho)d\varrho. \qquad (35)
\end{aligned}
$$

From (35), we have

$$
\begin{aligned}
\Lambda^{\varsigma}(\eta,x,\mu,\omega,\chi)&=\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\left[\chi'(\mu^{\varrho}x^{1-\varrho})-\frac{M+m}{2}\right]d\varrho\\
&\qquad-\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\left[\chi'(\eta^{\varrho}x^{1-\varrho})-\frac{M+m}{2}\right]d\varrho, \qquad (36)
\end{aligned}
$$

where $\Lambda^{\varsigma}(\eta,x,\mu,\omega,\chi)$ is determined as it is in (34). After applying absolute value on both sides of (36) and making use of the fact that $m\le\chi'(x)\le M$ holds for all $x\in J$, we have come at the following conclusion:

$$
\begin{aligned}
&\left|\Lambda^{\varsigma}(\eta,x,\mu,\omega,\chi)\right|\\
&\le\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 \left|p_1(\varrho)\right|\left|\chi'(\mu^{\varrho}x^{1-\varrho})-\frac{M+m}{2}\right|d\varrho\\
&+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 \left|p_2(\varrho)\right|\left|\chi'(\eta^{\varrho}x^{1-\varrho})-\frac{M+m}{2}\right|d\varrho\\
&\le\frac{(M-m)\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma)}\int_0^1\left|\int_{\varrho}^1(1-\rho)^{\varsigma-1}\omega(\mu^{\rho}x^{1-\rho})d\rho\right|d\varrho\\
&+\frac{(M-m)\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma)}\int_0^1\left|\int_{\varrho}^1(1-\rho)^{\varsigma-1}\omega(\eta^{\rho}x^{1-\rho})d\rho\right|d\varrho\\
&\qquad\le\frac{(M-m)\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma+2)}\|\omega\|_{[x,\mu],\infty}+\frac{(M-m)\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma+2)}\|\omega\|_{[\eta,x],\infty}. \qquad (37)
\end{aligned}
$$

The proof is completed. □

Corollary 3.2. *When we choose $x=\sqrt{\eta\mu}$ in Theorem 3.1, we get the following:*

$$
\left|J_{\sqrt{\eta\mu}^-}^{\varsigma}\omega\chi(\eta)+J_{\sqrt{\eta\mu}^+}^{\varsigma}\omega\chi(\mu)-\left[J_{\sqrt{\eta\mu}^-}^{\varsigma}\omega(\eta)+J_{\sqrt{\eta\mu}^+}^{\varsigma}\omega(\mu)\right]\chi\left(\sqrt{\eta\mu}\right)\right|\le\frac{(M-m)\left(\ln\frac{\mu}{\eta}\right)^{\varsigma+1}}{2^{\varsigma+1}\Gamma(\varsigma+2)}\|\omega\|_{[\eta,\mu],\infty}. \qquad (38)
$$

Corollary 3.3. *Letting* $\omega(u) = \frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ *in Theorem 3.1, we get the following:*

$$\left| J^{\varsigma}_{x^-}\chi(\eta) + J^{\varsigma}_{x^+}\chi(\mu) - \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma} + \left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma}}{\Gamma(\varsigma+1)}\chi(x) - \frac{(M+m)\left[\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1} - \left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}\right]}{2\Gamma(\varsigma+2)} \right| \leq \frac{(M-m)\left[\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1} + \left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}\right]}{2\Gamma(\varsigma+2)}. \quad (39)$$

Moreover, if we take $x = \sqrt{\eta\mu}$, *we obtain*

$$\left| \frac{2^{\varsigma-1}\Gamma(\varsigma+1)}{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma}} \left(J^{\varsigma}_{\sqrt{\eta\mu}^-}\chi(\eta) + J^{\varsigma}_{\sqrt{\eta\mu}^+}\chi(\mu) \right) - \chi\left(\sqrt{\eta\mu}\right) \right| \leq \frac{(M-m)\left(\ln\frac{\mu}{\eta}\right)}{4(\varsigma+1)}. \quad (40)$$

Corollary 3.4. *Suppose that* $\varsigma = 1$ *in Theorem 3.1, we get the following inequality:*

$$\left| \int_{\eta}^{\mu} \omega(u)\chi(u)du - \left(\int_{\eta}^{\mu} \omega(u)du\right)\chi(x) - \frac{(M+m)\left[\left(\ln\left(\frac{\mu}{x}\right)\right)^{2} - \left(\ln\left(\frac{x}{\eta}\right)\right)^{2}\right]}{2} \left(\int_{\varrho}^{1} (\omega(\eta^{\rho}x^{1-\rho}) + \omega(\mu^{\rho}x^{1-\rho}))d\rho \right) d\varrho \right| \leq \frac{(M-m)}{4} \left(\left(\ln\left(\frac{\mu}{x}\right)\right)^{2} \|\omega\|_{[x,\mu],\infty} + \left(\ln\left(\frac{x}{\eta}\right)\right)^{2} \|\omega\|_{[\eta,x],\infty} \right). \quad (41)$$

Moreover, for $x = \sqrt{\eta\mu}$, *we get*

$$\left| \int_{\eta}^{\mu} \omega(u)\chi(u)du - \left(\int_{\eta}^{\mu} \omega(u)du\right)\chi\left(\sqrt{\eta\mu}\right) \right| \leq \frac{(M-m)\left(\ln\frac{\mu}{\eta}\right)^{2}}{8} \|\omega\|_{[\eta,\mu],\infty}. \quad (42)$$

Corollary 3.5. *Taking* $\omega(u) = \frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ *in Corollary 3.4, we get the following inequality:*

$$\left| \frac{1}{\ln\frac{\mu}{\eta}} \int_{\eta}^{\mu} \chi(u)du - \chi(x) - \frac{(M+m)\left[\left(\ln\left(\frac{\mu}{x}\right)\right)^{2} - \left(\ln\left(\frac{x}{\eta}\right)\right)^{2}\right]}{4\left(\ln\frac{\mu}{\eta}\right)} \right| \leq \frac{\left[\left(\ln\left(\frac{\mu}{x}\right)\right)^{2} + \left(\ln\left(\frac{x}{\eta}\right)\right)^{2}\right](M-m)}{4\left(\ln\frac{\mu}{\eta}\right)}. \quad (43)$$

Taking $x = \sqrt{\eta\mu}$, *then*

$$\left| \frac{1}{\ln\frac{\mu}{\eta}} \int_{\eta}^{\mu} \chi(u)du - \chi\left(\sqrt{\eta\mu}\right) \right| \leq \frac{\left(\ln\frac{\mu}{\eta}\right)(M-m)}{8}. \quad (44)$$

Recalling that a function $\chi : J \to \mathbb{R}$ is σ-H-Hölder (Hölder condition, see [8]), we are now ready to present our next result.

$$|\chi(\eta) - \chi(\mu)| \leq H|\eta - \mu|^{\sigma}$$

holds for all $\eta, \mu \in J^{\circ}$, for some $H > 0$ and $\sigma \in (0, 1]$.

Theorem 3.6. *Let the conditions of Lemma 2.1 hold and suppose that χ' satisfies σ-H-Hölder condition for some $H > 0$ and $\sigma \in (0, 1]$, then*

$$\left|\Xi^{\varsigma}(\eta, x, \mu, \omega, \chi)\right| \leq H\left(\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+\sigma+1}}{(\varsigma+\sigma+1)\Gamma(\varsigma+1)}\|\omega\|_{[x,\mu],\infty} + \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+\sigma+1}}{(\varsigma+\sigma+1)\Gamma(\varsigma+1)}\|\omega\|_{[\eta,x],\infty}\right), \quad (45)$$

where

$$\Xi^{\varsigma}(\eta, x, \mu, \omega, \chi) := J^{\varsigma}_{x^-}\omega\chi(\eta) + J^{\varsigma}_{x^+}\omega\chi(\mu) - \left[J^{\varsigma}_{x^-}\omega(\eta) + J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x) - \frac{\chi'(\mu)\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1} - \chi'(\eta)\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma)}\int_0^1 (p_1(\varrho) + p_2(\varrho))\, d\varrho, \quad (46)$$

where p_1, p_2 are defined as in (3) *and* (4)*, respectively.*

Proof. Applying Lemma 2.1, we have

$$\begin{aligned}
&J^{\varsigma}_{x^-}\omega\chi(\eta) + J^{\varsigma}_{x^+}\omega\chi(\mu) - \left[J^{\varsigma}_{x^-}\omega(\eta) + J^{\varsigma}_{x^+}\omega(\mu)\right]\chi(x)\\
&= \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\chi'(\mu^{\varrho}x^{1-\varrho})d\varrho - \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\chi'(\eta^{\varrho}x^{1-\varrho})d\varrho\\
&= \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\left[\chi'(\mu^{\varrho}x^{1-\varrho}) - \chi'(\mu) + \chi'(\mu)\right]d\varrho\\
&- \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\left[\chi'(\eta^{\varrho}x^{1-\varrho}) - \chi'(\eta) + \chi'(\eta)\right]d\varrho\\
&= \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\left(\chi'(\mu^{\varrho}x^{1-\varrho}) - \chi'(\mu)\right)d\varrho\\
&- \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\left(\chi'(\eta^{\varrho}x^{1-\varrho}) - \chi'(\eta)\right)d\varrho\\
&\quad + \frac{\chi'(\mu)\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma)}\int_0^1 p_1(\varrho)d\varrho - \frac{\chi'(\eta)\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma)}\int_0^1 p_2(\varrho)d\varrho. \quad (47)
\end{aligned}$$

From (47), we get

$$\Xi^{\varsigma}(\eta, x, \mu, \omega, \chi) = \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_1(\varrho)\left(\chi'(\mu^{\varrho}x^{1-\varrho}) - \chi'(\mu)\right)d\varrho - \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 p_2(\varrho)\left(\chi'(\eta^{\varrho}x^{1-\varrho}) - \chi'(\eta)\right)d\varrho, \quad (48)$$

where $\Xi^{\varsigma}(\eta, x, \mu, \omega, \chi)$ is defined as in (46). Applying absolute value on both sides of (48) and σ-H-Hölder property of χ', we obtain

$$\left|\Xi^{\varsigma}(\eta, x, \mu, \omega, \chi)\right| \leq \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 \left|p_1(\varrho)\right|\left|\chi'(\mu^{\varrho}x^{1-\varrho}) - \chi'(\mu)\right|d\varrho + \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{\Gamma(\varsigma)}\int_0^1 \left|p_2(\varrho)\right|\left|\chi'(\eta^{\varrho}x^{1-\varrho}) - \chi'(\eta)\right|d\varrho$$

$$\leq H\left(\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+\sigma+1}}{\Gamma(\varsigma+1)}\|\omega\|_{[x,\mu],\infty}+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+\sigma+1}}{\Gamma(\varsigma+1)}\|\omega\|_{[\eta,x],\infty}\right)\int_0^1(1-\varrho)^{\varsigma+\sigma}d\varrho$$

$$=H\left(\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+\sigma+1}}{(\varsigma+\sigma+1)\Gamma(\varsigma+1)}\|\omega\|_{[x,\mu],\infty}+\frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+\sigma+1}}{(\varsigma+\sigma+1)\Gamma(\varsigma+1)}\|\omega\|_{[\eta,x],\infty}\right). \quad (49)$$

The proof is completed. □

Corollary 3.7. *If we take $x=\sqrt{\eta\mu}$ in Theorem 3.6, we get the following inequality:*

$$\left|J^{\varsigma}_{\sqrt{\eta\mu}^-}\omega\chi(\eta)+J^{\varsigma}_{\sqrt{\eta\mu}^+}\omega\chi(\mu)-\left[J^{\varsigma}_{\sqrt{\eta\mu}^-}\omega(\eta)+J^{\varsigma}_{\sqrt{\eta\mu}^+}\omega(\mu)\right]\chi\left(\sqrt{\eta\mu}\right)\right|\leq H\frac{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma+\sigma+1}\|\omega\|_{[\eta,\mu],\infty}}{2^{\varsigma+\sigma}(\varsigma+\sigma+1)\Gamma(\varsigma+1)}$$
$$+\frac{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma+1}(\chi'(\mu)-\chi'(\eta))}{2^{\varsigma+2}\Gamma(\varsigma)}\int_0^1(p_1(\varrho)+p_2(\varrho))\,d\varrho. \quad (50)$$

Corollary 3.8. *Let $\omega(u)=\frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ be in Theorem 3.6, we get the following result:*

$$\left|J^{\varsigma}_{x^-}\chi(\eta)+J^{\varsigma}_{x^+}\chi(\mu)-\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma}+\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma}}{\Gamma(\varsigma+1)}\chi(x)\right|\leq H\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+\sigma+1}+\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+\sigma+1}}{(\varsigma+\sigma+1)\Gamma(\varsigma+1)}$$
$$+\frac{\chi'(\mu)\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1}-\chi'(\eta)\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma+2)}. \quad (51)$$

Moreover, if we take $x=\sqrt{\eta\mu}$, we obtain

$$\left|\frac{2^{\varsigma-1}\Gamma(\varsigma+1)}{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma}}\left(J^{\varsigma}_{\sqrt{\eta\mu}^-}\chi(\eta)+J^{\varsigma}_{\sqrt{\eta\mu}^+}\chi(\mu)\right)-\chi\left(\sqrt{\eta\mu}\right)\right|\leq H\frac{\left(\ln\frac{\mu}{\eta}\right)^{\sigma+1}}{2^{1+\sigma}(\varsigma+\sigma+1)}+\frac{\left(\ln\frac{\mu}{\eta}\right)(\chi'(\mu)-\chi'(\eta))}{8(\varsigma+1)}. \quad (52)$$

Corollary 3.9. *If we take $\varsigma=1$ in Theorem 3.6, we get the following inequality:*

$$\left|\int_\eta^\mu\int_\eta(u)\chi(u)du-\left(\int_\eta^\mu\omega(u)du\right)\chi(x)\right|$$
$$\leq H\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\sigma+2}\|\omega\|_{[x,\mu],\infty}+\left(\ln\left(\frac{x}{\eta}\right)\right)^{\sigma+2}\|\omega\|_{[\eta,x],\infty}}{\sigma+2}$$
$$+\frac{\chi'(\mu)\left(\ln\left(\frac{\mu}{x}\right)\right)^{2}-\chi'(\eta)\left(\ln\left(\frac{x}{\eta}\right)\right)^{2}}{2}\int_0^1\left(\int_\varrho^1(\omega(\eta^\rho x^{1-\rho})+\omega(\mu^\rho x^{1-\rho}))d\rho\right)d\varrho. \quad (53)$$

Moreover, for $x=\sqrt{\eta\mu}$, we get

$$\left|\int_\eta^\mu\omega(u)\chi(u)du-\left(\int_\eta^\mu\omega(u)du\right)\chi\left(\sqrt{\eta\mu}\right)\right|\leq H\frac{\left(\ln\frac{\mu}{\eta}\right)^{\sigma+2}\|\omega\|_{[\eta,\mu],\infty}}{2^{\sigma+1}(\sigma+2)}+\frac{\chi'(\mu)\left(\ln\frac{\mu}{\eta}\right)^{2}-\chi'(\eta)\left(\ln\frac{\mu}{\eta}\right)^{2}}{8}$$
$$\times\int_0^1\left(\int_\varrho^1(\omega(\eta^\rho x^{1-\rho})+\omega(\mu^\rho x^{1-\rho}))d\rho\right)d\varrho. \quad (54)$$

Corollary 3.10. *Suppose that* $\omega(u) = \frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ *in Corollary 54, we get the following:*

$$\left|\frac{1}{\ln\frac{\mu}{\eta}}\int_\eta^\mu \chi(u)du - \chi(x)\right| \leq H\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\sigma+2} + \left(\ln\left(\frac{x}{\eta}\right)\right)^{\sigma+2}}{(\sigma+2)\left(\ln\frac{\mu}{\eta}\right)} + \frac{\chi'(\mu)\left(\ln\left(\frac{\mu}{x}\right)\right)^2 - \chi'(\eta)\left(\ln\left(\frac{x}{\eta}\right)\right)^2}{4\left(\ln\frac{\mu}{\eta}\right)}. \quad (55)$$

Moreover, if we take $x = \sqrt{\eta\mu}$*, we have*

$$\left|\frac{1}{\ln\frac{\mu}{\eta}}\int_\eta^\mu \chi(u)du - \chi\left(\sqrt{\eta\mu}\right)\right| \leq H\frac{\left(\ln\frac{\mu}{\eta}\right)^{\sigma+2}}{2^{\sigma+1}(\sigma+2)} + \frac{(\chi'(\mu)-\chi'(\eta))\left(\ln\frac{\mu}{\eta}\right)}{16}. \quad (56)$$

Corollary 3.11. *Suppose that the function* χ' *is a L-Lipschitzian function* ($H = L$ *and* $\sigma = 1$) *in Theorem 3.6, we get the following inequality:*

$$\left|\Xi^\varsigma(\eta, x, \mu, \omega, \chi)\right| \leq L\left(\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+2}}{(\varsigma+2)\Gamma(\varsigma+1)}\|\omega\|_{[x,\mu],\infty} + \frac{\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+2}}{(\varsigma+2)\Gamma(\varsigma+1)}\|\omega\|_{[\eta,x],\infty}\right), \quad (57)$$

where Ξ^ς *is defined by* (46).

Corollary 3.12. *Letting* $x = \sqrt{\eta\mu}$ *in Corollary 3.11 gives the following inequality:*

$$\begin{aligned}&\left|J^\varsigma_{\sqrt{\eta\mu}^-}\omega\chi(\eta) + J^\varsigma_{\sqrt{\eta\mu}^+}\omega\chi(\mu) - \left[J^\varsigma_{\sqrt{\eta\mu}^-}\omega(\eta) + J^\varsigma_{\sqrt{\eta\mu}^+}\omega(\mu)\right]\chi\left(\sqrt{\eta\mu}\right)\right| \\ &\leq L\frac{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma+2}}{2^{\varsigma+1}(\varsigma+2)\Gamma(\varsigma+1)}\|\omega\|_{[\eta,\mu],\infty} + \frac{\left(\ln\frac{\mu}{\eta}\right)^{\varsigma+1}(\chi'(\mu)-\chi'(\eta))}{2^{\varsigma+2}\Gamma(\varsigma)}\int_0^1 (p_1(\varrho)+p_2(\varrho))\,d\varrho. \quad (58)\end{aligned}$$

Corollary 3.13. *Taking* $\omega(u) = \frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ *in Corollary 3.11, we get the following inequality:*

$$\begin{aligned}&\left|J^\varsigma_{x^-}\chi(\eta) + J^\varsigma_{x^+}\chi(\mu) - \frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^\varsigma + \left(\ln\left(\frac{x}{\eta}\right)\right)^\varsigma}{\Gamma(\varsigma+1)}\chi(x)\right| \\ &\leq L\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+2} + \left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+2}}{(\varsigma+2)\Gamma(\varsigma+1)} + \frac{\chi'(\mu)\left(\ln\left(\frac{\mu}{x}\right)\right)^{\varsigma+1} - \chi'(\eta)\left(\ln\left(\frac{x}{\eta}\right)\right)^{\varsigma+1}}{2\Gamma(\varsigma+2)}. \quad (59)\end{aligned}$$

Moreover, if we take $x = \sqrt{\eta\mu}$*, we have*

$$\left|\frac{2^{\varsigma-1}\Gamma(\varsigma+1)}{\left(\ln\frac{\mu}{\eta}\right)^\varsigma}\left(J^\varsigma_{\sqrt{\eta\mu}^-}\chi(\eta) + J^\varsigma_{\sqrt{\eta\mu}^+}\chi(\mu)\right) - \chi\left(\sqrt{\eta\mu}\right)\right| \leq L\frac{\left(\ln\frac{\mu}{\eta}\right)^2}{4(\varsigma+2)} + \frac{\left(\ln\frac{\mu}{\eta}\right)(\chi'(\mu)-\chi'(\eta))}{8(\varsigma+1)}. \quad (60)$$

Corollary 3.14. *Letting* $\varsigma = 1$ *in Corollary 3.11, we get the following inequality:*

$$\begin{aligned}&\left|\int_\eta^\mu \omega(u)\chi(u)du - \left(\int_\eta^\mu \omega(u)du\right)\chi(x)\right| \leq L\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^3\|\omega\|_{[x,\mu],\infty} + \left(\ln\left(\frac{x}{\eta}\right)\right)^3\|\omega\|_{[\eta,x],\infty}}{3} \\ &+ \frac{\chi'(\mu)\left(\ln\left(\frac{\mu}{x}\right)\right)^2 - \chi'(\eta)\left(\ln\left(\frac{x}{\eta}\right)\right)^2}{2}\int_0^1\left(\int_\varrho^1 (\omega(\eta^\rho x^{1-\rho}) + \omega(\mu^\rho x^{1-\rho}))d\rho\right)d\varrho. \quad (61)\end{aligned}$$

Moreover, for $x = \sqrt{\eta\mu}$, we obtain

$$\left|\int_{\eta}^{\mu}\omega(u)\chi(u)du-\left(\int_{\eta}^{\mu}\omega(u)du\right)\chi\left(\sqrt{\eta\mu}\right)\right|\leq L\frac{\left(\ln\frac{\mu}{\eta}\right)^{3}}{24}\|\omega\|_{[\eta,\mu],\infty}$$
$$+\frac{\chi'(\mu)\left(\ln\frac{\mu}{\eta}\right)^{2}-\chi'(\eta)\left(\ln\frac{\mu}{\eta}\right)^{2}}{8}\int_{0}^{1}\left(\int_{\varrho}^{1}(\omega(\eta^{\rho}\left(\sqrt{\eta\mu}\right)^{1-\rho})+\omega(\mu^{\rho}\left(\sqrt{\eta\mu}\right)^{1-\rho}))d\rho\right)d\varrho. \quad (62)$$

Corollary 3.15. *If we choose $\omega(u)=\frac{1}{\ln\left(\frac{\mu}{\eta}\right)}$ in Corollary 3.14, we get the following inequality:*

$$\left|\frac{1}{\ln\frac{\mu}{\eta}}\int_{\eta}^{\mu}\chi(u)du-\chi(x)\right|\leq L\frac{\left(\ln\left(\frac{\mu}{x}\right)\right)^{3}+\left(\ln\left(\frac{x}{\eta}\right)\right)^{3}}{3\left(\ln\frac{\mu}{\eta}\right)}+\frac{\chi'(\mu)\left(\ln\left(\frac{\mu}{x}\right)\right)^{2}-\chi'(\eta)\left(\ln\left(\frac{x}{\eta}\right)\right)^{2}}{4\left(\ln\frac{\mu}{\eta}\right)}. \quad (63)$$

Moreover, for $x = \sqrt{\eta\mu}$, we get

$$\left|\frac{1}{\ln\frac{\mu}{\eta}}\int_{\eta}^{\mu}\chi(u)du-\chi\left(\sqrt{\eta\mu}\right)\right|\leq L\frac{\left(\ln\frac{\mu}{\eta}\right)^{3}}{24}+\frac{(\chi'(\mu)-\chi'(\eta))\left(\ln\frac{\mu}{\eta}\right)}{16}.$$

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