Filomat 37:18 (2023), 5969–5977
https://doi.org/10.2298/FIL2318969K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Gradient Ricci-harmonic solitons on doubly warped product manifolds

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Abstract. We give necessary and sufficient conditions for doubly warped product manifolds to be gradient Ricci-harmonic solitons. We also give a physical application for this kind of solitons.

1. Introduction

The notion of gradient Ricci-harmonic soliton is defined by Müller in [25]. Let (M, g) and $(N, \overline{g})$ be (pseudo)-Riemannian manifolds, $h : M \to \mathbb{R}$ a smooth function and $\phi : (M, g) \to (N, \overline{g})$ a smooth map. $\left((M, g), (N, \overline{g}), \phi, h, \lambda\right)$ is called a *gradient Ricci-harmonic soliton* (briefly GRHS), if it satisfies the coupled elliptic system

$$\begin{cases} Ric + Hessh - \alpha \nabla\phi \otimes \nabla\phi = \lambda g, \\ \tau_g \phi - g(\nabla\phi, \nabla h) = 0, \end{cases} \tag{1}$$

where α is a positive constant, $\lambda \in \mathbb{R}$, Ric is the Ricci curvature of (M, g) and $\tau_g \phi = tr_g \left(\nabla d\phi\right)$ is the tension field of ϕ. GRHS is called shrinking, steady or expanding depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If h occurs as a constant, GRHS is called trivial. The case α is a constant and ϕ is a smooth function was first studied by List in [22]. It is clear that the concept of gradient Ricci-harmonic soliton is a generalization of the concept of gradient Ricci soliton [23]. For some research about gradient Ricci solitons see ([10], [11], [12] and [31]). The map ϕ is called a harmonic map flow. A gradient Ricci soliton is a Ricci-harmonic soliton, if ϕ is a constant map [15]. If h is constant and ϕ is harmonic, then (1) defines a harmonic-Einstein metric, which is a natural generalization of Einstein metric [15].

In [25], Müller defined the notion of the Ricci-harmonic soliton. Complete non-compact gradient shrinking Ricci-harmonic solitons were considered by Yang and Shen in [35]. A classification of compact gradient Ricci harmonic solitons was given in [15] by Guo, Philipowski and Thalmaier. In [29], a lower diameter bound for compact domain manifolds of shrinking Ricci-harmonic solitons was studied by Tadano. In [30], Tadano studied gradient shrinking almost Ricci-harmonic solitons on a compact domain. In [36],

2020 *Mathematics Subject Classification*. Primary 53C21; Secondary 53C50, 53C25.
Keywords. Gradient Ricci-harmonic soliton, warped product, doubly warped product, space-time.
Received: 20 April 2022; Accepted: 07 February 2023
Communicated by Ljubica Velimirović
This work is supported by 1002 Scientific and Technological Research Projects Funding Program of The Scientific and Technological Research Council of Türkiye (TÜBİTAK) under the project number 121F196.
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Zhu studied on the relation between Ricci-harmonic solitons and Ricci solitons. In [34], Wu and Zhang studied the shrinking gradient Ricci-harmonic soliton and obtained a logarithmic Sobolev inequality. In [4], Abolarinwa obtained necessary and sufficient conditions for a gradient shrinking almost Ricci-harmonic soliton on a compact domain to be almost harmonic-Einstein. In [5], Abolarinwa, Oladejo and Salawu introduced some entropy formulas and gave a classification of gradient almost soliton for Ricci-harmonic flow. In [6], Batista, Adriano and Tokura considered gradient Ricci-harmonic soliton with the structure of warped product manifolds. In [3], the notion of rigidity for harmonic-Ricci solitons was given by Anselli and some characterizations of rigidity were obtained. In [19], the first author considered gradient Ricci-harmonic solitons with the structure of multiply warped product manifolds and gave some physical applications. For more information on solitons with the structure of warped product manifolds and their generalizations, see also ([8], [14], [18], [20], [21] and [26]). By a motivation from the above studies, in the present paper, with the structure of doubly warped product manifolds, we consider gradient Ricci-harmonic solitons. We obtain some characterizations for this kind of solitons. We also give a physical application.

2. Gradient Ricci-harmonic solitons

In this section, we consider gradient Ricci-harmonic solitons with the structure of doubly warped product (briefly DWP).

The notion of warped product was defined by Bishop and O'Neill in [9]. Warped products are important in a variety of physical applications, including general relativity, string theory and supergravity theories. As a generalization of the notion of warped product, the notion of doubly warped product was defined by Ünal in [33].

Let (M_1, g_1) and (M_2, g_2) be (pseudo)-Riemannian manifolds, respectively and $f_1 : M_1 \to (0, \infty)$ and $f_2 : M_2 \to (0, \infty)$ be positive smooth functions. The doubly warped product manifold (briefly DWPM) (M, g) is the product manifold $M = _{f_2} M_1 \times_{f_1} M_2$ endowed with the metric tensor

$$g = f_2^2 g_1 \oplus f_1^2 g_2. \tag{2}$$

The functions f_1 and f_2 are called warping functions of the doubly warped product manifold [33].

Throughout this work, we use the following notations:

Notation 2.1. *Let $M = _{f_2} M_1 \times_{f_1} M_2$ be a DWPM endowed with the metric tensor $g = f_2^2 g_1 \oplus f_1^2 g_2$.*

1) *The manifolds M_1 and M_2 have dimensions m_1 and m_2, where $m = m_1 + m_2$.*
2) *We denote the Ricci curvatures of M, M_1 and M_2 by Ric, Ric_{M_1} and Ric_{M_2}, respectively.*
3) *We denote the Hessian operators on M, M_1 and M_2 by $Hess$, $Hess_{M_1}$ and $Hess_{M_2}$, respectively.*
4) *We denote the Laplacians on M, M_1 and M_2 by Δ, Δ_{M_1} and Δ_{M_2}, respectively.*
5) *We denote the gradients on M, M_1 and M_2 by ∇, ∇_{M_1} and ∇_{M_2}, respectively.*

Let $M = _{f_2} M_1 \times_{f_1} M_2$ be a DWPM endowed with the metric tensor $g = f_2^2 g_1 \oplus f_1^2 g_2$, where $f_1 : M_1 \to (0, \infty)$ and $f_2 : M_2 \to (0, \infty)$ are positive smooth functions. Let $\phi : M \to \mathbb{R}$ be a harmonic map flow and denote the components of the ϕ and h on M_i (for $i = 1, 2$) by ϕ_{M_i} and h_{M_i}, respectively. That is, $\phi_{M_i} : M_i \to \mathbb{R}$ and $h_{M_i} : M_i \to \mathbb{R}$. So firstly, we have:

Theorem 2.2. $(M = _{f_2} M_1 \times_{f_1} M_2, g, h, \phi, \lambda)$ *is a GRHS if and only if the functions $f_1, f_2, h, \phi, \lambda$ satisfy:*

(1) *If $\phi = \phi_{M_1}$ and $h = h_{M_1}$, then*

$$\begin{cases} Ric_{M_1} - \frac{m_2}{f_1} Hess_{M_1}(f_1) + Hess_{M_1} h_{M_1} - \frac{\alpha}{f_2^4} \nabla_{M_1}\phi_{M_1} \otimes \nabla_{M_1}\phi_{M_1} \\ \qquad = \left[\lambda f_2^2 + (m_1 - 1) \frac{\|\nabla_{M_2} f_2\|^2_{M_2}}{f_1^2} + \frac{f_2}{f_1^2} \Delta_{M_2} f_2 \right] g_1, \\ \qquad\qquad \Delta_{w_1}\phi_{M_1} = 0 \text{ in } M_1, \end{cases} \tag{3}$$

where $\Delta_{w_1} = \Delta_{M_1} - g_1(\nabla_{M_1}, \nabla_{M_1} w_1)$, $w_1 = h_{M_1} - m_2 log(f_1)$ and

M_2 is Einstein with $Ric_{M_2} = \mu_1 g_2$, where

$$\mu_1 = \lambda f_1^2 + \frac{f_1}{f_2^2}\Delta_{M_1} f_1 + (m_2 - 1)\frac{\left\|\nabla_{M_1} f_1\right\|_{M_1}^2}{f_2^2} - \frac{f_1}{f_2^2}\nabla_{M_1} h_{M_1}(f_1) + \frac{m_2}{f_2}\beta_1, \tag{4}$$

when $Hess_{M_2}(f_2) = \beta_1 g_2$.

(2) If $\phi = \phi_{M_1}$ and $h = h_{M_2}$, then

$$\begin{cases} Ric_{M_1} - \frac{m_2}{f_1} Hess_{M_1}(f_1) - \frac{\alpha}{f_2^4}\nabla_{M_1}\phi_{M_1} \otimes \nabla_{M_1}\phi_{M_1} \\ = \left[\lambda f_2^2 + (m_1 - 1)\frac{\left\|\nabla_{M_2} f_2\right\|_{M_2}^2}{f_1^2} - \frac{f_2}{f_1^2}\nabla_{M_2} h_{M_2}(f_2) + \frac{f_2}{f_1^2}\Delta_{M_2} f_2\right] g_1, \\ \Delta_{w_2}\phi_{M_1} = 0 \text{ in } M_1, \end{cases} \tag{5}$$

where $\Delta_{w_2} = \Delta_{M_1} + g_1(\nabla_{M_1}, \nabla_{M_1} w_2)$, $w_2 = m_2 log(f_1)$ and

M_2 is Einstein with $Ric_{M_2} = \mu_2 g_2$, where

$$\mu_2 = \lambda f_1^2 + \frac{f_1}{f_2^2}\Delta_{M_1} f_1 + (m_2 - 1)\frac{\left\|\nabla_{M_1} f_1\right\|_{M_1}^2}{f_2^2} + \left(\frac{m_2}{f_2} - f_1^2\right)\beta_1, \tag{6}$$

when $h = h_{M_2} = f_2$ and $Hess_{M_2}(f_2) = \beta_1 g_2$.

(3) If $\phi = \phi_{M_2}$ and $h = h_{M_1}$, then M_1 is Einstein with $Ric_{M_1} = \mu_3 g_1$, where

$$\mu_3 = \lambda f_2^2 + \frac{f_2}{f_1^2}\Delta_{M_2} f_2 + (m_1 - 1)\frac{\left\|\nabla_{M_2} f_2\right\|_{M_2}^2}{f_1^2} + \left(\frac{m_2}{f_1} - 1\right)\beta_2, \tag{7}$$

when $h = h_{M_1} = f_1$ and $Hess_{M_1}(f_1) = \beta_2 g_1$ and

$$\begin{cases} Ric_{M_2} - \frac{m_1}{f_2} Hess_{M_2}(f_2) - \frac{\alpha}{f_1^4}\nabla_{M_2}\phi_{M_2} \otimes \nabla_{M_2}\phi_{M_2} \\ = \left[\lambda f_1^2 + (m_2 - 1)\frac{\left\|\nabla_{M_1} f_1\right\|_{M_1}^2}{f_2^2} + \frac{f_1}{f_2^2}\Delta_{M_1} f_1 - \frac{f_1}{f_2^2}\nabla_{M_1} h_{M_1}(f_1)\right] g_2, \\ \Delta_{w_3}\phi_{M_2} = 0 \text{ in } M_2, \end{cases} \tag{8}$$

where $\Delta_{w_3} = \Delta_{M_2} + g_2(\nabla_{M_2}, \nabla_{M_2} w_3)$, $w_3 = m_1 log(f_2)$.

(4) If $\phi = \phi_{M_2}$ and $h = h_{M_2}$, then M_1 is Einstein with $Ric_{M_1} = \mu_4 g_1$, where

$$\mu_4 = \lambda f_2^2 + \frac{f_2}{f_1^2}\Delta_{M_2} f_2 + (m_1 - 1)\frac{\left\|\nabla_{M_2} f_2\right\|_{M_2}^2}{f_1^2} - \frac{f_2}{f_1^2}\nabla_{M_2} h_{M_2}(f_2) + \frac{m_2}{f_1}\beta_2, \tag{9}$$

when $Hess_{M_1}(f_1) = \beta_2 g_1$ and

$$\begin{cases} Ric_{M_2} - \frac{m_1}{f_2} Hess_{M_2}(f_2) + f_1^2 Hess_{M_2} h_{M_2} - \frac{\alpha}{f_1^4}\nabla_{M_2}\phi_{M_2} \otimes \nabla_{M_2}\phi_{M_2} \\ = \left[\lambda f_1^2 + (m_2 - 1)\frac{\left\|\nabla_{M_1} f_1\right\|_{M_1}^2}{f_2^2} + \frac{f_1}{f_2^2}\Delta_{M_1} f_1\right] g_2, \\ \Delta_{w_4}\phi_{M_2} = 0 \text{ in } M_2, \end{cases} \tag{10}$$

where $\Delta_{w_4} = \Delta_{M_2} - g_2(\nabla_{M_2}, \nabla_{M_2} w_4)$, $w_4 = h_{M_2} - m_1 log(f_2)$.

Proof. (1) Let $({}_{f_2}M_1 \times_{f_1} M_2, g, h, \phi, \lambda)$ be a GRHS with $\phi = \phi_{M_1}$ and $h = h_{M_1}$. From Proposition 2.3.1, Theorem 2.5.2, Proposition 2.6.2 in [32] and the first equation of (1) with $\phi = \phi_{M_1}$ and $h = h_{M_1}$, we have the first equation of (3) for $X, Y \in \chi(M_1)$. Using Proposition 2.5.3 in [32] and the second equation of (1) with $\phi = \phi_{M_1}$ and $h = h_{M_1}$, we obtain

$$\Delta_{w_1}\phi_{M_1} = 0,$$

where $\Delta_{w_1} = \Delta_{M_1} - g_1(\nabla_{M_1}, \nabla_{M_1} w_1)$, $w_1 = h_{M_1} - m_2 log(f_1)$. Similarly, from Theorem 2.5.2 in [32] and the first equation of (1) with $\phi = \phi_{M_1}$ and $h = h_{M_1}$, we find

$$Ric_{M_2}(U,V) - \left[\frac{f_1}{f_2^2}\Delta_{M_1} f_1 + (m_2-1)\frac{\left\|\nabla_{M_1} f_1\right\|^2_{M_1}}{f_2^2}\right] g_2(U,V) - \frac{m_1}{f_2} Hess_{M_2} f_2(U,V)$$

$$= \lambda f_1^2 g_2(U,V) - Hessh(U,V) \tag{11}$$

for $U, V \in \chi(M_2)$. From the definition of the Hessian of a function, we write

$$Hessh(U,V) = \frac{f_1}{f_2^2}\nabla_{M_1} h_{M_1}(f_1)\, g_2(U,V). \tag{12}$$

When $Hess_{M_2}(f_2) = \beta_1 g_2$, substituting the equation (12) in (11), we obtain M_2 is an Einstein manifold with

$$Ric_{M_2} = \left[\lambda f_1^2 + \frac{f_1}{f_2^2}\Delta_{M_1} f_1 + (m_2-1)\frac{\left\|\nabla_{M_1} f_1\right\|^2_{M_1}}{f_2^2} - \frac{f_1}{f_2^2}\nabla_{M_1} h_{M_1}(f_1) + \frac{m_2}{f_2}\beta_1\right] g_2.$$

For the cases $\phi = \phi_{M_1}$ and $h = h_{M_2}$, $\phi = \phi_{M_2}$ and $h = h_{M_1}$, $\phi = \phi_{M_2}$ and $h = h_{M_2}$, by using the same method, (2), (3) and (4) are proven. This completes the proof of the theorem. □

Let $\left(M = {}_{f_2}\left(\mathbb{R}^{m_1}, \upsilon^{-2} g_{\mathbb{R}}\right) \times_{f_1} \left(\mathbb{R}^{m_2}, \tau^{-2} g_{\mathbb{R}}\right), g = \frac{f_2^2}{\upsilon^2} g_{\mathbb{R}} + \frac{f_1^2}{\tau} g_{\mathbb{R}}\right)$ be a DWPM, where $\left(\mathbb{R}^{m_1}, \upsilon^{-2} g_{\mathbb{R}}\right)$ and $\left(\mathbb{R}^{m_2}, \tau^{-2} g_{\mathbb{R}}\right)$ are conformal to m_1-dimensional and m_2-dimensional pseudo-Euclidean spaces, respectively and $(g_{\mathbb{R}})_{i,j} = \epsilon_i \delta_{i,j}$, $\epsilon_i = \pm 1$ is the canonical pseudo-Riemannian metric. For an arbitrary choice of non-zero vectors $a = (a_1, a_2, ..., a_{m_1})$ and $b = (b_{m_1+1}, b_{m_1+2}, ..., b_{m_1+m_2})$, we define the functions

$$\xi(x_1, x_2, ..., x_{m_1}) = \sum_{i=1}^{m_1} a_i x_i, a_i \in \mathbb{R}$$

and

$$\zeta(x_{m_1+1}, x_{m_1+2}, ..., x_{m_1+m_2}) = \sum_{j=m_1+1}^{m_2} b_j x_j, b_j \in \mathbb{R},$$

where $x = (x_1, ..., x_{m_1}) \in \mathbb{R}^{m_1}$ and $y = (x_{m_1+1}, ..., x_{m_1+m_2}) \in \mathbb{R}^{m_2}$.

Now we consider a GRHS on $M = {}_{f_2}\left(\mathbb{R}^{m_1}, \upsilon^{-2} g_{\mathbb{R}}\right) \times_{f_1} \left(\mathbb{R}^{m_2}, \tau^{-2} g_{\mathbb{R}}\right)$ with $\phi = \phi_{M_2}$ and $h = h_{M_1}$. So we can state the following theorem:

Theorem 2.3. *$({}_{f_2}\mathbb{R}^{m_1} \times_{f_1} \mathbb{R}^{m_2}, g = f_2^2 \upsilon^{-2} g_{\mathbb{R}} + f_1^2 \tau^{-2} g_{\mathbb{R}}, h = h_{M_1}, \phi = \phi_{M_2}, \lambda)$ is a GRHS with non-constant harmonic map flow ϕ such that $f_1 = f_1 \circ \xi$, $h = h \circ \xi$, $\upsilon = \upsilon \circ \xi$ are defined in $\left(\mathbb{R}^{m_1}, \upsilon^{-2} g_{\mathbb{R}}\right)$ and $\phi = \phi \circ \zeta$, $f_2 = f_2 \circ \zeta$, $\tau = \tau \circ \zeta$ are defined in $\left(\mathbb{R}^{m_2}, \tau^{-2} g_{\mathbb{R}}\right)$ if and only if the functions f_1, f_2, h, υ, ϕ, τ satisfy the following equalities:*

$$(m_1 - 2)\frac{\upsilon''}{\upsilon} = 0, \tag{13}$$

$$\left[\upsilon\upsilon'' - (m_1 - 1)(\upsilon')^2\right] \|a\|^2 + \left[(m_2 - 1)(f_2')^2 + f_2 f_2'' - (m_2 - 2)\frac{\tau'}{\tau} f_2 f_2'\right]\left(\frac{\tau}{f_1}\right)^2 \|b\|^2$$
$$= \lambda f_2^2 + (\frac{m_2}{f_1} - 1)\beta_2, \tag{14}$$

$$(m_2 - 2)\frac{\tau''}{\tau} - m_1 \frac{f_2''}{f_2} - 2m_1 \frac{f_2'}{f_2}\frac{\tau'}{\tau} - \frac{\alpha}{f_1^4}\left(\phi'\right)^2 = 0, \tag{15}$$

$$\left[\tau\tau'' - (m_2 - 1)(\tau')^2 + m_1 \tau\tau' \frac{f_2'}{f_2}\right] \|b\|^2$$
$$-\left[\left\{(m_2 - 1)\left(\frac{f_1'}{f_1}\right)^2 - \frac{f_1 f_1'}{f_2^2} h' + \frac{f_1 f_1''}{f_2^2}\right\} \upsilon^2 - (m_1 - 2)\frac{f_1 f_1'}{f_2^2}\upsilon\upsilon'\right] \|a\|^2 = \lambda f_1^2 \tag{16}$$

and

$$\left[\phi''\tau^2 + \left(m_1 \frac{f_2'}{f_2}\tau^2 - (m_2 - 2)\tau\tau'\right)\phi'\right] \|b\|^2 = 0. \tag{17}$$

Proof. For an arbitrary choice of non-zero vectors $a = (a_1, a_2, ..., a_{m_1})$ and $b = (b_{m_1+1}, b_{m_1+2}, ..., b_{m_1+m_2})$, we consider the functions

$$\xi(x_1, x_2, ..., x_{m_1}) = a_1 x_1 + a_2 x_2 + ... + a_{m_1} x_{m_1}$$

and

$$\zeta(x_{m_1+1}, x_{m_1+2}, ..., x_{m_1+m_2}) = b_{m_1+1} x_{m_1+1} + b_{m_1+2} x_{m_1+2} + ... + b_{m_1+m_2} x_{m_1+m_2},$$

where $x = (x_1, ..., x_{m_1}) \in \mathbb{R}^{m_1}$ and $y = (x_{m_1+1}, ..., x_{m_1+m_2}) \in \mathbb{R}^{m_2}$. Assume that $h(\xi)$, $f_1(\xi)$ and $\upsilon(\xi)$ (respectively $f_2(\zeta)$, $\tau(\zeta)$ and $\phi(\zeta)$) are some functions of ξ (respectively of ζ), where $\xi : \mathbb{R}^{m_1} \to \mathbb{R}$ and $\zeta : \mathbb{R}^{m_2} \to \mathbb{R}$. Then, we have

$$\begin{array}{lll} h_{x_i} = h' a_i, & \upsilon_{x_i} = \upsilon' a_i, & \phi_{x_i} = \phi' b_i, \\ h_{x_i x_j} = h'' a_i a_j, & \upsilon_{x_i x_j} = \upsilon'' a_i a_j, & \phi_{x_i x_j} = \phi'' b_i b_j, \\ (f_1)_{x_i} = (f_1)' a_i, & (f_2)_{x_i} = (f_2)' b_i, & \tau_{x_i} = \tau' b_i, \\ (f_1)_{x_i x_j} = (f_1)'' a_i a_j, & (f_2)_{x_i x_j} = (f_2)'' b_i b_j, & \tau_{x_i x_j} = \tau'' b_i b_j. \end{array} \tag{18}$$

The Ricci curvatures with conformal metrics $g_{M_1} = \upsilon^{-2} g_{\mathbb{R}}$ and $g_{M_2} = \tau^{-2} g_{\mathbb{R}}$ are given by

$$Ric_{M_1} = \frac{1}{\upsilon^2}\left\{(m_1 - 2)\upsilon Hess_{g_{\mathbb{R}}}(\upsilon) + \left[\upsilon \Delta_{g_{\mathbb{R}}} \upsilon - (m_1 - 1)\left\|\nabla_{g_{\mathbb{R}}} \upsilon\right\|^2\right] g_{\mathbb{R}}\right\} \tag{19}$$

and

$$Ric_{M_2} = \frac{1}{\tau^2}\left\{(m_2 - 2)\tau Hess_{g_{\mathbb{R}}}(\tau) + \left[\tau \Delta_{g_{\mathbb{R}}} \tau - (m_2 - 1)\left\|\nabla_{g_{\mathbb{R}}} \tau\right\|^2\right] g_{\mathbb{R}}\right\}, \tag{20}$$

respectively [7]. From

$$\begin{array}{lll} \left(Hess_{g_{\mathbb{R}}}(\upsilon)\right)_{i,j} = \upsilon'' a_i a_j, & \Delta_{g_{\mathbb{R}}} \upsilon = \upsilon'' \|a\|^2, & \left\|\nabla_{g_{\mathbb{R}}} \upsilon\right\|^2 = (\upsilon')^2 \|a\|^2, \\ \left(Hess_{g_{\mathbb{R}}}(\tau)\right)_{i,j} = \tau'' b_i b_j, & \Delta_{g_{\mathbb{R}}} \tau = \tau'' \|b\|^2, & \left\|\nabla_{g_{\mathbb{R}}} \tau\right\|^2 = (\tau')^2 \|b\|^2, \end{array} \tag{21}$$

we find

$$(Ric_{M_1})_{i,j} = \frac{1}{\upsilon}(m_1 - 2)\upsilon''\left(a_i a_j\right) \tag{22}$$

for $\forall i \neq j = 1, 2, ..., m_1$ and

$$(Ric_{M_2})_{i,j} = \frac{1}{\tau}(m_2 - 2)\tau'' \left(b_i b_j\right) \tag{23}$$

for $\forall i \neq j = m_1 + 1, m_1 + 2, ..., m_2$. Similarly, from the equation (21), we obtain

$$(Ric_{M_1})_{i,i} = \frac{1}{\upsilon^2} \left\{(m_1 - 2)\upsilon\upsilon'' (a_i)^2 + \left[\upsilon\upsilon'' \|a\|^2 - (m_1 - 1)(\upsilon')^2 \|a\|^2\right] \epsilon_i\right\} \tag{24}$$

for $\forall i = 1, 2, ..., m_1$ and

$$(Ric_{M_2})_{i,i} = \frac{1}{\tau^2} \left\{(m_2 - 2)\tau\tau'' (b_i)^2 + \left[\tau\tau'' \|b\|^2 - (m_2 - 1)(\tau')^2 \|b\|^2\right] \epsilon_i\right\} \tag{25}$$

for $\forall i = m_1 + 1, m_1 + 2, ..., m_2$. From the definition of Hessian function, we have

$$(Hess_{M_1}(h))_{ij} = h'' a_i a_j + \left(2a_i a_j - \delta_{ij}\epsilon_i \|a\|^2\right) \frac{\upsilon'}{\upsilon^{-1}} h' \tag{26}$$

and

$$(Hess_{M_2}(f_2))_{ij} = f_2'' b_i b_j + \left(2b_i b_j - \delta_{ij}\epsilon_i \|b\|^2\right) \frac{\tau'}{\tau^{-1}} f_2'. \tag{27}$$

Then, the Laplacian of f_1 with conformal metric g_1 is

$$\Delta_{M_1} f_1 = \upsilon^2 \|a\|^2 \left[f_1'' - (m_1 - 2)\frac{\upsilon'}{\upsilon^{-1}} f_1'\right] \tag{28}$$

and the Laplacian of f_2 with conformal metric g_2 is

$$\Delta_{M_2} f_2 = \tau^2 \|b\|^2 \left[f_2'' - (m_2 - 2)\frac{\tau'}{\tau^{-1}} f_2'\right]. \tag{29}$$

Furthermore, we find

$$\begin{cases} \nabla_{M_1} f_1 (h) = \upsilon^2 \|a\|^2 f_1' h', \\ \left\|\nabla_{M_1} f_1\right\|^2 = \upsilon^2 \|a\|^2 \left(f_1'\right)^2, \\ \left\|\nabla_{M_2} f_2\right\|^2 = \tau^2 \|b\|^2 \left(f_2'\right)^2, \\ \left(\nabla_{M_2}\phi \otimes \nabla_{M_2}\phi\right)_{ij} = \left(\phi'\right)^2 b_i b_j. \end{cases} \tag{30}$$

Substituting the equations (22), (29) and (30) for $i \neq j$ in (7) and $Ric_{M_1} = \mu_3 g_1$, we obtain

$$\left[(m_1 - 2)\frac{\upsilon''}{\upsilon}\right] a_i a_j = 0. \tag{31}$$

From the equation (31), if there exist i, j for $i \neq j$ such that $a_i a_j \neq 0$, then we obtain (13). Similarly, using the equations (24), (29) and (30) for $i = j$ in (7) and $Ric_{M_1} = \mu_3 g_1$, we find (14).

Furthermore, using the equations (23), (27) and (30) for $i \neq j$ in (8), we have

$$\left[(m_2 - 2)\frac{\tau''}{\tau} - m_1 \frac{f_2''}{f_2} - 2m_1 \frac{f_2'}{f_2}\frac{\tau'}{\tau} - \frac{\alpha}{f_1^4}\left(\phi'\right)^2\right] b_i b_j = 0. \tag{32}$$

From the equation (32), if there exist i, j for $i \neq j$ such that $b_i b_j \neq 0$, then we obtain (15). Similarly, replacing the equations (25), (27), (28) and (30) for $i = j$ in (8), we find (16). Finally, using (29) and (30) in the second equation of (8), we obtain (17). This completes the proof. □

So we can state the following theorem for $\phi = \phi_{M_2}$ and $h = h_{M_2}$:

Theorem 2.4. $({}_{f_2}\mathbb{R}^{m_1} \times_{f_1} \mathbb{R}^{m_2}, g = f_2^2 \upsilon^{-2} g_{\mathbb{R}} + f_1^2 \tau^{-2} g_{\mathbb{R}}, h = h_{M_2}, \phi = \phi_{M_2}, \lambda)$ *is a GRHS with non-constant harmonic map flow* ϕ *such that* $f_1 = f_1 \circ \xi$, $\upsilon = \upsilon \circ \xi$, $h = h \circ \zeta$ *are defined in* $\left(\mathbb{R}^{m_1}, \upsilon^{-2} g_{\mathbb{R}}\right)$ *and* $\phi = \phi \circ \zeta$, $f_2 = f_2 \circ \zeta$, $\tau = \tau \circ \zeta$ *are defined in* $\left(\mathbb{R}^{m_2}, \tau^{-2} g_{\mathbb{R}}\right)$ *if and only if the functions* $f_1, f_2, h, \upsilon, \phi, \tau$ *satisfy the following equalities:*

$$(m_1 - 2)\frac{\upsilon''}{\upsilon} = 0, \tag{33}$$

$$\left[\upsilon\upsilon'' - (m_1 - 1)(\upsilon')^2\right] \|a\|^2 + \left[(m_2 - 1)\left(\frac{f_2'}{f_1}\right)^2 \tau^2 - \frac{f_2}{f_1^2} h' f_2' \tau^2 \right.$$
$$\left. + \frac{f_2 f_2''}{f_1^2}\tau^2 - (m_2 - 2)\tau\tau' \frac{f_2 f_2'}{f_1^2}\right] \|b\|^2 = \lambda f_2^2 + \frac{m_2}{f_1}\beta_2, \tag{34}$$

$$(m_2 - 2)\frac{\tau''}{\tau} - m_1 \frac{f_2''}{f_2} - 2m_1 \frac{f_2'}{f_2}\frac{\tau'}{\tau} + f_1^2 h' + 2 f_1^2 h' \frac{\tau'}{\tau} - \frac{\alpha}{f_1^4}\left(\phi'\right)^2 = 0, \tag{35}$$

$$\left[\tau\tau'' - (m_2 - 1)(\tau')^2 + m_1 \tau\tau' \frac{f_2'}{f_2} - f_1^2 h' \tau\tau'\right] \|b\|^2$$
$$- \left[\left\{(m_2 - 1)\left(\frac{f_1'}{f_1}\right)^2 + \frac{f_1 f_1''}{f_2^2}\right\}\upsilon^2 - (m_1 - 2)\frac{f_1 f_1'}{f_2^2}\upsilon\upsilon'\right] \|a\|^2 = \lambda f_1^2 \tag{36}$$

and

$$\left[\phi'' + m_1 \frac{f_2'}{f_2}\phi' - (m_2 - 2)\frac{\tau'}{\tau}\phi' - h'\phi'\right] \|b\|^2 = 0. \tag{37}$$

Proof. Similar to the proof of Theorem 2.3, the proof is obtained. □

3. A Physical Application

Doubly warped space-time is a very nice example of Lorentzian DWPM. This space-time is often utilized to solve Einstein's field equations precisely. The geometric properties of doubly warped space-times have been studied by many authors. For example, see ([2], [24], [27], [28]). In this section, we consider gradient Ricci-harmonic soliton with the structure of doubly warped space-times.

Let (M_2, g_2) be a m_2-dimensional Riemannian manifold and I an open, connected interval furnished with the negative definite metric $\left(-dt^2\right)$. Let $f_1 : I \to (0, \infty)$ and $f_2 : M_2 \to (0, \infty)$ be positive smooth functions. Doubly warped space-time $M = {}_{f_2} I \times_{f_1} M_2$ is the product manifold $I \times M_2$ furnished with the metric tensor

$$g = f_2^2(-dt^2) \oplus f_1^2 g_2, \tag{38}$$

(see [33]). If f_2 is a constant, then doubly warped space-time (M, g) turns into a generalized Robertson–Walker space-time. Furthermore, if f_1 is a constant, then doubly warped space-time (M, g) becomes a standard static space-time [1].

Let $\phi : M \to \mathbb{R}$ be a harmonic map flow. Then we can give the following corollary:

Corollary 3.1. $(M = {}_{f_2} I \times_{f_1} M_2, g = f_2^2(-dt^2) \oplus f_1^2 g_2, h, \phi, \lambda)$ *is a GRHS if and only if the functions* $f_1, f_2, h, \phi, \lambda$ *satisfy:*

(1) *If* $\phi = \phi_I$ *and* $h = h_I$, *then*

$$\begin{cases} m_2 \frac{f_1''}{f_1} - h_I'' + \frac{\alpha}{f_2^4}\left(\phi_I'\right)^2 = \left[\lambda f_2^2 + (m_1 - 1)\frac{\left\|\nabla_{M_2} f_2\right\|_{M_2}^2}{f_1^2} + \frac{f_2}{f_1^2}\Delta_{M_2} f_2\right] \\ \phi_I'' - \phi_I' h_I' + m_2 \frac{f_1'}{f_1}\phi_I' = 0 \end{cases} \tag{39}$$

and M_2 *is Einstein with* $Ric_{M_2} = \mu_1 g_2$, *where*

$$\mu_1 = \lambda f_1^2 - \frac{f_1}{f_2^2} f_1'' - (m_2 - 1)\left(\frac{f_1'}{f_2}\right)^2 - \frac{f_1}{f_2^2} f_1' h_I' + \frac{m_2}{f_2}\beta_1, \tag{40}$$

when $Hess_{M_2}(f_2) = \beta_1 g_2$.

(2) *If* $\phi = \phi_I$ *and* $h = h_{M_2}$, *then*

$$\begin{cases} m_2 \frac{f_1''}{f_1} + \frac{\alpha}{f_2^4}\left(\phi_I'\right)^2 = \left[\lambda f_2^2 + (m_1 - 1)\frac{\left\|\nabla_{M_2} f_2\right\|_{M_2}^2}{f_1^2} - \frac{f_2}{f_1^2}\nabla_{M_2} h_{M_2}(f_2) + \frac{f_2}{f_1^2}\Delta_{M_2} f_2\right] \\ \phi_I'' + m_2 \frac{f_1'}{f_1}\phi_I' = 0 \end{cases} \tag{41}$$

and M_2 *is Einstein with* $Ric_{M_2} = \mu_2 g_2$, *where*

$$\mu_2 = \lambda f_1^2 - \frac{f_1}{f_2^2} f_1'' - (m_2 - 1)\left(\frac{f_1'}{f_2}\right)^2 + \left(\frac{m_2}{f_2} - f_1^2\right)\beta_1, \tag{42}$$

when $h = h_{M_2} = f_2$ *and* $Hess_{M_2}(f_2) = \beta_1 g_2$.

(3) *If* $\phi = \phi_{M_2}$ *and* $h = h_I$, *then*

$$\lambda f_2^2 + \frac{f_2}{f_1^2}\Delta_{M_2} f_2 + (m_1 - 1)\frac{\left\|\nabla_{M_2} f_2\right\|_{M_2}^2}{f_1^2} + \frac{m_2}{f_1} f_1'' + h_I'' = 0 \tag{43}$$

and

$$\begin{cases} Ric_{M_2} - \frac{m_1}{f_2} Hess_{M_2}(f_2) - \frac{\alpha}{f_1^4}\nabla_{M_2}\phi_{M_2} \otimes \nabla_{M_2}\phi_{M_2} \\ \quad = \left[\lambda f_1^2 - (m_2 - 1)\left(\frac{f_1'}{f_2}\right)^2 - \frac{f_1}{f_2^2} f_1'' + \frac{f_1}{f_2^2} f_1' h_I'\right] g_2, \\ \Delta_{w_3}\phi_{M_2} = 0 \text{ in } M_2, \end{cases} \tag{44}$$

where $\Delta_{w_3} = \Delta_{M_2} + g_2\left(\nabla_{M_2}, \nabla_{M_2} w_3\right)$, $w_3 = m_1 log\left(f_2\right)$.

(4) *If* $\phi = \phi_{M_2}$ *and* $h = h_{M_2}$, *then*

$$\lambda f_2^2 + \frac{f_2}{f_1^2}\Delta_{M_2} f_2 + (m_1 - 1)\frac{\left\|\nabla_{M_2} f_2\right\|_{M_2}^2}{f_1^2} - \frac{f_2}{f_1^2}\nabla_{M_2} h_{M_2}(f_2) + m_2\frac{f_1''}{f_1} = 0 \tag{45}$$

and

$$\begin{cases} Ric_{M_2} - \frac{m_1}{f_2} Hess_{M_2}(f_2) + f_1^2 Hess_{M_2} h_{M_2} - \frac{\alpha}{f_1^4}\nabla_{M_2}\phi_{M_2} \otimes \nabla_{M_2}\phi_{M_2} \\ \quad = \left[\lambda f_1^2 + (m_2 - 1)\left(\frac{f_1'}{f_2}\right)^2 - \frac{f_1}{f_2^2} f_1''\right] g_2, \\ \Delta_{w_4}\phi_{M_2} = 0 \text{ in } M_2, \end{cases} \tag{46}$$

where $\Delta_{w_4} = \Delta_{M_2} - g_2\left(\nabla_{M_2}, \nabla_{M_2} w_4\right)$, $w_4 = h_{M_2} - m_1 log\left(f_2\right)$.

Proof. By substituting $\nabla_I f_1 = -f_1'$, $Hess_I f_1(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = f_1''$, $\Delta_I f_1 = -f_1''$, $g_I(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1$, $g_I(\nabla_I f_1, \nabla_I f_1) = -\left(f_1'\right)^2$ in Theorem 2.2, we obtain the equations (39)-(46). This completes the proof. □

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