Filomat 37:19 (2023), 6373–6386
https://doi.org/10.2298/FIL2319373A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the pseudo semi-Browder essential spectra and application to 2×2 block operator matrices

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Abstract. In the present paper, we introduce and study the pseudo semi-Browder essential spectra of bounded linear operators in a Banach space. We start by defining the pseudo semi-Browder operators and we prove the stability of these operators under commuting Riesz operator perturbations. Then, we apply the obtained results to study the stability of the pseudo semi-Browder essential spectra. We show as well the relation between the pseudo semi-Browder spectrum of the sum of two bounded linear operators and the pseudo semi-Browder spectrum of each of these operators. As an application, we study the pseudo semi-Browder spectra of 2×2 block operator matrices.

1. Introduction

In the literature, there are several applications in science and engineering that are based on eigenvalues problems. The main two object of interest when dealing with such applications are, to extract (determine) and localize the eigenvalues. Due to the insufficiency of the classical spectral analysis to achieve both aims, given that the latter approach can determine but cannot localize the eigenvalues in question, researchers resorted to another tools, namely the concept of pseudo spectrum that was first introduced by Varah [15]. As a matter of fact, this technic has been relayed on in more than one occasion (see [7, 9, 10, 14]) and in many different fields of mathematical physics. For instance, it was used in engineering (e.g. electrical), where the eigenvalues may determine the accuracy of power system at the national level or the frequency response of an amplifier. Also, in aeronautics, where the eigenvalues may determine if the flow is laminar or turbulent over a plane wing. In ecology, the eigenvalues may determine if a food web will settle into a steady equilibrium. At last, in chemistry, the eigenvalues may determine the states of energy in stable hydrogen atom.

Throughout this paper, let X be a Banach spaces and let $\mathcal{C}(X)$ be the set of all closed densely defined linear operators from X. We denote by $\mathcal{L}(X)$ the space of all bounded linear operators from X into X and by $\mathcal{K}(X)$ the subspace of compact operators from X into X. For $A \in \mathcal{C}(X)$, we write $D(T)$ for the domain, $N(A) \subset X$ for the null space and $R(A) \subset X$ for the range of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $\mathrm{N}(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $\mathrm{R}(A)$ in X. Let $\sigma(A)$ (resp. $\rho(A)$) denote

2020 *Mathematics Subject Classification*. Primary 47A53, 47A55 ; Secondary 47A13.
Keywords. Pseudo spectrum; Pseudo semi-Browder essential spectrum; Compact operator; Riesz operator.
Received: 17 October 2022; Accepted: 28 February 2023
Communicated by Snežana Živković Zlatanović
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the spectrum (resp. the resolvent set) of A. The definition of pseudo spectrum of a closed densely linear operator A for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{\lambda \in \mathbb{C} : \left\|(\lambda - A)^{-1}\right\| > \frac{1}{\varepsilon}\right\}. \tag{1}$$

By convention, we write $\left\|(\lambda - A)^{-1}\right\| = \infty$ if $(\lambda - A)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(A)$. In [7], Davies defined another equivalent of the pseudo spectrum, one that is in terms of perturbations of the spectrum. In fact for $A \in C(X)$, we have that

$$\sigma_\varepsilon(A) := \bigcup_{\|D\|<\varepsilon} \sigma(A + D). \tag{2}$$

An operator $A \in \mathcal{L}(X)$ is called upper(resp. lower) semi Fredholm operator if the range $R(A)$ is closed and $\alpha(A) < \infty$ (resp. $\beta(A) < \infty$) . We denote by $\Phi_+(X)$ (resp. $\Phi_-(X)$) the set of upper (resp. lower) semi-Fredholm operators. A is called semi-Fredholm if it is lower or upper semi Fredholm operator and it is called Fredholm operator if it is upper and lower semi Fredholm operator. We denote respectively by $\Phi_\pm(X)$ and $\Phi(X)$ the set of semi Fredholm and Fredholm operator. Then we have

$$\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X) \text{ and } \Phi(X) = \Phi_+(X) \cap \Phi_-(X).$$

If A is a semi-Fredholm operator, then the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. Clearly, if $A \in \Phi(X)$ then $i(A) < \infty$. If $A \in \Phi_+(X)\backslash\Phi(X)$ then $i(A) = -\infty$ and if $A \in \Phi_-(X)\backslash\Phi(X)$ then $i(A) = +\infty$. Let $R \in \mathcal{L}(X), R$ is said to be a Riesz operator if $\Phi_R(X) = \mathbb{C}\backslash\{0\}$ and we denote by $\mathcal{R}(X)$ the set of Riesz operators. Recall that for $A \in \mathcal{L}(X)$, the ascent, $\mathrm{a}(A)$, and the descent, $\mathrm{d}(A)$, are defined by

$$\mathrm{a}(A) = \inf\left\{n \geq 0 : \mathrm{N}(A^n) = \mathrm{N}\left(A^{n+1}\right)\right\}, \ \mathrm{d}(A) = \inf\left\{n \geq 0 : \mathrm{R}(A^n) = \mathrm{R}\left(A^{n+1}\right)\right\}.$$

If no such n exists, then $\mathrm{a}(A) = \infty$ (resp. $\mathrm{d}(A) = \infty$). An operator A is called upper semi-Browder if $A \in \Phi_+(X)$, $i(A) \leq 0$ and $a(A) < \infty$ and A is called lower semi-Browder if $A \in \Phi_-(X)$, $i(A) \geq 0$ and $d(A) < \infty$. Let $\mathcal{B}_+(X)$ (resp. $\mathcal{B}_-(X)$) denote the set of upper (resp. lower) semi-Browder operators. An operator in a Banach space is called semi-Browder if it is upper semi-Browder or lower semi-Browder and we denoted by $\mathcal{B}_\pm(X)$ the set of semi-Browder operator. An operator $A \in \mathcal{L}(X)$ is called Browder if it is both upper semi-Browder and lower semi-Browder, that is, $A \in \Phi(X), i(A) = 0$, $\mathrm{a}(A) < \infty$ and $\mathrm{d}(A) < \infty$. Let $\mathcal{B}(X)$ and $\mathcal{B}_\pm(X)$ be respectively the sets of Browder and semi-Brower operators, then we have that,

$$\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X) \text{ and } \mathcal{B}_\pm(X) = \mathcal{B}_+(X) \cup \mathcal{B}_-(X).$$

The corresponding spectra of an operator $A \in \mathcal{L}(X)$ are defined as follows:

$$\begin{aligned}
\sigma_{\mathcal{B}_+}(A) &:= \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}_+(X)\} \text{ -the upper semi-Browder essential spectrum ,}\\
\sigma_{\mathcal{B}_-}(A) &:= \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}_-(X)\} \text{ -the lower semi-Browder essential spectrum ,}\\
\sigma_{\mathcal{B}_\pm}(A) &:= \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}_\pm(X)\} \text{ -the semi-Browder essential spectrum ,}\\
\sigma_{\mathcal{B}}(A) &:= \{\lambda \in \mathbb{C} : A - \lambda \notin \mathcal{B}(X)\} \text{ -the Browder essential spectrum.}
\end{aligned}$$

For further information on the family of Fredholm operator and Browder operator we refer the reader to [11, 13].

Rakočević in [12] characterized the Browder essential spectrum for $A \in \mathcal{L}(X)$ by the following equality:

$$\sigma_{\mathcal{B}4}(A) = \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma(A + K). \tag{3}$$

Inspired by the notion of pseudospectra, Ammar and Jeribi in their works [3–5], aimed to extend these results for the essential pseudo-spectra of bounded linear operators on a Banach space and give the definitions of pseudo-Fredholm operator as follows: for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo-upper (resp. lower) semi-Fredholm operator if $A + D$ is an upper (resp. lower) semi-Fredholm operator and it is called a pseudo semi-Fredholm operator if $A + D$ is a semi-Fredholm operator. A is called a pseudo-Fredholm operator if $A + D$ is a Fredholm operator. They are noted by $\Phi^{\varepsilon}(X)$ the set of pseudo-Fredholm operators, by $\Phi^{\varepsilon}_{\pm}(X)$ the set of pseudo-semi-Fredholm operator and by $\Phi^{\varepsilon}_{+}(X)$ (resp. $\Phi^{\varepsilon}_{-}(X)$) the set of pseudo-upper semi-Fredholm (resp. lower semi-Fredholm) operator. A complex number λ is in $\Phi^{\varepsilon}_{\pm A}$, Φ^{ε}_{+A}, Φ^{ε}_{-A} or Φ^{ε}_{A} if $\lambda - A$ is in $\Phi^{\varepsilon}_{\pm}(X)$, $\Phi^{\varepsilon}_{+}(X)$, $\Phi^{\varepsilon}_{-}(X)$ or $\Phi^{\varepsilon}(X)$. Let the following essential pseudospectra:

$$\begin{aligned}
\sigma_{e1,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{+}(X)\} = \mathbb{C}\backslash\Phi^{\varepsilon}_{+A},\\
\sigma_{e2,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{-}(X)\} = \mathbb{C}\backslash\Phi^{\varepsilon}_{-A},\\
\sigma_{e3,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{\pm}(X)\} = \mathbb{C}\backslash\Phi^{\varepsilon}_{\pm A},\\
\sigma_{e4,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}(X)\} = \mathbb{C}\backslash\Phi^{\varepsilon}_{A},\\
\sigma_{e5,\varepsilon}(A) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{\varepsilon}(A+K).
\end{aligned}$$

F. Abdmouleh et al. defined in [2] the notion of pseudo Browder essential spectrum for densely closed linear operators in the Banach space as follows:

$$\sigma_{B,\varepsilon}(A) = \sigma_B(A) \cup \left\{\lambda \in \mathbb{C} : \|R_B(\lambda, A)\| > \frac{1}{\varepsilon}\right\},$$

where $R_B(\lambda, A) = \left((\lambda - A)|_{K_\lambda}\right)^{-1}(I - P_\lambda) + P_\lambda$, being P_λ the Riesz projection, K_λ a kernel of P_λ and R_λ a range of P_λ. By convention, we write $\|R_B(\lambda, A)\| = \infty$ if $R_B(\lambda, T)$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma_B(A)$. Also in [2], the authors characterized the pseudo Browder essential spectrum for bounded linear operator in Banach space by

$$\sigma_{B,\varepsilon}(A) = \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_B(A + D). \tag{4}$$

This notions of pseudospectra and the essential pseudospectra is an interesting subject by itself since these pseudospectra carry more information than spectra, especially about the transient instead of just the asymptotic behaviour of dynamical systems. Also, they have better convergence and approximation properties than spectra. These include the existence of approximate eigenvalues far from the spectrum, the instability of the spectrum even under small perturbations. The analysis of pseudospectra and essential pseudospectra has been performed in order to determine and localize the spectrum of operators, hence leading to many applications of the pseudospectra.

F. Abdmouleh and B. Elgabeur in [2] defined the concept of pseudo left (right)-Fredholm and pseudo left(right)-Browder operator, as well as their spectra in Banach spaces. These spectra are called the pseudo left (right)-Fredholm spectra, and pseudo left (right)-Browder essential spectra, and they are denoted as follows:

$$\begin{aligned}
\sigma_{\mathcal{B}l,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \mathcal{B}^{\varepsilon}_{l}(X)\right\},\\
\sigma_{\mathcal{B}r,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \mathcal{B}^{\varepsilon}_{r}(X)\right\},\\
\sigma_{el,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{l}(X)\right\},\\
\sigma_{er,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{r}(X)\right\},
\end{aligned}$$

where, $\mathcal{B}^{\varepsilon}_{l}(X)$ (resp. $\mathcal{B}^{\varepsilon}_{r}(X)$) the set of left (resp. right)-pseudo Browder operators and we denote by $\Phi^{\varepsilon}_{l}(X)$ (resp. $\Phi^{\varepsilon}_{r}(X)$) the set of left (resp. right)-pseudo Fredholm operators.

In this paper, motivated by the works done in [1–5], we will continue by introducing new essential pseudospectra for bounded linear operators in Banach space . We study the pseudo-semi Browder operator and the pseudo-semi Browder spectra for bounded linear operators in Banach space. Our aim is, to give some properties of the pseudo semi-Browder spectra. We start by showing the stability of pseudo semi-Browder spectra under Riesz operator perturbations in Banach space. One of the basic questions consists in, characterizing the relation between the pseudo semi-Browder spectra of the sum of two bounded linear operators and the pseudo semi-Browder spectrum of each of these operators. In the last section of the paper we shall apply the results described above to study the pseudo essential spectra of 2×2 block operator matrices in Banach space.

The paper is organized as follows. In Section 2, we give the definitions and the properties of the pseudo semi-Browder operators and the pseudo semi-Browder spectra of bounded linear operators in Banach space and we move on to study the stability of these spectra under commuting Riesz operator perturbations. In section 3, we show the relation between the pseudo semi-Browder spectra of the sum of two bounded linear operators and the pseudo semi-Browder spectrum of each of these operators. Finally in section 4, as an application we study the pseudo semi-Browder spectra of 2×2 block operator matrix.

2. Pseudo semi-Browder spectra

We start this section by giving the necessary definitions of the pseudo semi-Browder operators and their corresponding spectra.

Definition 2.1. *Let $\varepsilon > 0$ and $A \in \mathcal{L}(X)$.*

(*i*) *A is called a pseudo-upper (resp. lower) semi-Browder operator if $A + D$ is an upper (resp. lower) semi-Browder operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $AD = DA$.*

(*ii*) *A is called a pseudo-semi-Browder operator if $A + D$ is a semi-Browder operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $AD = DA$.*

(*iii*) *A is called a pseudo-Browder operator if $A + D$ is a Browder operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $AD = DA$.*

We denote by $\mathcal{B}^{\varepsilon}(X)$ the set of pseudo-Browder operators, by $\mathcal{B}^{\varepsilon}_{\pm}(X)$ the set of pseudo semi-Browder operators and by $\mathcal{B}^{\varepsilon}_{+}(X)$ (resp. $\mathcal{B}^{\varepsilon}_{-}(X)$) the sets of pseudo-upper semi-Browder (resp. lower semi-Browder) operators. A complex number λ is in $\mathcal{B}^{\varepsilon}_{\pm A}$, $\mathcal{B}^{\varepsilon}_{+A}$, $\mathcal{B}^{\varepsilon}_{-A}$ or $\mathcal{B}^{\varepsilon}_{A}$ if $\lambda - A$ is in $\mathcal{B}^{\varepsilon}_{\pm}(X)$, $\mathcal{B}^{\varepsilon}_{+}(X)$, $\mathcal{B}^{\varepsilon}_{-}(X)$ or $\mathcal{B}^{\varepsilon}(X)$. In this paper, we are concerned with the following pseudo semi-Browder essential spectra.

Definition 2.2. *Let $\varepsilon > 0$ and $A \in \mathcal{L}(X)$, we define the following sets:*

- *Pseudo upper semi-Browder essential spectrum,* $\sigma_{\mathcal{B}1,\varepsilon}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \mathcal{B}^{\varepsilon}_{+}(X)\} = \mathbb{C} \backslash \mathcal{B}^{\varepsilon}_{+A}$
- *Pseudo lower semi-Browder essential spectrum,* $\sigma_{\mathcal{B}2,\varepsilon}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \mathcal{B}^{\varepsilon}_{-}(X)\} = \mathbb{C} \backslash \mathcal{B}^{\varepsilon}_{-A}$
- *Pseudo semi-Browder essential spectrum,* $\sigma_{\mathcal{B}3,\varepsilon}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \mathcal{B}^{\varepsilon}_{\pm}(X)\} = \mathbb{C} \backslash \mathcal{B}^{\varepsilon}_{\pm A}$
- *Pseudo Browder essential spectrum,* $\sigma_{\mathcal{B}4,\varepsilon}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \mathcal{B}^{\varepsilon}(X)\} = \mathbb{C} \backslash \mathcal{B}^{\varepsilon}_{A}$
- *Essential pseudo Browder spectrum.* $\sigma_{\varepsilon,e}(A) := \displaystyle\bigcap_{\substack{K \in \mathcal{K}(X),\\ AK = KA}} \sigma_{\varepsilon}(A + K).$

Note that if ε tends to 0, we recover the usual definition of the semi-Browder essential spectra of a bounded linear operator A.

Proposition 2.3. *Let $A \in \mathcal{L}(X)$ and consider $i \in \{1, \ldots, 4\}$.*

(1) $\sigma_{\mathcal{B}i,\varepsilon}(A) \subset \sigma_{\varepsilon}(A)$.

(2) $\displaystyle\bigcap_{\varepsilon > 0} \sigma_{\mathcal{B}i,\varepsilon}(A) = \sigma_{\mathcal{B}i}(A)$.

(3) *If* $\varepsilon_1 < \varepsilon_2$ *then* $\sigma_{\mathcal{B}i,\varepsilon_1}(A) \subset \sigma_{\mathcal{B}i,\varepsilon_2}(A)$.

(4) *Both pseudo semi-Browder essential and the pseudo Browder essential spectra can be ordered as follows*

$$\sigma_{\mathcal{B}3,\varepsilon}(A) = \sigma_{\mathcal{B}1,\varepsilon}(A) \cap \sigma_{\mathcal{B}2,\varepsilon}(A) \subseteq \sigma_{\mathcal{B}4,\varepsilon}(A) = \sigma_{\mathcal{B}1,\varepsilon}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(A).$$

.

Proof. (1) For $i \in \{1, 2\}$ and by Definition 2.2 we have that

$$\sigma_{\mathcal{B}1,\varepsilon}(A) = \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}1}(A+D) \text{ and } \sigma_{\mathcal{B}2,\varepsilon}(A) = \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}2}(A+D).$$

Since $\sigma_{\mathcal{B}1}(A+D) \subset \sigma_{\mathcal{B}4}(A+D)$ and $\sigma_{\mathcal{B}2}(A+D) \subset \sigma_{\mathcal{B}4}(A+D)$ and using [1] we get

$$\sigma_{\mathcal{B}1,\varepsilon}(A) \subset \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}4}(A+D) \subset \sigma_\varepsilon(A) \text{ and } \sigma_{\mathcal{B}2,\varepsilon}(A) \subset \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}4}(A+D) \subset \sigma_\varepsilon(A).$$

With the same raison for $(i = 3, 4)$.

(2) For $i \in \{1, \ldots, 4\}$ we have that

$$\bigcap_{\varepsilon>0} \sigma_{\mathcal{B}i,\varepsilon}(A) = \bigcap_{\varepsilon>0} \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}i}(A+D) = \sigma_{\mathcal{B}i}(A).$$

(3) If $\varepsilon_1 < \varepsilon_2$, then for all $i \in \{1, \ldots, 4\}$, we have that

$$\bigcup_{\substack{\|D\|<\varepsilon_1,\\ AD=DA}} \sigma_{\mathcal{B}i}(A+D) \subset \bigcup_{\substack{\|D\|<\varepsilon_2,\\ AD=DA}} \sigma_{\mathcal{B}i}(A+D).$$

(4) For all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $AD = DA$, we have

$$\sigma_{\mathcal{B}3}(A+D) = \sigma_{\mathcal{B}1}(A+D) \cap \sigma_{\mathcal{B}2}(A+D) \text{ and } \sigma_{\mathcal{B}4}(A+D) = \sigma_{\mathcal{B}1}(A+D) \cup \sigma_{\mathcal{B}2}(A+D).$$

Then

$$\begin{aligned}
\sigma_{\mathcal{B}3,\varepsilon}(A) = \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}3}(A+D) &= \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}1}(A+D) \cap \sigma_{\mathcal{B}2}(A+D) \\
&= \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}1}(A+D) \bigcap \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}2}(A+D) \\
&= \sigma_{\mathcal{B}1,\varepsilon}(A) \cap \sigma_{\mathcal{B}2,\varepsilon}(A). \\
\sigma_{\mathcal{B}4,\varepsilon}(A) = \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}4}(A+D) &= \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}1}(A+D) \cup \sigma_{\mathcal{B}2}(A+D) \\
&= \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}1}(A+D) \bigcup \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}2}(A+D). \\
&= \sigma_{\mathcal{B}1,\varepsilon}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(A).
\end{aligned}$$

□

Proposition 2.4. *Let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$, then the following hold:*

(i) $\sigma_{\varepsilon,e}(A) \subset \sigma_{\varepsilon}(A)$.

(ii) $\bigcap_{\varepsilon>0} \sigma_{\varepsilon,e}(A) = \sigma_{\mathcal{B}4}(A)$.

(iii) *If* $\varepsilon_1 < \varepsilon_2$ *then* $\sigma_B(A) \subset \sigma_{\varepsilon_1,e}(A) \subset \sigma_{\varepsilon_2,e}(A)$.

(iv) $\sigma_{\varepsilon,e}(A + K) = \sigma_{\varepsilon,e}(A)$ *for all* $K \in K(X)$ *satisfying* $AK = KA$.

Proof. (i) Let $\lambda \in \sigma_{\varepsilon,e}(A)$ then, $\lambda \in \sigma_{\varepsilon}(A + K)$ for all $K \in K(X)$ and $AK = KA$. By choosing $K = 0$, one obtains $\lambda \in \sigma_{\varepsilon}(A)$.

(ii) Indeed one has:

$$\bigcap_{\varepsilon>0} \sigma_{\varepsilon,e}(A) = \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma(A + K) = \sigma_{\mathcal{B}4}(A).$$

(iii) Let $\lambda \in \sigma_{\mathcal{B}4}(A)$, then by Equation 3, we obtain $\lambda \in \sigma(A + K)$ for all $K \in \mathcal{K}(X)$ and $AK = KA$. Since $\sigma(A + K) \subset \sigma_{\varepsilon}(A + K)$, then $\lambda \in \sigma_{\varepsilon}(A + K)$ for all $K \in \mathcal{K}(X)$ and $KA = AK$. Hence we get

$$\lambda \in \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma_{\varepsilon}(A + K) = \sigma_{\varepsilon,e}(A).$$

If $\varepsilon_1 < \varepsilon_2$ then $\sigma_{\varepsilon_1}(A + K) \subset \sigma_{\varepsilon_2}(A + K)$ for all $K \in \mathcal{K}(X)$ and $KA = AK$. Therefore,

$$\bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma_{\varepsilon_1}(A + K) \subset \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma_{\varepsilon_2}(A + K).$$

(iv)

$$\sigma_{\varepsilon,e}(A + K) = \bigcap_{\substack{K'\in\mathcal{K}(X),\\ (A+K)K'=K'(A+K)}} \sigma_{\varepsilon_1}(A + K + K').$$

Let us choose $K_1 = K + K'$, we have that $K_1 \in \mathcal{K}(X)$ and $AK_1 = A(K + K') = AK + AK' = KA + K'A = (K + K')A = K_1A$. We infer that

$$\sigma_{\varepsilon,e}(A + K) = \bigcap_{\substack{K_1\in\mathcal{K}(X),\\ A+K_1=K_1A}} \sigma_{\varepsilon_1}(A + K_1) = \sigma_{\varepsilon,e}(A).$$

In the following theorem we present the relation between the essential pseudo Browder spectrum and the pseudo essential Browder spectra for bounded linear operator A in the Banach space X.
□

In the following theorem we present the relation between the essential pseudo Browder spectrum and the pseudo essential Browder spectra for bounded linear operator A in the Banach space X.

Theorem 2.5. *Let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then*

(1) $\sigma_{\varepsilon,e}(A) \subset \sigma_{\mathcal{B}4,\varepsilon}(A)$.

(2) $\sigma_{\mathcal{B}4,\varepsilon}(A) \subset \sigma_{\varepsilon,e}(A)$.

Proof. (1) Let $\lambda \in \sigma_{\varepsilon,e}(A)$, then $\lambda \in \bigcap\limits_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma_\varepsilon(A+K)$. We obtain that

$$\lambda \in \sigma_\varepsilon(A+K) \;\; \forall K \in \mathcal{K}(X),\; AK = KA.$$

By Equation (1), we have that $\lambda \in \sigma(A+K) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \left\|(A+K-\lambda I)^{-1}\right\| > \dfrac{1}{\varepsilon}\right\}$.
So there are two possible cases:

- Case 1: If $\lambda \in \sigma(A+K)$ for all $K \in \mathcal{K}(X)$ and $AK = KA$. Then

$$\lambda \in \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma(A+K) = \sigma_{\mathcal{B}4}(A) \subset \sigma_{\mathcal{B}4,\varepsilon}(A).$$

- Case 2: If $\lambda \in \sigma_{\varepsilon,e}(A)$ and $\lambda \notin \sigma(A+K)$. Then we have

$$\|(A+K-\lambda I)^{-1}\| > \frac{1}{\varepsilon}, \;\; \forall K \in \mathcal{K}(X), AK = KA. \tag{5}$$

We infer there exist a nonzero vector $f \in X$ such that $\|(A+K-\lambda I)\| \leq \varepsilon\|f\|$. Let $\psi \in X^*$ such that $\|\psi\| = 1$ and $\psi(f) = 1$. Let us define the rank one operator $D : X \longrightarrow X$ by $Dg = -\psi(g)\,(A+K-\lambda)\,f$. We see immediately that $\|D\| < \varepsilon$ and $AD = DA$. Furthermore $(A+D+K-\lambda)f = 0$, then $(A+D+K-\lambda)$ is not invertible. Therefore

$$\lambda \in \bigcup_{\substack{\|D\|<\varepsilon,\\ AD=DA}} \sigma_{\mathcal{B}4}(A+D) = \sigma_{\mathcal{B}4,\varepsilon}(A).$$

(2) Let $\lambda \in \sigma_{\mathcal{B},\varepsilon}(A)$, then by Equation (4) there are two cases:

- Case 1: Let $\lambda \in \sigma_{\mathcal{B}4,\varepsilon}(A)$ and $\lambda \in \sigma_{\mathcal{B}4}(A)$ we have that

$$\lambda \in \sigma_{\mathcal{B}4}(A) = \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma(A+K) \subset \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma_\varepsilon(A+K).$$

In this case we obtain

$$\sigma_{\mathcal{B}4,\varepsilon}(A) \subset \sigma_{\varepsilon,e}(A).$$

- Case 2: Let $\lambda \in \sigma_{\mathcal{B}4,\varepsilon}(A)$ and $\lambda \notin \sigma_{\mathcal{B}4}(A)$ then we have that $\lambda \in \left\{\lambda \in \mathbb{C}, \|R_{\mathcal{B}}(A,\lambda)\| > \dfrac{1}{\varepsilon}\right\}$. Suppose that $\lambda \notin \left\{\lambda \in \mathbb{C}, \|(A+K-\lambda I)^{-1}\| > \dfrac{1}{\varepsilon}\right\}$, then we obtain that $\|(A+K-\lambda I)^{-1}\| \leq \dfrac{1}{\varepsilon}$. The operator $A-\lambda$ can read as follows:

$$A-\lambda = (A+K-\lambda)\left(I-(A+K-\lambda)^{-1}K\right).$$

The fact that, $\left\|(A+K-\lambda)^{-1}K\right\| \leq \|(A+K-\lambda)^{-1}\|\|K\| < 1$ implies that $I-(A+K-\lambda)^{-1}K$ is an invertible operator and

$$(A-\lambda)^{-1} = \left(I-(A+K-\lambda)^{-1}K\right)^{-1}(A+K-\lambda)^{-1}.$$

However, $KA = AK$, then

$$(I-(A+K-\lambda)^{-1}K)^{-1} = \sum_{n=0}^{+\infty}\left((-1)(A+K-\lambda)^{-1}\right)^n K^n.$$

Then $\left\|\left(I + (A + K - \lambda)^{-1}K\right)^{-1}\right\| < \dfrac{\varepsilon}{\varepsilon + \|K\|}$. Consequently

$$\left\|(A - \lambda)^{-1}\right\| \leq \frac{\varepsilon\|(A + K - \lambda)^{-1}\|}{\varepsilon + \|K\|} \leq \frac{1}{\varepsilon}.$$

Finally, we obtain that

$$\lambda \notin \left\{\lambda \in \mathbb{C} \text{ such that } \|R_{\mathcal{B}}(A, \lambda)\| > \frac{1}{\varepsilon}\right\}. \tag{6}$$

Since $\lambda \notin \sigma_{\mathcal{B}4}(A)$ then by Equation (6) we deduce that $\lambda \notin \sigma_{\mathcal{B}4,\varepsilon}(A)$.

□

3. Stability of the Pseudo Essential Spectrum

Now, once the spectra are determined, we are able to study the stability of pseudo semi-Browder spectra by Riesz operator perturbations satisfying some assumptions.

Theorem 3.1. *Let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$ then we have ,that*

(i) *If $R \in \mathcal{R}(X)$ and $AR = RA$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $DR = RD$ we have*

$$\sigma_{\mathcal{B}i,\varepsilon}(A) = \sigma_{\mathcal{B},\varepsilon}(A + R), \quad i \in \{1, \dots, 4\}.$$

(ii) *If $K \in \mathcal{K}(X)$ and $AK = KA$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $DK = KD$ we have*

$$\sigma_{\mathcal{B}i,\varepsilon}(A) = \sigma_{\mathcal{B}i,\varepsilon}(A + K), \quad i \in \{1, \dots, 4\}.$$

Proof. (i) We start by proving $\sigma_{\mathcal{B}1,\varepsilon}(A + R) \subset \sigma_{\mathcal{B}1,\varepsilon}(A)$. Let $\lambda \notin \sigma_{\mathcal{B}1,\varepsilon}(A)$ then $\lambda - A \in \mathcal{B}^{\varepsilon}_{+}(X)$ we infer that

$$\lambda - (A + D) \in \mathcal{B}_{+}(X) \ \ \forall \|D\| < \varepsilon \text{ and } AD = DA.$$

Since $(A + D)R = R(A + D)$, Rakočević [12] which ensure that

$$\lambda - (A + R + D) \in \mathcal{B}_{+}(X), \ \ \forall \|D\| < \varepsilon \text{ and } (A + R)D = D(A + R).$$

we get $\lambda \notin \sigma_{\mathcal{B}1,\varepsilon}(A + R)$. A similar raison we obtain $\sigma_{\mathcal{B}1,\varepsilon}(A + R) \subset \sigma_{\mathcal{B}1,\varepsilon}(A)$.

We prove know $\sigma_{\mathcal{B}2,\varepsilon}(A + R) \subset \sigma_{\mathcal{B}2,\varepsilon}(A)$. let $\lambda \notin \sigma_{\mathcal{B}2,\varepsilon}(A)$ then $\lambda - A \in \mathcal{B}^{\varepsilon}_{-}(X)$, we obtain that

$$\lambda - (A + D) \in \mathcal{B}_{-}(X), \ \ \forall \|D\| < \varepsilon \text{ and } AD = DA.$$

Since $(A + D)R = R(A + D)$ then by Rakočević [12] we have that

$$\lambda - (A + R + D) \in \mathcal{B}_{-}(X) \ \ \forall \|D\| < \varepsilon \text{ and } (A + R)D = D(A + R).$$

we get $\lambda \notin \sigma_{\mathcal{B}2,\varepsilon}(A + R)$. A similar raison, we show that $\sigma_{\mathcal{B}2,\varepsilon}(A + R) \subset \sigma_{\mathcal{B}2,\varepsilon}(A)$.

We deduce from Proposition 2.3 (4) that

$$\sigma_{\mathcal{B}3,\varepsilon}(A + R) = \sigma_{\mathcal{B}1,\varepsilon}(A + R) \cup \sigma_{\mathcal{B}2,\varepsilon}(A + R) = \sigma_{\mathcal{B}1,\varepsilon}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(A) = \sigma_{\mathcal{B}3,\varepsilon}(A),$$

and

$$\sigma_{\mathcal{B}4,\varepsilon}(A + R) = \sigma_{\mathcal{B}1,\varepsilon}(A + R) \cap \sigma_{\mathcal{B}2,\varepsilon}(A + R) = \sigma_{\mathcal{B}1,\varepsilon}(A) \cap \sigma_{\mathcal{B}2,\varepsilon}(A) = \sigma_{\mathcal{B}4,\varepsilon}(A).$$

(ii) Since $\mathcal{K}(X) \subset \mathcal{R}(X)$ and using (i), we obtain the result.

□

In the following theorem, we examine the stability of the essential pseudo spectrum under Riesz operator perturbations.

Theorem 3.2. *Let $\varepsilon > 0, A \in \mathcal{L}(X)$ and $R \in \mathcal{R}(X)$ such that $R(A+K) = (A+K)R$ for all $K \in \mathcal{K}(X)$. If $\|R\| < \varepsilon$ then there exist $\varepsilon_0, \varepsilon_1$ such that $0 < \varepsilon_0 < \varepsilon < \varepsilon_1$ satisfying*

$$\sigma_{\varepsilon_0,e}(A+R) \subset \sigma_{\varepsilon,e}(A) \subset \sigma_{\varepsilon_1,e}(A+R).$$

Proof. Let $\lambda \in \sigma_{\varepsilon,e}(A) = \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma_\varepsilon(A+K)$. If $\lambda \in \sigma(A+K)$, for all $K \in \mathcal{K}(X)$ and $AK = KA$, then

$$\lambda \in \bigcap_{\substack{K\in\mathcal{K}(X),\\ AK=KA}} \sigma(A+K) = \sigma_{\mathcal{B}4}(A).$$

First, we prove that there exists ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and $\sigma_{\varepsilon_0,e}(A+R) \subset \sigma_{\varepsilon,e}(A)$. For that let $\lambda \notin \left\{\lambda \in \mathbb{C}, \text{ such that } \|(A+K-\lambda I)^{-1}\| > \dfrac{1}{\varepsilon},\ \forall K \in \mathcal{K}(X)\right\}$, so $\|(A+K-\lambda I)^{-1}\| \le \dfrac{1}{\varepsilon}$.

By writing $A+K+\lambda$ in the form

$$A+K+R-\lambda = (A+K-\lambda)\left(I+(A+K-\lambda)^{-1}R\right),$$

and due to the fact that $\|(A+K-\lambda I)^{-1}R\| \le \|(A+K-\lambda I)^{-1}\|\|R\| < 1$, one gets that $\left(I+(A+K-\lambda)^{-1}R\right)$ is an invertible operator and its inverse is written as follows:

$$(A+K+R-\lambda)^{-1} = \left(I+(A+K-\lambda)^{-1}R\right)^{-1}(A+K-\lambda)^{-1}.$$

Using the fact that $R(A+K) = (A+K)R$, we have that

$$\left(I+(A+K-\lambda)^{-1}R\right)^{-1} = \sum_{n=0}^{+\infty}\left((-1)(A+K-\lambda)^{-1}\right)^n.$$

Which implies that

$$(A+K+R-\lambda)^{-1} = \sum_{n=0}^{+\infty}\left((-1)(A+K-\lambda)^{-1}\right)^n R^n.$$

Since

$$\left\|\left(I+(A+K-\lambda)^{-1}R\right)^{-1}\right\| \le \frac{\varepsilon}{\varepsilon-\|R\|},$$

this shows that $\left\|(A+K+R-\lambda)^{-1}\right\| \le \dfrac{\left\|(A+K-\lambda)^{-1}\right\|\varepsilon}{\varepsilon-\|R\|}$. Consequently, we infer that

$$\left\|(A+K+R-\lambda)^{-1}\right\| \le \frac{1}{\varepsilon-\|R\|}.$$

By taking $\varepsilon_0 = \varepsilon - \|R\|$ then $0 < \varepsilon_0 < \varepsilon$ and $\lambda \notin \sigma_{\varepsilon_0,e}(A+R)$, we conclude that, there exists ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and

$$\sigma_{\varepsilon_0,e}(A+R) \subseteq \sigma_{\varepsilon,e}(A).$$

Hence we prove that there exists ε_1 such that $0 < \varepsilon < \varepsilon_1$ and $\sigma_{\varepsilon,e}(A) \subset \sigma_{\varepsilon_1,e}(A+R)$. Let $\varepsilon_1 = \varepsilon + \|R\|$, then for $\lambda \notin \left\{\lambda \in \mathbb{C}, \left\|(A+K+R)^{-1}\right\| > \frac{1}{\varepsilon_1}\right\}$ implies $\left\|(A+K+R)^{-1}\right\| \le \varepsilon_1^{-1}$. By writing $A+K-\lambda$ in the following form

$$A+K-\lambda = (A+K+R-\lambda)\left(I-(A+K+R-\lambda)^{-1}R\right), \tag{7}$$

and by the fact that, $\left\|(A+K+R-\lambda)^{-1}R\right\| < 1$ one gets that $I-(A+K+R-\lambda)^{-1}R$ is an invertible operator and using Equation (7) we obtain,

$$(A+K-\lambda)^{-1} = \left(I-(A+K+R-\lambda)^{-1}R\right)^{-1}(A+K+R-\lambda)^{-1}.$$

However, $(R+K)A = A(R+K)$, then $\left\|\left(I+(A+K+R-\lambda)^{-1}R\right)^{-1}\right\| < \dfrac{\varepsilon_1}{\varepsilon_1 - \|R\|}$. Consequently

$$\left\|(A+K-\lambda)^{-1}\right\| \leq \frac{1}{\varepsilon}.$$

Then one can conclude that, there exists ε_1 such that $0 < \varepsilon < \varepsilon_1$ satisfying

$$\sigma_{\varepsilon,e}(A+R) \subseteq \sigma_{\varepsilon_1,e}(A).$$

□

4. Pseudo semi-Browder spectra of the sum of two operators

This section devoted to the study of the pseudo semi-Browder spectra of the sum of two bounded linear operators by exhibiting its relation with the pseudo semi-Browder spectrum of each of these operator.

Theorem 4.1. *Let A, B and D three operators in $\mathcal{L}(X)$ such that $AB = BA$, $AD = DA$ and $A(B+D) \in \mathcal{R}(X)$ and let $\varepsilon > 0$. Then*

$$\sigma_{\mathcal{B}1,\varepsilon}(A+B)\backslash\{0\} = \left[\sigma_{\mathcal{B}1}(A) \cup \sigma_{\mathcal{B}1,\varepsilon}(B)\right]\backslash\{0\}.$$

Proof. For $\lambda \in \mathbb{C}$, we can write

$$(\lambda - A)(\lambda - B - D) = A(B+D) + \lambda(\lambda - A - B - D), \tag{8}$$

and

$$(\lambda - B - D)(\lambda - A) = (B+D)A + \lambda(\lambda - A - B - D). \tag{9}$$

Suppose that $\lambda \neq 0$ such that $\lambda \notin \sigma_{\mathcal{B}1}(A) \cup \sigma_{\mathcal{B}1,\varepsilon}(B)$ then $(\lambda - A) \in \mathcal{B}_1(X)$ and $(\lambda - B) \in \mathcal{B}_+^{\varepsilon}(X)$, hence

$$(\lambda - A) \in \mathcal{B}_+(X) \text{ and } (\lambda - B - D) \in \mathcal{B}_+(X),\ \forall \|D\| < \varepsilon,\ BD = DB. \tag{10}$$

Since $A(B+D) = (B+D)A$, then

$$(\lambda - A)(\lambda - B - D) = (\lambda - B - D)(\lambda - A). \tag{11}$$

By Equations (10), (11) and [[8], Theorem7.9.2, page 276], we obtain that

$$(\lambda - A)(\lambda - B - D) \in \mathcal{B}_+(X),\ \forall \|D\| < \varepsilon \text{ and } BD = DB.$$

So, applying Equation (8), we deduce that

$$A(B+D) + \lambda(\lambda - A - B - D) \in \mathcal{B}_+(X). \tag{12}$$

Since $AB = BA$, $AD = DA$ and $BD = DB$, then

$$A(B+D)(\lambda - A - B - D) = (\lambda - A - B - D)A(B+D). \tag{13}$$

We have $A(B+D) \in \mathcal{R}(X)$ then by Rakočević [12] and Equations (12), (13), we obtain that

$$\lambda - A - B - D \in \mathcal{B}_+(X)\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } (A+B)D = D(A+B).$$

By Definition 2.1, we obtain that $\lambda \notin \sigma_{\mathcal{B}1,\varepsilon}(A+B)\backslash\{0\}$. Therefore

$$\sigma_{\mathcal{B}1,\varepsilon}(A+B)\backslash\{0\} \subset \left[\sigma_{\mathcal{B}1}(A) \cup \sigma_{\mathcal{B}1,\varepsilon}(B)\right]\backslash\{0\}.$$

Conversely, let $\lambda \notin \sigma_{\mathcal{B}1,\varepsilon}(A+B)\backslash\{0\}$, then $(\lambda - A - B) \in \mathcal{B}^{\varepsilon}_{+}(X)$ and by Definition 2.1 we obtain that

$$(\lambda - A - B - D) \in \mathcal{B}_{+}(X)\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } (\lambda - A - B)D = D(\lambda - A - B).$$

Since $A(B+D) \in \mathcal{R}(X)$ and $(B+D)A \in \mathcal{R}(X)$ then by using Rakočević [12] and Equation (13) we obtain that

$$A(B+D) + \lambda(\lambda - A - B - D) \in \mathcal{B}_{+}(X) \textit{ and } (B+D)A + \lambda(\lambda - A - B - D) \in \mathcal{B}_{+}(X).$$

Using Equations (8) and (9) we get

$$(\lambda - A)(\lambda - B - D) \in \mathcal{B}_{+}(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \mathcal{B}_{+}(X).$$

By Equations (11) and [[8], Theorem 7.9.2, page 276], we obtain that

$$\lambda - A \in \mathcal{B}_{+}(X) \text{ and } (\lambda - B - D) \in \mathcal{B}_{+}(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon,\ (\lambda - B)D = D(\lambda - B).$$

Hence $\lambda \notin \left[\sigma_{\mathcal{B}1}(A) \cup \sigma_{\mathcal{B}1,\varepsilon}(B)\right]\backslash\{0\}$. Therefore

$$\sigma_{\mathcal{B}1,\varepsilon}(A+B)\backslash\{0\} = \left[\sigma_{\mathcal{B}1}(A) \cup \sigma_{\mathcal{B}1,\varepsilon}(B)\right]\backslash\{0\}.$$

□

Theorem 4.2. *Let A, B and D three operators in $\mathcal{L}(X)$ such that $AB = BA$, $AD = DA$ and $A(B+D) \in \mathcal{R}(X)$ and let $\varepsilon > 0$. Then*

$$\sigma_{\mathcal{B}2,\varepsilon}(A+B)\backslash\{0\} = \left[\sigma_{\mathcal{B}2}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(B)\right]\backslash\{0\}.$$

Proof. Suppose that $\lambda \neq 0$ such that $\lambda \notin \sigma_{\mathcal{B}_{-}}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(B)$. Then $(\lambda - A) \in \mathcal{B}_{-}(X)$ and $(\lambda - B) \in \mathcal{B}^{\epsilon}_{-}(X)$, hence $(\lambda - A) \in \mathcal{B}_{-}(X)$ and $(\lambda - B - D) \in \mathcal{B}_{-}(X)$ for all $\|D\| < \varepsilon$ and $BD = DB$. As a consequence we get that Equations (11) and $\Big[$ [8], Theorem 7.9.2, page 276$\Big]$, yields

$$(\lambda - A)(\lambda - B - D) \in \mathcal{B}_{-}(X).$$

Using Equation (8) we obtain $A(B+D) + \lambda(\lambda - A - B - D) \in \mathcal{B}_{-}(X)$. Since $A(B+D) \in \mathcal{R}(X)$ then, by Equation (13) and Rakočević [12] we obtain that

$$\lambda - A - B - D \in \mathcal{B}_{-}(X)\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } (A+B)D = D(A+B),$$

Applying Definition 2.1, we get that $\lambda \notin \sigma_{\mathcal{B}2,\varepsilon}(A+B)\backslash\{0\}$. Therefore

$$\sigma_{\mathcal{B}2,\varepsilon}(A+B)\backslash\{0\} \subset \left[\sigma_{\mathcal{B}2}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(B)\right]\backslash\{0\}.$$

To prove the inverse, let $\lambda \notin \sigma_{\mathcal{B}2,\varepsilon}(A+B)\backslash\{0\}$, then $(\lambda - A - B) \in \mathcal{B}^{\varepsilon}_{-}(X)$. We deduce from Definition 2.1, that

$$(\lambda - A - B - D) \in \mathcal{B}_{-}(X)\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } (A+B)D = D(A+B).$$

Since $A(B+D) \in \mathcal{R}(X)$ and $(B+D)A \in \mathcal{R}(X)$, it follows from Rakočević [12] and Equation (13) that

$$A(B+D) + \lambda(\lambda - A - B - D) \in \mathcal{B}_{-}(X) \textit{ and } (B+D)A + \lambda(\lambda - A - B - D) \in \mathcal{B}_{-}(X).$$

We can apply Equations (8) and (9) we infer that

$$(\lambda - A)(\lambda - B - D) \in \mathcal{B}_{-}(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \mathcal{B}_{-}(X).$$

Due to Equation (11) and $\Big[$ [8], Theorem 7.9.2, page 276 $\Big]$, we obtain that

$$\lambda - A \in \mathcal{B}_{-}(X) \text{ and } (\lambda - B - D) \in \mathcal{B}_{-}(X)\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } (\lambda - B)D = D(\lambda - B).$$

Then we have $\lambda \notin \left[\sigma_{\mathcal{B}1}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(B)\right]\backslash\{0\}$. Therefore

$$\sigma_{\mathcal{B}2,\varepsilon}(A+B)\backslash\{0\} = \left[\sigma_{\mathcal{B}2}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(B)\right]\backslash\{0\}.$$

□

We derive from the previous results the following Corollary.

Corollary 4.3. *Let A, B and D three operators in $\mathcal{L}(X)$ such that $AB = BA$, $AD = DA$ and $A(B + D) \in \mathcal{R}(X)$ and let $\varepsilon > 0$. Then*

(*i*) $\sigma_{\mathcal{B}3,\varepsilon}(A + B)\backslash\{0\} = \left[(\sigma_{\mathcal{B}3}(A) \cup \sigma_{\mathcal{B}3,\varepsilon}(B)) \cup (\sigma_{\mathcal{B}1}(A) \cap \sigma_{\mathcal{B}2,\varepsilon}(B)) \cup (\sigma_{\mathcal{B}2}(A) \cap \ \sigma_{\mathcal{B}1,\varepsilon}(B))\right] \backslash\{0\}$.

(*ii*) $\sigma_{\mathcal{B}4,\varepsilon}(A + B)\backslash\{0\} = \left[\sigma_{\mathcal{B}4}(A) \cup \sigma_{\mathcal{B}4,\varepsilon}(B)\right] \backslash\{0\}$.

Proof. (*i*) Let $A \in \mathcal{L}(X)$, then by Proposition 2.3 one gets

$$\sigma_{\mathcal{B}3}(A) = \sigma_{\mathcal{B}1}(A) \cap \sigma_{\mathcal{B}2}(A) \text{ and } \sigma_{\mathcal{B}3,\varepsilon}(A) = \sigma_{\mathcal{B}1,\varepsilon}(A) \cap \sigma_{\mathcal{B}2,\varepsilon}(A).$$

So from Proposition 2.3, we deduce that

$$\sigma_{\mathcal{B}3}(A + B) = \sigma_{\mathcal{B}1}(A + B) \cap \sigma_{\mathcal{B}2}(A + B) \text{ and } \sigma_{\mathcal{B}3,\varepsilon}(A + B) = \sigma_{\mathcal{B}1,\varepsilon}(A + B) \cap \sigma_{\mathcal{B}2,\varepsilon}(A + B). \tag{14}$$

The proof is concluded by using Theorem(4.1), Theorem(4.2) and Equation (14).

(*ii*) Applying Proposition 2.3, one gets

$$\sigma_{\mathcal{B}4}(A + B) = \sigma_{\mathcal{B}1}(A + B) \cup \sigma_{\mathcal{B}2}(A + B) \text{ and } \sigma_{\mathcal{B}4,\varepsilon}(A + B) = \sigma_{\mathcal{B}1,\varepsilon}(A + B) \cup \sigma_{\mathcal{B}2,\varepsilon}(A + B). \tag{15}$$

The proof is concluded by using Theorem(4.1), Theorem(4.2) and Equation (15).

□

5. Application to block operator matrices

The purpose of this section is to apply the main result of sections and to study the semi pseudo Browder spectra of 2×2 block operator matrices

Theorem 5.1. *Let X_1 and X_2 be two Banach spaces and consider the 2×2 block operator matrix defined on $X_1\times X_2$ by*

$$\mathcal{M}_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$, $C \in \mathcal{L}(X_1, X_2)$ and let $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $T_i \in \mathcal{L}(X_i)$, $i = 1, 2$. If $AT_1 = T_1A$, $AT_2 = T_2A$ and $CT_2 = T_1C$ then

(*i*) $\sigma_{\mathcal{B}i,\varepsilon}(\mathcal{M}_C) \subset \sigma_{\mathcal{B}i,\varepsilon}(A) \cup \sigma_{\mathcal{B}i,\varepsilon}(B), i = 1, 2, 4.$

(*ii*) $\sigma_{\mathcal{B}3,\varepsilon}(\mathcal{M}_C) \subset \left[(\sigma_{\mathcal{B}3,\varepsilon}(A) \cup \sigma_{\mathcal{B}3,\varepsilon}(B)) \cup (\sigma_{\mathcal{B}1,\varepsilon}(A) \cap \sigma_{\mathcal{B}2,\varepsilon}(B)) \cup (\sigma_{\mathcal{B}1,\varepsilon}(B) \cap \sigma_{\mathcal{B}2,\varepsilon}(A))\right]$.

Proof. (*i*) The expression $\lambda - M_C - T$ can read as follows:

$$\lambda - M_C - T = \begin{pmatrix} I & 0 \\ 0 & \lambda - B - T_2 \end{pmatrix} \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda - A - T_1 & 0 \\ 0 & I \end{pmatrix}. \tag{16}$$

If $\lambda \notin \sigma_{\mathcal{B}1,\varepsilon}(A) \cup \sigma_{\mathcal{B}1,\varepsilon}(B)$ (resp. $\sigma_{\mathcal{B}2,\varepsilon}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(B)$) then $\lambda - A \in \mathcal{B}^{\varepsilon}_{+}(X_1)$ (resp. $\mathcal{B}^{\varepsilon}_{-}(X_1)$) and $\lambda - B \in \mathcal{B}^{\varepsilon}_{+}(X_2)$ (resp. $\mathcal{B}^{\varepsilon}_{-}(X_2)$) Since $\|T\| < \varepsilon$ and $AT_1 = T_1A$ and $AT_2 = T_2A$ then $\|T_1\| < \varepsilon$ and $\|T_2\| < \varepsilon$, so $\lambda - A - T_1 \in \mathcal{B}_{+}(X)$ (resp. $\mathcal{B}^{\varepsilon}_{-}(X_1)$) and $\lambda - B - T_2 \in \mathcal{B}_{+}(X)$ (resp. $\mathcal{B}^{\varepsilon}_{-}(X_1)$). Using [[16], Theorem 2.7], we infer that

$$\begin{pmatrix} \lambda - A - T_1 & 0 \\ 0 & I \end{pmatrix} \in \mathcal{B}_{+}(X_1 \times X_2) \text{ (resp. } \mathcal{B}_{-}(X_1 \times X_2))$$

and

$$\begin{pmatrix} I & 0 \\ 0 & \lambda - B - T_2 \end{pmatrix} \in \mathcal{B}_+ (X_1 \times X_2) \,(\text{resp. } \mathcal{B}_-(X_1 \times X_2))$$

Since $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible, then by Equation(16) we obtain that

$$\lambda - M_C - T \in \mathcal{B}_+(X_1) \,(\text{resp. } \mathcal{B}_-(X_1)) \,,\ \forall \, \|T\| < \varepsilon \text{ and } M_C T = T M_C.$$

Thus $\lambda - M_C \in \mathcal{B}_+^{\varepsilon} (X_1 \times X_2) \,(\text{resp. } \mathcal{B}_-^{\varepsilon}(X_1 \times X_2))$. Hence

$$\sigma_{\mathcal{B}1,\varepsilon} (M_C) \subset \sigma_{\mathcal{B}1,\varepsilon}(A) \cup \sigma_{\mathcal{B}1,\varepsilon}(B) \text{ and } \sigma_{\mathcal{B}2,\varepsilon} (M_C) \subset \sigma_{\mathcal{B}2,\varepsilon}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(B). \tag{17}$$

Using the same approach for $\sigma_{\mathcal{B}4,\varepsilon} (M_C)$ and $\sigma_{\mathcal{B}4,\varepsilon} (M_C)$, we obtain the following result

$$\sigma_{\mathcal{B}4,\varepsilon} (M_C) \subset \sigma_{\mathcal{B}4,\varepsilon}(A) \cup \sigma_{\mathcal{B}4,\varepsilon}(B).$$

(*ii*) We have $\sigma_{\mathcal{B}3,\varepsilon} (M_C) = \sigma_{\mathcal{B}1,\varepsilon} (M_C) \cap \sigma_{\mathcal{B}2,\varepsilon} (M_C)$ and by Equation (17) we obtain that

$$\sigma_{\mathcal{B}3,\varepsilon} (M_C) \subset \left[(\sigma_{\mathcal{B}3,\varepsilon}(A) \cup \sigma_{\mathcal{B}3,\varepsilon}(B)) \cup (\sigma_{\mathcal{B}1,\varepsilon}(A) \cap \sigma_{\mathcal{B}2,\varepsilon}(B)) \cup (\sigma_{\mathcal{B}1,\varepsilon}(B) \cap \sigma_{\mathcal{B}2,\varepsilon}(A))\right].$$

□

Theorem 5.2. *Let X_1 and X_2 be two Banach spaces and consider the 2×2 block operator matrix defined on $X_1 \times X_2$ by*

$$\mathfrak{L} := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2, X_1)$, $C \in \mathcal{L}(X_1, X_2)$ and $D \in \mathcal{L}(X_2)$. Let $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $T_i \in \mathcal{L}(X_i)$ and $\|T\| < \varepsilon$. If $\begin{cases} CB \in \mathcal{B}(X_2) \\ C(A + T_1) \in \mathcal{B}(X_1, X_2) \\ AT_1 = T_1 A,\ CT_1 = T_2 C \\ BT_2 = T_2 B,\ DT_2 = T_2 D \end{cases}$

Then

(*i*) $\sigma_{\mathcal{B}i,\varepsilon}(\mathfrak{L})\backslash\{0\} \subseteq [\sigma_{\mathcal{B}i,\varepsilon}(A) \cup \sigma_{\mathcal{B}i,\varepsilon}(D)] \backslash\{0\}, i = 1, 2, 4.$

(*ii*) $\sigma_{\mathcal{B}3,\varepsilon}(\mathfrak{L})\backslash\{0\} \subseteq \left[(\sigma_{\mathcal{B}3,\varepsilon}(A) \cup \sigma_{\mathcal{B}3,\varepsilon}(D)) \cup (\sigma_{\mathcal{B}1,\varepsilon}(A) \cap \sigma_{\mathcal{B}2,\varepsilon}(D)) \cup (\sigma_{\mathcal{B}1,\varepsilon}(D) \cap \sigma_{\mathcal{B}2,\varepsilon}(A))\right].$

Proof. (*i*) For $\lambda \in \mathbb{C}\backslash\{0\}$, we can write $(\lambda - \mathfrak{L} - T)$ in the following form

$$(\lambda - \mathfrak{L} - T) = \frac{1}{\lambda} \begin{pmatrix} 0 & 0 \\ -C(A + T_1) & -CB \end{pmatrix} + \begin{pmatrix} I & 0 \\ -\frac{C}{\lambda} & I \end{pmatrix} \begin{pmatrix} \lambda - A - T_1 & -B \\ 0 & \lambda - D - T_2 \end{pmatrix}. \tag{18}$$

Let $\lambda \notin \sigma_{\mathcal{B}1,\varepsilon}(A) \cup \sigma_{\mathcal{B}1,\varepsilon}(D)$ (resp. $\sigma_{\mathcal{B}2,\varepsilon}(A) \cup \sigma_{\mathcal{B}2,\varepsilon}(D)$), Using [[16], Theorem 2.7], we obtain

$$\begin{pmatrix} \lambda - A - T_1 & B \\ 0 & \lambda - D - T_2 \end{pmatrix} \in \mathcal{B}_+ (X_1 \times X_2) \,(\text{resp. } \mathcal{B}_- (X_1 \times X_2)).$$

Since $\begin{pmatrix} I & 0 \\ -\frac{C}{\lambda} & I \end{pmatrix}$ is invertible then we obtain that

$$\begin{pmatrix} I & 0 \\ -\frac{C}{\lambda} & I \end{pmatrix} \begin{pmatrix} \lambda - A - T_1 & B \\ 0 & \lambda - D - T_2 \end{pmatrix} \in \mathcal{B}(X_1 \times X_2) \tag{19}$$

Since $CB \in \mathcal{B}_+(X_2)$ (resp. $\mathcal{B}_-(X_2)$) and $C(A+T_1) \in \mathcal{B}_+(X_1 \times X_2)$ (resp. $\mathcal{B}_-(X_1 \times X_2)$), then one gets that

$$\begin{pmatrix} 0 & 0 \\ -C(A+T_1) & -CB \end{pmatrix} \in \mathcal{B}_+(X_1 \times X_2) \, (\text{rep. } \mathcal{B}_-(X_1 \times X_2)) \tag{20}$$

Applying Equations (18), (19) and (20) we obtain that

$$(\lambda - \mathfrak{L} - T) \in \mathcal{B}_+(X) \, (\text{resp. } \mathcal{B}_-(X)) \quad \forall \, \|T\| < \varepsilon \textit{ and } T\mathcal{L} = \mathcal{L}T.$$

Definition 2.2 states that $\lambda \notin \sigma_{\mathcal{B}1,\varepsilon}(\mathfrak{L})$ (resp. $\sigma_{\mathcal{B}2,\varepsilon}(\mathfrak{L})$).

The proof for $i = 4$ follows the same approach of that used for $i = 1, 2$.

(ii) For $i = 3$, it suffice to show that $\sigma_{\mathcal{B}3,\varepsilon}(\mathfrak{L}) = \sigma_{\mathcal{B}1,\varepsilon}(\mathfrak{L}) \cap \sigma_{\mathcal{B}2,\varepsilon}(\mathfrak{L})$, the proof is afterwards concluded using (i).

□

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