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Convergence of certain subsequences of the power sequence in a Banach algebra

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Abstract. We consider the existence of limits $\lim\limits_{n\to\infty} a^{qn}$ and $\lim\limits_{n\to\infty} a^{q^n}$, where a is an arbitrary element of a complex Banach algebra with the unit, and q is an integer, $q \geq 2$.

1. Introduction

If B is a complex square matrix, it is useful to know conditions that imply the existence of the limit $\lim\limits_{n\to\infty} B^n$ (see [3], [14]). Koliha extended these results for Banach space operators [9] and elements in Banach algebras [10]. If a is an element of a Banach algebra, then $(a^n)_n$ is a power sequence. In this article we will consider special subsequences of the power sequence and thus extend some results from [3] to complex unital Banach algebras. Precisely, we will determine the equivalent conditions for the convergence of the sequences $(a^{qn})_n$ and $(a^{q^n})_n$, where $q \in \mathbb{N}$ and $q \geq 2$. Our results extend the well-known results of Koliha [9], [10], and Chen and Hartwig [3].

Let $\mathcal{A}$ be a complex Banach algebra with the unit 1. We use $\mathcal{A}^{-1}$, $\mathcal{A}^{\bullet}$, $\mathcal{A}^{Nil}$ and $\mathcal{A}^{qNil}$, respectively, to denote the set of all invertible, idempotent, nilpotent, and quasinilpotent elements in $\mathcal{A}$. If $a \in \mathcal{A}$, let $\sigma(a)$, $r(a)$, $\mathrm{comm}(a)$ and $\mathrm{comm}^2(a)$, respectively, denote the spectrum, spectral radius, commutant, and double commutant of a.

If $p \in \mathcal{A}^{\bullet}$, then $p\mathcal{A}p$ is a Banach subalgebra of $\mathcal{A}$ with the unit p. If $ap = pa$, then $\sigma^{p\mathcal{A}p}(ap)$ is the spectrum of ap in the Banach algebra $p\mathcal{A}p$.

If $S \subset \mathbb{C}$, then we use $\mathrm{acc}\, S$ and $\mathrm{iso}\, S$, respectively, for the set of all points of accumulation and the set of all isolated points of S.

If $a \in \mathcal{A}$ and $\lambda \notin \mathrm{acc}\,\sigma(a)$, then $a^{\pi,\lambda}$ is the spectral projection of a corresponding to the spectral set $\{\lambda\}$. Obviously, $a^{\pi,\lambda} = 0$ if and only if $\lambda \notin \sigma(a)$. If $0 \notin \mathrm{acc}\,\sigma(a)$, then it is common to write $a^{\pi} \equiv a^{\pi,0}$.

Let $a \in \mathcal{A}$. The element $a^d \in \mathcal{A}$ is the generalized Drazin inverse of a, if the following hold:

$$aa^d = a^d a,\ a^d a a^d = a^d,\ a(aa^d - 1) \in \mathcal{A}^{qNil}.$$

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If the generalized Drazin inverse a^d of a exists, then it is unique and the notation is justified.

It is well-known that for given $a \in \mathcal{A}$ there exists $a^d \in \mathcal{A}$ if and only if $0 \notin \operatorname{acc}\sigma(a)$. In this case $a(1-a^\pi)$ is invertible in the algebra $(1-a^\pi)\mathcal{A}(1-a^\pi)$, and aa^π is quasinilpotent in the algebra $a^\pi\mathcal{A}a^\pi$. The ordinary inverse of $a(1-a^\pi)$ in $(1-a^\pi)\mathcal{A}(1-a^\pi)$ is the generalized Drazin inverse of a in $\mathcal{A}$. The relation between the generalized Drazin inverse a^d and the spectral idempotent a^π is given by $a^\pi = 1 - aa^d$. The element a^d can be obtained by the analytic functional calculus. Take $f(z) = 0$ for z around $\{0\}$, and take $f(z) = 1/z$ for z around $\sigma(a) \setminus \{0\}$. Then f is analytic in a neighborhood of $\sigma(a)$ and $f(a) = a^d$. Thus, we have $a^d \in \operatorname{comm}^2(a)$. $\mathcal{A}^d$ is the usual notation for the set of all generalized Drazin invertible elements in $\mathcal{A}$. Obviously, $\mathcal{A}^{-1} \subset \mathcal{A}^d$ and if $a \in \mathcal{A}^{-1}$ then $a^d = a^{-1}$. Also, $\mathcal{A}^{qNil} \subset \mathcal{A}^d$ and if $a \in \mathcal{A}^{qNil}$ then $a^d = 0$. The last statement follows from the uniqueness of the generalized Drazin inverse in the case when it exists.

From $\mathcal{A}^{Nil} \subset \mathcal{A}^{qNil}$ we see that $a(aa^d - 1) \in \mathcal{A}^{Nil}$ is stronger condition then $a(aa^d - 1) \in \mathcal{A}^{qNil}$. For given $a \in \mathcal{A}$, the element a^D is the Drazin inverse of a, if the following hold:

$$aa^D = a^D a,\ a^D a a^D = a^D,\ a^{n+1}a^D = a^n \text{ for some } n \in \mathbb{N}_0.$$

The Drazin inverse a^D of a is unique in the case when it exists. The smallest n in previous definition is the Drazin index of a, and it is denoted by $\operatorname{ind}(a)$. The set of all Drazin invertible elements in $\mathcal{A}$ is denoted by $\mathcal{A}^D$. Thus, $\mathcal{A}^D \subset \mathcal{A}^d$ and if $a \in \mathcal{A}^D$ then $a^D = a^d$. Obviously, $\operatorname{ind}(a) = 0$ if and only if $a \in \mathcal{A}^{-1}$ and in this case $a^{-1} = a^D$.

If $a \in \mathcal{A}$ and $\operatorname{ind}(a) \leq 1$, then the Drazin inverse of a is known as the group inverse of a, denoted by $a^\#$. In this case $a^\#$ is the unique element in $\mathcal{A}$ that satisfies conditions:

$$aa^\# a = a, \quad a^\# a a^\# = a^\#, \quad aa^\# = a^\# a.$$

The set of all group invertible elements in $\mathcal{A}$ is denoted by $\mathcal{A}^\#$.

If $a \in \mathcal{A}^d$, then $\operatorname{ind}(a) \leq 1$ if and only if $aa^\pi = 0$. Also, $\mathcal{A}^{qNil} \cap \mathcal{A}^\# = \{0\}$.

Further results about the Drazin and group inverse, as well as spectral idempotents, can be found in [4–7, 10].

Recall that the mapping $x \mapsto \sigma(x)$ is upper semi-continuous in every element $a \in \mathcal{A}$, i.e. for every $a \in \mathcal{A}$ and all $\epsilon > 0$ there exists $\delta > 0$, such that for all $x \in \mathcal{A}$ if $\|x - a\| < \delta$ then $\sigma(x) \subset \sigma(a) + D(0;\epsilon)$. Here $D(0;\epsilon) = \{z \in \mathbb{C} : |z| < \epsilon\}$. Moreover, if a commutes with every $x \in \mathcal{A}$ then the mapping $x \mapsto \sigma(x)$ is continuous in a, i.e. for all $\epsilon > 0$ there exists $\delta > 0$, such that for all $x \in \mathcal{A}$ if $\|x-a\| < \delta$, then $\sigma(x) \subset \sigma(a) + D(0,\epsilon)$ and $\sigma(a) \subset \sigma(x) + D(0,\epsilon)$.

The paper is organized as follows. Section 2 contains mostly known results. Sometimes we offer a different proof, which is simpler than the original one. In Section 3, we consider certain subsequences of the basic power sequence. We give necessary and sufficient conditions such that $\lim\limits_{n\to\infty} a^{qn} = d$ exists, where q is an integer $q \geq 2$, and give the explicit form of d. The final result of this limit has the same form as the one obtained by Chen and Hartwig. Naturally, we use a different method to prove it. In this section we also consider the existence and the form of the limit $\lim\limits_{n\to\infty} a^{q^n}$ for the same q. The final form of this limit is not established by Chen and Hartwig, so this is its first appearance in any structure that allows the Drazin inverse. Again, a different approach is used in order to prove the main result of this paper.

2. Auxiliary results

In this section we present auxiliary results. We start with elementary complex limit results, which are originally proved by Chen and Hartwig.

Lemma 2.1. ([3], pages 211-212) *Let $z \in \mathbb{C}$, $|z| = 1$, and $q \in \mathbb{N}$, $q \geq 2$. Then $\lim\limits_{n\to\infty} z^{q^n} = 1$ if and only if $z^{q^K} = 1$ for some $K \in \mathbb{N}$.*

Proof. We suggest the original proof of Chen and Hartwig. □

Lemma 2.2. ([3], pages 212-213) *Let $z \in \mathbb{C}$, $|z| = 1$, and $q \in \mathbb{N}$, $q \geq 2$. Then $\lim_{n\to\infty} z^{q^n}$ exists if and only if $z^{(q-1)q^K} = 1$ for some $K \in \mathbb{N}$.*

Proof. We offer a more direct proof than the one in [3]. Let $\lim_{n\to\infty} z^{q^n} = w$, so we get $|w| = 1$. Then $w^q = \left(\lim_{n\to\infty} z^{q^n}\right)^q = \lim_{n\to\infty} z^{q^{n+1}} = w$, implying that $w^q = w$. Thus, $w^{q-1} = 1$. We conclude that $\lim_{n\to\infty} z^{(q-1)q^n} = 1$. From Lemma 2.1 we get that $z^{(q-1)q^K} = 1$ for some $K \in \mathbb{N}$. □

The following characterization of the generalized Drazin invertibility is proved by Koliha.

Theorem 2.3. ([7]) *Let $a \in \mathcal{A}$. Then the following statements are equivalent:*
(1) $0 \notin \operatorname{acc} \sigma(a)$.
(2) *There exists some $p \in \mathcal{A}^{\bullet} \cap \operatorname{comm}(a)$ such that $ap \in \mathcal{A}^{qNil}$ and $a + p \in \mathcal{A}^{-1}$.*
Moreover, if (1) *or* (2) *holds, then $p = a^{\pi}$.*

We prove the following small observation, which is actually the justification of notations.

Lemma 2.4. *Let $a \in \mathcal{A}$ and $\lambda \notin \operatorname{acc} \sigma(a)$. Then $a^{\pi,\lambda} = (a - \lambda)^{\pi}$.*

Proof. Since $\lambda \notin \operatorname{acc} \sigma(a)$, then $0 \notin \operatorname{acc} \sigma(a - \lambda)$. There exists $r_0 > 0$ such that if $r \in (0, r_0)$ then the graph γ^* of the circle $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$, surrounds 0 and it does not surround any other point from $\sigma(a - \lambda) \setminus \{0\}$.

Let $\Gamma(t) = \lambda + re^{it}$, $t \in [0, 2\pi]$. Then Γ^* surrounds λ and it does not surround any other point from $\sigma(a) \setminus \{\lambda\}$. We have

$$\begin{aligned} a^{\pi,\lambda} &= \frac{1}{2\pi i} \int_{\Gamma} (\mu - a)^{-1} d\mu = \frac{1}{2\pi i} \int_0^{2\pi} \left(\lambda + re^{it} - a\right)^{-1} re^{it} i\, dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \left(re^{it} - (a - \lambda)\right)^{-1} re^{it} i\, dt = \frac{1}{2\pi i} \int_{\gamma} (\mu - (a - \lambda))^{-1} d\mu \\ &= (a - \lambda)^{\pi} \end{aligned}$$

□

Remark 2.5. *Note that, if $\lambda \notin \operatorname{acc} \sigma(a)$, then $(a - \lambda)^{\pi} = (\lambda - a)^{\pi}$. Indeed, since $(-x)^d = (-1)x^d$, then $(a - \lambda)^{\pi} = 1 - (a - \lambda)(a - \lambda)^d = 1 - (\lambda - a)(\lambda - a)^d = (\lambda - a)^{\pi}$.*

If $a \in \mathcal{A}$ and $\lambda \in \operatorname{iso} \sigma(a)$, $\mathcal{A}_1 = a^{\pi,\lambda} \mathcal{A} a^{\pi,\lambda}$ and $\mathcal{A}_2 = (1 - a^{\pi,\lambda}) \mathcal{A} (1 - a^{\pi,\lambda})$, then $\sigma^{\mathcal{A}_1}(aa^{\pi,\lambda}) = \{\lambda\}$ and $\sigma^{\mathcal{A}_2}\left(a(1 - a^{\pi,\lambda})\right) = \sigma(a) \setminus \{\lambda\}$.

We prove the following interesting result, which is obviously a local generalization of the Gelfand-Mazur theorem.

Lemma 2.6. *Let $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, $\sigma(a) = \{\lambda\}$ and $\operatorname{ind}(\lambda - a) \leq 1$. Then $a = \lambda 1$.*

Proof. Since $\sigma(a) = \{\lambda\}$, we get $a - \lambda \in \mathcal{A}^{qNil} \cap \mathcal{A}^{\#}$. By the uniqueness of the generalized Drazin inverse, we have $(a - \lambda)^{\#} = 0$, so $a - \lambda = 0$. □

The following result is classical and easy to verify.

Theorem 2.7. *If $a \in \mathcal{A}$, then the following statements are equivalent:*
(1) $\lim_{n\to\infty} a^n = 0$.
(2) $\sum_{n=1}^{\infty} a^n$ *is norm-convergent in $\mathcal{A}$.*
(3) $r(a) < 1$.

We are interested in a more general situation.

The element $a \in \mathcal{A}$ is convergent, if $\lim_{n\to\infty} a^n$ exists [9]. The set of all convergent elements in $\mathcal{A}$ is denoted by $\mathcal{A}^{con}$.

The following result is proved by Koliha. For the completeness we give a more direct proof here.

Theorem 2.8. ([10], [12]) *If $a \in \mathcal{A}$, then the following statements are equivalent:*
(1) $a \in \mathcal{A}^{con}$.
(2) $\sigma(a) \setminus \{1\} \subset D(0;1)$ *and* $\mathrm{ind}(1-a) \leq 1$.
Moreover, if (1) *or* (2) *holds, then* $\lim_{n\to\infty} a^n = (1-a)^\pi$.

Proof. (1) $\Longrightarrow$ (2): Let $p = \lim_{n\to\infty} a^n$. We immediately get $p = p^2$, $ap = pa = p$ and $(1-a)p = 0$.

Let $\lambda \in \sigma(a)$. From the continuity of the spectrum (in this commutative case) and from $\sigma(p) \subset \{0,1\}$, we conclude that $\lim_{n\to\infty} \lambda^n = 0$ or $\lim_{n\to\infty} \lambda^n = 1$. If $\lim_{n\to\infty} \lambda^n = 0$, then $|\lambda| < 1$. If $\lim_{n\to\infty} \lambda^n = 1$, then $|\lambda| = 1$ and consequently $\lambda = 1$. Thus, $\sigma(a) \setminus \{1\} \subset D(0;1)$.

Notice that for every $n \in \mathbb{N}$ we have $(a-p)^n = a^n - p$. Thus, $\lim_{n\to\infty}(a-p)^n = 0 = \lim_{n\to\infty}(p-a)^n$. From Theorem 2.7 we conclude that $\sigma(p-a) \subset D(0;1)$. Then $0 \notin 1 + \sigma(p-a) = \sigma(1-a+p)$ and $1-a+p \in \mathcal{A}^{-1}$.

By Theorem 2.3 it follows that $\mathrm{ind}(1-a) \leq 1$ and $p = (1-a)^\pi$.

(2) $\Longrightarrow$ (1): Let $\mathcal{A}_1 = a^{\pi,1}\mathcal{A}a^{\pi,1}$ and $\mathcal{A}_2 = (1-a^{\pi,1})\mathcal{A}(1-a^{\pi,1})$. Then $\sigma^{\mathcal{A}_1}(aa^{\pi,1}) = \{1\}$ and $\sigma^{\mathcal{A}_2}\Big(a(1-a^{\pi,1})\Big) = \sigma(a) \setminus \{1\}$. From Lemma 2.6 we have that $aa^{\pi,1} = a^{\pi,1}$. From Theorem 2.7 we have that $\lim_{n\to\infty} a^n(1-a^{\pi,1}) = 0$. Thus, $a = a(1-a^{\pi,1}) + aa^{\pi,1} = a(1-a^{\pi,1}) + a^{\pi,1}$. It follows that $\lim_{n\to\infty} a^n = a^{\pi,1} = (1-a)^\pi$ from Lemma 2.4 and Remark 2.5. $\square$

3. Convergence of certain subsequences of the power sequence

In this section, we consider the convergence behavior of some particular subsequences of $(a^n)_n$. We will generalize Theorem 1 from [3]. Also, results from [9] and [10] will be extended.

First, we prove the following result.

Theorem 3.1. *Let $q \in \mathbb{N}$ and $q \geq 2$, $b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, such that $b^q = 1$ and $\lambda^q = 1$. Then*

$$p = \frac{1}{q\lambda^{q-1}}(\lambda^{q-1}1 + \lambda^{q-2}b + \cdots + \lambda b^{q-2} + b^{q-1}) \in \mathcal{A}^\bullet \cap \mathrm{comm}(b),$$

$(b - \lambda 1)p = 0$ *and* $b - \lambda 1 + p \in \mathcal{A}^{-1}$.

Proof. Let

$$p_1 = \lambda^{q-1}1 + \lambda^{q-2}b + \lambda^{q-3}b^2 + \cdots + \lambda^2 b^{q-3} + \lambda b^{q-2} + b^{q-1}.$$

Since $\lambda^q = 1$ and $b^q = 1$, we get

$$\begin{aligned} p_1^2 &= \lambda^{q-1}\left(\lambda^{q-1}1 + \lambda^{q-2}b + \lambda^{q-3}b^2 + \cdots + \lambda^2 b^{q-3} + \lambda b^{q-2} + b^{q-1}\right) \\ &+ \lambda^{q-1}\left(\lambda^{q-2}b + \lambda^{q-3}b^2 + \cdots + b^{q-1} + \lambda^{q-1}1\right) \\ &+ \lambda^{q-1}\left(\lambda^{q-3}b^2 + \lambda^{q-4}b^3 + \cdots + b^{q-1} + \lambda^{q-1}1 + \lambda^{q-2}b\right) \\ &\vdots \\ &+ \lambda^{q-1}\left(b^{q-1} + \lambda^{q-1}1 + \lambda^{q-2}b + \lambda^{q-3}b^2 + \cdots + \lambda^2 b^{q-3} + \lambda b^{q-2}\right) \\ &= q\lambda^{q-1}p_1. \end{aligned}$$

Hence, we conclude that $p = q^{-1}\lambda^{-q+1}p_1$ is an idempotent. Obviously, p commutes with b. Notice that

$$0 = b^q - 1 = (b - \lambda 1)(b - \lambda_1 1)\cdots(b - \lambda_{q-1}1) = (b - \lambda 1)p_1,$$

where $\sqrt[q]{1} = \{\lambda, \lambda_1, \ldots, \lambda_{q-1}\}$. Hence, $(b - \lambda 1)p = 0$. Notice that $b^q = 1$ implies $(\sigma(b))^q = \{1\}$ and $\sigma(b) \subset \{\lambda, \lambda_1, \ldots, \lambda_{q-1}\}$. Consider the polynomial

$$P(z) = z - \lambda + \frac{1}{q\lambda^{q-1}}(z - \lambda_1)\cdots(z - \lambda_{q-1}).$$

Then $P(b) = b - \lambda 1 + p$ and

$$\begin{aligned}\sigma(b - \lambda 1 + p) = P(\sigma(b)) &\subset \left\{P(\mu) : \mu \in \{\lambda, \lambda_1, \ldots, \lambda_{q-1}\}\right\} \\ &= \left\{\frac{1}{q\lambda^{q-1}}(\lambda - \lambda_1)\cdots(\lambda - \lambda_{q-1}), \lambda_1 - \lambda, \ldots, \lambda_{q-1} - \lambda\right\}.\end{aligned}$$

Since $\lambda \neq \lambda_1, \ldots, \lambda_{q-1}$, we conclude that $0 \notin \sigma(b - \lambda 1 + p)$, so $b - \lambda 1 + p$ is invertible. □

Now we formulate and prove the following result, extending Theorem 1 (a) in [3]. Our proof is essentially different from the proof in [3], where the Jordan form of a complex square matrix is used.

Theorem 3.2. *Let $b \in \mathcal{A}$, and let $q \in \mathbb{N}$, $q \geq 2$. The following statements are equivalent:*
(1) $\lim_{n\to\infty} b^{qn}$ *exists.*
(2) *If $\lambda \in \sigma(b)$ then: $|\lambda| < 1$, or $\lambda^q = 1$ and* $\mathrm{ind}(b - \lambda 1) \leq 1$.
If (1) or (2) holds, then $\lim_{n\to\infty} b^{qn} = 1 - (1 - b^q)(1 - b^q)^{\#}$.

Proof. (1) $\Longrightarrow$ (2): From Theorem 2.8 it follows that $\lim_{n\to\infty} b^{qn}$ exists if and only if for all $\mu \in \sigma(b^q)$ either $|\mu| < 1$, or $\mu = 1$ and $\mathrm{ind}(b^q - 1) \leq 1$. Since $\mu \in \sigma(b^q)$ if and only if there exists $\lambda \in \sigma(b)$ such that $\lambda^q = \mu$, we get that for all $\lambda \in \sigma(b)$ either $|\lambda| < 1$, or $\lambda^q = 1$ and $\mathrm{ind}(b^q - 1) \leq 1$.

Let $\lambda \in \sigma(b)$ and $\lambda^q = 1$. Since $1 \notin \mathrm{acc}\,\sigma(b^q)$ we get $\lambda \notin \mathrm{acc}\,\sigma(b)$. Let $e = (b^q - 1)^\pi$. Since $\mathrm{ind}(b^q - 1) \leq 1$, we get that $(b^q - 1)(1 - e)$ is invertible in the Banach algebra $(1 - e)\mathcal{A}(1 - e)$ and $(b^q - 1)e = 0$. Since b commutes with $b^q - 1$, we conclude that b commutes with e. Now we have that

$$(b^q - 1)(1 - e) = (b - \lambda 1)(b^{q-1} + \lambda b^{q-2} + \cdots + \lambda^{q-2}b + \lambda^{q-1}1)(1 - e)$$

is invertible in $(1 - e)\mathcal{A}(1 - e)$. From the commutativity of b and e we deduce that $(b - \lambda 1)(1 - e)$ is invertible in the algebra $(1 - e)\mathcal{A}(1 - e)$. On the other hand,

$$0 = (b^q - 1)e = (b - \lambda 1)(b^{q-1} + \lambda b^{q-2} + \cdots + \lambda^{q-2}b + \lambda^{q-1}1)e$$

and $b^q e = e$. According to Theorem 3.1 we know that

$$p = q^{-1}\lambda^{-q+1}(b^{q-1} + \lambda b^{q-2} + \cdots + \lambda^{q-2}b + \lambda^{q-1}1)e$$

is an idempotent in $e\mathcal{A}e$, p commutes with $(b - \lambda 1)e$, $(b - \lambda 1)pe = 0$ and $(b - \lambda 1 + p)e$ is invertible in $e\mathcal{A}e$. From Theorem 2.3 we conclude that p is the spectral idempotent of $(b - \lambda 1)e$ in $e\mathcal{A}e$ corresponding to $\{0\}$, and hence $\mathrm{ind}((b - \lambda 1)e) \leq 1$. We also know that $(b - \lambda 1)(1 - p)e$ is invertible in $(1 - p)e\mathcal{A}e(1 - p)$. Now, $pe = ep = p$ and we easily verify $(1 - p)(1 - e) = (1 - e)$. Since $(b - \lambda 1)(1 - e)$ is invertible in $(1 - e)\mathcal{A}(1 - e)$, there exists some $c \in (1 - e)\mathcal{A}(1 - e)$ such that

$$(b - \lambda 1)(1 - e)c(1 - e) = (1 - e)c(1 - e)(b - \lambda 1) = 1 - e.$$

Since $(b - \lambda 1)e(1 - p)$ is invertible in $(1 - p)e\mathcal{A}e(1 - p)$, there exists some $d \in (1 - p)e\mathcal{A}e(1 - p)$ such that

$$(b - \lambda 1)e(1 - p)de(1 - p) = e(1 - p)de(1 - p)(b - \lambda 1) = e(1 - p).$$

Now we compute

$$\begin{aligned}&\Big[c(1-e)+de(1-p)\Big]\Big[(b-\lambda 1)(1-e)+(b-\lambda 1)e(1-p)\Big]=\\&=\Big[c(1-e)+de(1-p)\Big]\Big[(b-\lambda 1)(1-e)(1-p)+(b-\lambda 1)e(1-p)\Big]=\\&=(1-e)+e(1-p)=1-p.\end{aligned}$$

We conclude that

$$(b-\lambda 1)(1-e)(1-p)+(b-\lambda 1)e(1-p)=(b-\lambda 1)(1-p)$$

is invertible in $(1-p)\mathcal{A}(1-p)$. Also, $(b-\lambda 1)p=0$ holds, and hence $(b-\lambda 1)(1-p)=b-\lambda 1$. There exists some $f\in(1-p)\mathcal{A}(1-p)$ such that

$$(b-\lambda 1)(1-p)f(1-p)=1-p=(1-p)f(1-p)(b-\lambda 1).$$

Now it is easy to verify that $(1-p)f(1-p)=(b-\lambda 1)^{\#}$, thus implying $\mathrm{ind}(b-\lambda 1)\le 1$.

(2) $\Longrightarrow$ (1): Now suppose that for all $\lambda\in\sigma(b)$ either $|\lambda|<1$, or $\lambda^q=1$ and $\mathrm{ind}(b-\lambda 1)=1$. We only need to consider the q-th roots of 1: $\sqrt[q]{1}=\{\lambda,\lambda_1,\dots,\lambda_{q-1}\}$, $\mathrm{ind}(b-\lambda 1)\le 1$ and $\mathrm{ind}(b-\lambda_i 1)\le 1$ for all $i=1,\dots,q-1$. It can easily be seen that

$$(b^q-1)^{\#}=(b-\lambda 1)^{\#}(b-\lambda_1 1)^{\#}\cdots(b-\lambda_{q-1}1)^{\#}.$$

Hence, $\mathrm{ind}(b^q-1)\le 1$. From Theorem 2.8 it follows that $\lim\limits_{n\to\infty} b^{qn}$ exists and

$$\lim_{n\to\infty} b^{qn}=1-(1-b^q)(1-b^q)^{\#}.$$

□

Now we will consider the convergence of the sequence $(a^{q^n})_n$.

Theorem 3.3. *Let $a\in\mathcal{A}$ and $q\in\mathbb{N}$, $q\ge 2$. The following statements are equivalent:*

(1) $\lim\limits_{n\to\infty} a^{q^n}=c$ *exists.*

(2) *If $\lambda\in\sigma(a)$ then: $|\lambda|<1$, or there exists $K\in\mathbb{N}_0$ such that $\lambda^{(q-1)q^K}=1$ and* $\mathrm{ind}(a-\lambda 1)\le 1$.

If (1) *or* (2) *holds, then* $c^q=c=\sum\limits_{j=1}^{(q-1)q^K} a^{\pi,\lambda_j}$, $(\lambda_j^{(q-1)q^K}=1)$, *and* $\mathrm{ind}(c)\le 1$.

Proof. (1) $\Longrightarrow$ (2): We compute

$$c^q=\left(\lim_{n\to\infty} a^{q^n}\right)^q=\lim_{n\to\infty} a^{q^{n+1}}=c,$$

so $c(1-c^{q-1})=0$ and $\left(c^{q-1}\right)^2=c^{q-1}$. From the Spectral mapping theorem applied to the polynomial $P(z)=z(z^{q-1}-1)$, we conclude that $\sigma(c)\subset\{0\}\cup\{\lambda_1,\dots,\lambda_{q-1}\}$, where $\sqrt[q-1]{1}=\{\lambda_1,\dots,\lambda_{q-1}\}$.

Consider the spectrum of the element $c+1-c^{q-1}$:

$$\begin{aligned}\sigma(c+1-c^{q-1})&=\{1+\mu-\mu^{q-1}:\mu\in\sigma(c)\}\\&\subset\{1+\mu-\mu^{q-1}:\mu\in\{0,\lambda_1,\dots,\lambda_{q-1}\}=\{1,\lambda_1,\dots,\lambda_{q-1}\}.\end{aligned}$$

Hence, $c+1-c^{q-1}\in\mathcal{A}^{-1}$. From Theorem 2.3 we conclude that $1-c^{q-1}=c^{\pi}$ and $\mathrm{ind}(c)\le 1$.

Since c commutes with a, from the continuity of the spectrum in this commutative case and from Lemma 2.2 we conclude that for all $\lambda\in\sigma(a)$ either $|\lambda|<1$, or $|\lambda|=1$ and $\lambda^{q^K(q-1)}=1$ for some non-negative integer K.

(2) $\Longrightarrow$ (1): Let $M = (q-1)q^K$ and $\sqrt[M]{1} = \{\lambda_1, \dots, \lambda_M\}$. We have $a^{\pi,\lambda_1}a^{\pi,\lambda_k} = 0$ if $j \neq k$. Take $p = \sum_{j=1}^{M} a^{\pi,\lambda_j}$ and obtain $p \in \mathcal{A}^\bullet$. We have

$$a = a(1-p) + \sum_{j=1}^{M} aa^{\pi,\lambda_j}$$

and consequently

$$a^{q^n} = a^{q^n}(1-p) + \sum_{j=1}^{M} a^{q^n}a^{\pi,\lambda_j}$$

for every $n \in \mathbb{N}$. Let $\mathcal{A}_0 = (1-p)\mathcal{A}(1-p)$ and $\mathcal{A}_j = a^{\pi,\lambda_j}\mathcal{A}a^{\pi,\lambda_j}$ for $j = 1, \dots, M$. We have $\sigma^{\mathcal{A}_0}\Big(a(1-p)\Big) \subset D(0;1)$, so $\lim_{n\to\infty} a^{q^n}(1-p) = 0$. Also, $\sigma^{\mathcal{A}_j}\left(aa^{\pi,\lambda_j}\right) = \{\lambda_j\}$ for every j. Since $\mathrm{ind}(\lambda_j - a) \leq 1$, we get $aa^{\pi,\lambda_1} = \lambda_j a^{\pi,\lambda_j}$. Thus, $\lim_{n\to\infty} a^{q^n}a^{\pi,\lambda_j} = \lim_{n\to\infty} \lambda_j^{q^n} a^{\pi,j} = a^{\pi,j}$. We get

$$\lim_{n\to\infty} a^{q^n} = \sum_{j=1}^{(q-1)q^K} a^{\pi,\lambda_j}.$$

□

Remark 3.4. *It is important to mention that in Theorem 3.3 the result*

$$\lim_{n\to\infty} a^{q^n} = \sum_{j=1}^{(q-1)q^K} a^{\pi,\lambda_j}$$

is not proved in [3]. It seems that the form of this limit appears in the present article for the first time.

Remark 3.5. *We mention closely related topics. If $\mathcal{A}$ is the Banach algebra of operators on a Banach (or Hilbert) space, stable operators are defined in the following way [9], [2], [1]. Let H^+ denote the right open half plane of $\mathbb{C}$. An operator A is stable, if $\sigma(A) \subset H^+$. Stable and convergent operators are related by the Cayley transform: an operator A with $-1 \notin \sigma(A)$ is stable, if and only if its Cayley transform $T = (I-A)(I+A)^{-1}$ is convergent [9]. Various generalizations and applications of stable and convergent operators can be found in [1], [2], [8], [9], [11], and references cited there.*

Remark 3.6. *In [3] convergence properties of $\left(a^{qn}\right)_n$ and $\left(a^{q^n}\right)_n$ are used in the investigation of the well-known hyper-power iterative method.*

Finally, we prove one more result.

An element $c \in \mathcal{A}$ is generalized quasinilpotent, if $\lambda + (1+\lambda)^n \neq 0$ for every $n \in \mathbb{N}$ and every $\lambda \in \sigma(c)$. The set of all generalized quasinilpotent elements is denoted by $\mathcal{A}^{GqNil}$ and obviously $\mathcal{A}^{qNil} \subset \mathcal{A}^{GqNil}$.

Theorem 3.7. *If $c \in \mathcal{A}^{GqNil}$ and $1 + c \in \mathcal{A}^{con}$, then $c = 0$.*

Proof. Let $p = \lim_{n\to\infty}(1+c)^n$. Then $p^2 = p$ and p commutes with c. Also, $cp = (1+c-1)\lim_{n\to\infty}(1+c)^n = \lim_{n\to\infty}[(1+c)^{n+1} - (1+c)^n] = 0$. Denote by $x_n = c + (1+c)^n$ and notice that $\lim_{n\to\infty} x_n = c + p$. Then $\sigma(x_n) = \{\lambda + (1+\lambda)^n : \lambda \in \sigma(c)\}$. Since every x_n commutes with $c + p$, from the continuity of the spectrum in this commutative case, we conclude that $0 \notin \sigma(c+p)$ and $c + p \in \mathcal{A}^{-1}$. By Theorem 2.3 it follows that $p = c^\pi$. Since $cc^\pi = 0$, we get that $\mathrm{ind}(c) = 1$ and $0 = c^d = c^\#$, implying $c = 0$. □

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