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Strongly lacunary convergence of order β of difference sequences of fractional order in neutrosophic normed spaces

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Abstract. In this paper, we introduce the concept of strongly lacunary convergence of order β of difference sequences of fractional order in the neutrosophic normed spaces. We investigate a few fundamental properties of this new concept.

1. Introduction

The concept neutrosophy implies impartial knowledge of thought and then neutral describes the basic difference between neutral, fuzzy, intuitive fuzzy set and logic. The neutrosophic set (NS) was investigated by Smarandache [23] who defined the degree of indeterminacy (i) as independent component. In [24], neutrosophic logic was firstly examined. It is a logic where each proposition is determined to have a degree of truth (T), falsity (F), and indeterminacy (I). A Neutrosophic set (NS) is determined as a set where every component of the universe has a degree of T, F and I. In IFSs the 'degree of non-belongingness' is not independent but it is dependent on the 'degree of belongingness'. FSs can be thought as a remarkable case of an IFS where the 'degree of non-belongingness' of an element is absolutely equal to '1- degree of belongingness'. Uncertainty is based on the belongingness degree in IFSs, whereas the uncertainty in NS is considered independently from T and F values. Since no any limitations among the degree of T, F, I, NSs are actually more general than IFS. Neutrosophic soft linear spaces (NSLSs) were considered by Bera and Mahapatra [6]. Subsequently, in [7], the concept neutrosophic soft normed linear (NSNLS) was defined and the features of (NSNLS) were examined.

Kirişçi and Şimşek [12] defined new concept known as neutrosophic metric space (NMS) with continuous t-norms and continuous t-conorms. Some notable features of NMS have been examined. Neutrosophic normed space (NNS) and statistical convergence in NNS has been investigated by Kirişçi and Şimşek [13]. Neutrosophic set and neutrosophic logic has used by applied sciences and theoretical science such as decision making, robotics, summability theory.

In [14], lacunary statistical convergence of sequences in NNS was examined. Also, lacunary statistically Cauchy sequence in NNS was given and lacunary statistically completeness in connection with a neutrosophic normed space was presented. Kişi [15] defined lacunary ideal convergence and gave various results

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about lacunary ideal convergence in [15] and [16]. Some works related to this concept can be found [17],[18] and [19].

Definition 1.1. ([20]) *Let* $* : [0,1] \times [0,1] \to [0,1]$ *be an operation. When* $*$ *satisfies following situations, it is called continuous TN (Triangular norms (t-norms)). Take* $p, q, r, s \in [0,1]$

(i) $p * 1 = p$,
(ii) *If* $p \leq r$ *and* $q \leq s$, *then* $p * q \leq r * s$,
(iii) $*$ *is continuous,*
(iv) $*$ *associative and commutative.*

Definition 1.2. ([20]) *Let* $\diamond : [0,1] \times [0,1] \to [0,1]$ *be an operation. When* $\diamond$ *satisfies following situations, it is said to be continuous TC (Triangular conorms (t-conorms)).*

(i) $p \diamond 0 = p$,
(ii) *If* $p \leq r$ *and* $q \leq s$, *then* $p \diamond q \leq r \diamond s$,
(iii) $\diamond$ *is continuous,*
(iv) $\diamond$ *associative and commutative.*

Definition 1.3. ([13]) *Let F be a vector space,* $N = \{\langle u, G(u), B(u), Y(u)\rangle : u \in F\}$ *be a normed space* (NS) *such that* $N : F \times \mathbb{R}^+ \to [0,1]$. *While following conditions hold,* $V = (F, N, *, \diamond)$ *is called to be NNS. For each* $u, v \in F$ *and* $\lambda, \mu > 0$ *and for all* $\sigma \neq 0$,

(i) $0 \leq G(u,\lambda) \leq 1, 0 \leq B(u,\lambda) \leq 1, 0 \leq Y(u,\lambda) \leq 1, \forall \lambda \in \mathbb{R}^+$
(ii) $G(u,\lambda) + B(u,\lambda) + Y(u,\lambda) \leq 3, \forall \lambda \in \mathbb{R}^+$
(iii) $G(u,\lambda) = 1$ (*for* $\lambda > 0$) *iff* $u = 0$,
(iv) $G(\sigma u, \lambda) = G\left(u, \frac{\lambda}{|\sigma|}\right)$,
(v) $G(u,\mu) * G(v,\lambda) \leq G(u+v, \mu+\lambda)$
(vi) $G(u,.)$ *is non-decreasing continuous function,*
(vii) $\lim_{\lambda\to\infty} G(u,\lambda) = 1$,
($viii$) $B(u,\lambda) = 0$ (*for* $\lambda > 0$) *iff* $u = 0$,
(ix) $B(\sigma u, \lambda) = B\left(u, \frac{\lambda}{|\sigma|}\right)$,
(x) $B(u,\mu) \diamond B(v,\lambda) \geq B(u+v, \mu+\lambda)$
(xi) $B(u,.)$ *is non-increasing continuous function,*
(xii) $\lim_{\lambda\to\infty} B(u,\lambda) = 0$,
($xiii$) $Y(u,\lambda) = 0$ (*for* $\lambda > 0$) *iff* $u = 0$,
(xiv) $Y(\sigma u, \lambda) = Y\left(u, \frac{\lambda}{|\sigma|}\right)$,
(xv) $Y(u,\mu) \diamond Y(v,\lambda) \geq Y(u+v, \mu+\lambda)$
(xvi) $Y(u,.)$ *is non-increasing continuous function,*
($xvii$) $\lim_{\lambda\to\infty} Y(u,\lambda) = 0$,
($xviii$) *If* $\lambda \leq 0$, *then* $G(u,\lambda) = 0, B(u,\lambda) = 1$ *and* $Y(u,\lambda) = 1$.

Then $N = (G, B, Y)$ *is called Neutrosophic norm* (NN).

Definition 1.4. ([13]) *Let V be an NNS, the sequence* (x_k) *in* V, $\varepsilon \in (0,1)$ *and* $\lambda > 0$. *Then, the sequence* (x_k) *is converges to* ζ *iff there is* $N \in \mathbb{N}$ *such that* $G(x_k - \zeta, \lambda) > 1 - \varepsilon, B(x_k - \zeta, \lambda) < \varepsilon, Y(x_k - \zeta, \lambda) < \varepsilon$. *That is,* $\lim_{k\to\infty} G(x_k - \zeta, \lambda) = 1, \lim_{k\to\infty} B(x_k - \zeta, \lambda) = 0$ *and* $\lim_{k\to\infty} Y(x_k - \zeta, \lambda) = 0$ *as* $\lambda > 0$. *In this case, the sequence* (x_k) *is named a convergent sequence in V. The convergent in NNS is indicated by* $N - \lim x_k = \zeta$.

Definition 1.5. ([13]) *Let V be an NNS. For* $\lambda > 0$, $w \in F$ *and* $\varepsilon \in (0,1)$,

$$OB(w,\varepsilon,\lambda) = \{u \in F : G(w-u,\lambda) > 1-\varepsilon, B(w-u,\lambda) < \varepsilon, Y(w-u,\lambda) < \varepsilon\}$$

is called open ball with center w, *radius* ε.

Definition 1.6. ([13]) *The set* $A \subset F$ *is called neutrosophic-bounded* (NB) *in NNS V, if there exist* $\lambda > 0$, *and* $\varepsilon \in (0,1)$ *such that* $G(u,\lambda) > 1-\varepsilon$, $B(u,\lambda) < \varepsilon$ *and* $Y(u,\lambda) < \varepsilon$ *for each* $u \in A$.

Difference sequence spaces was defined by Kızmaz [11] and the concept was generalized by Et et al. ([8],[9]) as follows:

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one sequence $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m}, \tag{1}$$

$$y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. After then some properties of difference sequence spaces have been studied in ([1],[2],[9],[10],[22],[29]).

For a proper fraction α, we define a fractional difference operator $\Delta^\alpha : w \to w$ defined by

$$\Delta^\alpha(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}. \tag{2}$$

In particular, we have $\Delta^{\frac{1}{2}} x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \frac{21}{1024}x_{k+6} \cdots$

$\Delta^{-\frac{1}{2}} x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} + \frac{231}{1024}x_{k+6} \cdots$

$\Delta^{\frac{1}{3}} x_k = x_k - \frac{1}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{5}{81}x_{k+3} - \frac{10}{243}x_{k+4} - \frac{22}{729}x_{k+5} - \frac{154}{6561}x_{k+6} \cdots$

$\Delta^{\frac{2}{3}} x_k = x_k - \frac{2}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{4}{81}x_{k+3} - \frac{7}{243}x_{k+4} - \frac{14}{729}x_{k+5} - \frac{91}{6561}x_{k+6} \cdots$

By $\Gamma(r)$, we denote the Gamma function of a real number r and $r \notin \{0, -1, -2, -3, ...\}$. By the definition, it can be expressed as an improper integral as:

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt.$$

From the definition, it is observed that:

(i) For any natural number n, $\Gamma(n+1) = n!$,

(ii) For any real number n and $n \notin \{0, -1, -2, -3, ...\}$, $\Gamma(n+1) = n\Gamma(n)$,

(iii) For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!, \Gamma(4) = 3!, ...$

Without loss of generality, we assume throughout that the series defined in (2) is convergent. Moreover, if α is a positive integer, then the infinite sum defined in (2) reduces to a finite sum i.e., $\sum_{i=0}^{\alpha} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}$. In fact, this operator is generalized the difference operator introduced by Et and Çolak [8].

Recently, using fractional operator Δ^α (fractional order of α) Baliarsingh et al. ([4],[5],[21]) defined the sequence space $\Delta^\alpha(X)$ such as:

$$\Delta^\alpha(X) = \{x = (x_k) : (\Delta^\alpha x_k) \in X\},$$

where X is any sequence space.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([3],[25],[26],[27],[28]).

Fractional order difference sequence space has been an active field of research during the recent times. Many authors have introduced the difference sequence spaces of fractional order in different spaces. The motivation of the present paper is to define strongly lacunary convergence of order β of difference sequences of fractional order in NNS. The most important difference of our study while studying the lacunary convergence we used the T, F and I functions. The results obtained here are more general than the corresponding results for normed spaces.

2. Main Results

Definition 2.1. *Take an NNS V. Let θ be a lacunary sequence, $0 < \beta \leq 1$ and α be a proper fraction. The sequence $x = (x_k)$ is named to be $\Delta^{\alpha}-$ strongly lacunary convergent to $\zeta \in F$ of order β with regards to NN ($LC - NN$), if for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that*

$$\frac{1}{h_r^{\beta}} \sum_{k \in I_r} G(\Delta^{\alpha} x_k - \zeta, \lambda) > 1 - \varepsilon \text{ and } \frac{1}{h_r^{\beta}} \sum_{k \in I_r} B(\Delta^{\alpha} x_k - \zeta, \lambda) < \varepsilon, \frac{1}{h_r^{\beta}} \sum_{k \in I_r} Y(\Delta^{\alpha} x_k - \zeta, \lambda) < \varepsilon$$

for all $r \geq r_0$.

We indicate $(G, B, Y)_{\theta}^{\beta} - \lim \Delta^{\alpha} x = \zeta$. In case of $\theta = (2^r)$, $(G, B, Y)^{\beta} - \lim \Delta^{\alpha} x = \zeta$ is obtained.

Theorem 2.2. *Let V be an NNS. If x is $\Delta^{\alpha}-$ strongly lacunary convergent of order β with regards to NN, then $(G, B, Y)_{\theta}^{\beta} - \lim \Delta^{\alpha} x = \zeta$ is unique.*

Proof. Suppose that $(G, B, Y)_{\theta}^{\beta} - \lim \Delta^{\alpha} x = \zeta_1$, $(G, B, Y)_{\theta}^{\beta} - \lim \Delta^{\alpha} x = \zeta_2$ and $\zeta_1 \neq \zeta_2$. Given $\varepsilon > 0$, select $\rho \in (0,1)$ such that $(1 - \rho) * (1 - \rho) > 1 - \varepsilon$ and $\rho \diamond \rho < \varepsilon$. For each $\lambda > 0$, there is $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_r^{\beta}} \sum_{k \in I_r} G(\Delta^{\alpha} x_k - \zeta_1, \lambda) > 1 - \rho \text{ and } \frac{1}{h_r^{\beta}} \sum_{k \in I_r} B(\Delta^{\alpha} x_k - \zeta_1, \lambda) < \rho, \frac{1}{h_r^{\beta}} \sum_{k \in I_r} Y(\Delta^{\alpha} x_k - \zeta_1, \lambda) < \rho$$

for all $r \geq r_1$. Also, there is $r_2 \in \mathbb{N}$ such that

$$\frac{1}{h_r^{\beta}} \sum_{k \in I_r} G(\Delta^{\alpha} x_k - \zeta_2, \lambda) > 1 - \rho \text{ and } \frac{1}{h_r^{\beta}} \sum_{k \in I_r} B(\Delta^{\alpha} x_k - \zeta_2, \lambda) < \rho, \frac{1}{h_r^{\beta}} \sum_{k \in I_r} Y(\Delta^{\alpha} x_k - \zeta_2, \lambda) < \rho$$

for all $r \geq r_2$. Assume that $r_0 = \max \{r_1, r_2\}$. Then, for $r \geq r_0$, we can find a positive integer $m \in \mathbb{N}$ such that

$$G(\zeta_1 - \zeta_2, \lambda) \geq G\left(\Delta^{\alpha} x_m - \zeta_1, \frac{\lambda}{2}\right) * G\left(\Delta^{\alpha} x_m - \zeta_2, \frac{\lambda}{2}\right) > (1 - \rho) * (1 - \rho) > 1 - \varepsilon,$$

$$B(\zeta_1 - \zeta_2, \lambda) \leq B\left(\Delta^{\alpha} x_m - \zeta_1, \frac{\lambda}{2}\right) \diamond B\left(\Delta^{\alpha} x_m - \zeta_2, \frac{\lambda}{2}\right) < \rho \diamond \rho < \varepsilon,$$

and

$$Y(\zeta_1 - \zeta_2, \lambda) \leq Y\left(\Delta^{\alpha} x_m - \zeta_1, \frac{\lambda}{2}\right) \diamond Y\left(\Delta^{\alpha} x_m - \zeta_2, \frac{\lambda}{2}\right) < \rho \diamond \rho < \varepsilon.$$

Since $\varepsilon > 0$ is abritrary, we get $G(\zeta_1 - \zeta_2, \lambda) = 1$, $B(\zeta_1 - \zeta_2, \lambda) = 0$ and $Y(\zeta_1 - \zeta_2, \lambda) = 0$ for all $\lambda > 0$, which gives that $\zeta_1 = \zeta_2$. □

We give an example to denote the sequence $\Delta^{\alpha}-$ strongly lacunary convergence of order β in an *NNS*.

Example 2.3. *Let* $(F, \|.\|)$ *be a NNS. For all* $u, v, \alpha \in [0,1]$*, define* $u * v = uv$ *and* $u \diamond v = min\{u+v; 1\}$*. For all* $x \in F$ *and every* $\lambda > 0$*, we take* $G(\Delta^\alpha x, \lambda) = \frac{\lambda}{\lambda + \|\Delta^\alpha x\|}$, $B(\Delta^\alpha x, \lambda) = \frac{\|\Delta^\alpha x\|}{\lambda + \|\Delta^\alpha x\|}$ *and* $Y(\Delta^\alpha x, \lambda) = \frac{\|\Delta^\alpha x\|}{\lambda}$*. Then* V *is a NNS. We define a sequence* (x_k) *by*

$$\Delta^\alpha x_k = \begin{cases} 1, & if \quad k = t^2 (t \in \mathbb{N}) \\ 0, & otherwise \end{cases}.$$

Consider

$$A = \{k \in I_r : G(\Delta^\alpha x, \lambda) > 1 - \varepsilon \ \ and \ \ B(\Delta^\alpha x, \lambda) < \varepsilon, Y(\Delta^\alpha x, \lambda) < \varepsilon\}.$$

Then, for any $\lambda > 0$ *and for all* $\varepsilon \in (0,1)$*, the following set*

$$\begin{aligned} A &= \left\{k \in I_r : \frac{\lambda}{\lambda + \|\Delta^\alpha x_k\|} > 1 - \varepsilon, \ and \ \frac{\|\Delta^\alpha x_k\|}{\lambda + \|\Delta^\alpha x_k\|} < \varepsilon, \frac{\|\Delta^\alpha x_k\|}{\lambda} < \varepsilon\right\} \\ &= \left\{k \in I_r : \|\Delta^\alpha x_k\| \leq \frac{\lambda\varepsilon}{1-\varepsilon}, \ and \ \|\Delta^\alpha x_k\| < \lambda\varepsilon\right\} \\ &\subset \{k \in I_r : \|\Delta^\alpha x_k\| = 1\} = \left\{k \in I_r : k = t^2\right\} \end{aligned}$$

i.e.,

$$A_r(\varepsilon, \lambda) = \left\{r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k \in I_r} G(\Delta^\alpha x_k, \lambda) > 1 - \varepsilon \ and \ \frac{1}{h_r^\beta} \sum_{k \in I_r} B(\Delta^\alpha x_k, \lambda) < \varepsilon, \frac{1}{h_r^\beta} \sum_{k \in I_r} Y(\Delta^\alpha x_k, \lambda) < \varepsilon\right\}$$

will be a finite set.

Theorem 2.4. *Let* V *be an NNS. If* $(G, B, Y)_\theta^\beta - \lim \Delta^\alpha x = \zeta_1$ *and* $(G, B, Y)_\theta^\beta - \lim \Delta^\alpha y = \zeta_2$*, then* $(G, B, Y)_\theta^\beta - \lim \Delta^\alpha (x + y) = \zeta_1 + \zeta_2$ *and* $c \in F$*,* $(G, B, Y)_\theta^\beta - \lim \Delta^\alpha cx = c\zeta$*.*

Proof. For every $\lambda > 0$ and $\varepsilon \in (1,0)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r^\beta} \sum_{k \in I_r} G(\Delta^\alpha x_k - \zeta_1, \lambda) > 1 - \rho \text{ and } \frac{1}{h_r^\beta} \sum_{k \in I_r} B(\Delta^\alpha x_k - \zeta_1, \lambda) < \rho, \frac{1}{h_r^\beta} \sum_{k \in I_r} Y(\Delta^\alpha x_k - \zeta_1, \lambda) < \rho$$

for all $r \geq r_1$. Also, there is $r_2 \in \mathbb{N}$ such that

$$\frac{1}{h_r^\beta} \sum_{k \in I_r} G(\Delta^\alpha y_k - \zeta_2, \lambda) > 1 - \rho \text{ and } \frac{1}{h_r^\beta} \sum_{k \in I_r} B(\Delta^\alpha y_k - \zeta_2, \lambda) < \rho, \frac{1}{h_r^\beta} \sum_{k \in I_r} Y(\Delta^\alpha y_k - \zeta_2, \lambda) < \rho$$

for all $r \geq r_2$. Assume that $r_0 = \max\{r_1, r_2\}$. Now, for $r \geq r_0$ we get

$$\frac{1}{h_r^\beta} \sum_{k \in I_r} G(\Delta^\alpha (x_k + y_k) - (\zeta_1 + \zeta_2), \lambda) = \frac{1}{h_r^\beta} \sum_{k \in I_r} G(\Delta^\alpha x_k - \zeta_1 + \Delta^\alpha y_k - \zeta_2, \lambda)$$

$$\geqslant \frac{1}{h_r^\beta} \sum_{k \in I_r} G\left(\Delta^\alpha x_k - \zeta_1, \frac{\lambda}{2}\right) * G\left(\Delta^\alpha y_k - \zeta_2, \frac{\lambda}{2}\right)$$

$$> (1 - \rho) * (1 - \rho) > 1 - \varepsilon$$

and

$$\frac{1}{h_r^\beta} \sum_{k \in I_r} B(\Delta^\alpha (x_k + y_k) - (\zeta_1 + \zeta_2), \lambda) = \frac{1}{h_r^\beta} \sum_{k \in I_r} B(\Delta^\alpha x_k - \zeta_1 + \Delta^\alpha y_k - \zeta_2, \lambda)$$

$$\leqslant \frac{1}{h_r^\beta}\sum_{k\in I_r} B\left(\Delta^\alpha(x_k-\zeta_1),\frac{\lambda}{2}\right)\diamond B\left(\Delta^\alpha(y_k-\zeta_2),\frac{\lambda}{2}\right) < \rho\diamond\rho < \varepsilon.$$

Further,

$$\frac{1}{h_r^\beta}\sum_{k\in I_r} Y(\Delta^\alpha(x_k+y_k)-(\zeta_1+\zeta_2),\lambda) = \frac{1}{h_r^\beta}\sum_{k\in I_r} Y(\Delta^\alpha x_k-\zeta_1+\Delta^\alpha y_k-\zeta_2,\lambda)$$

$$\leqslant \frac{1}{h_r^\beta}\sum_{k\in I_r} Y\left(\Delta^\alpha(x_k-\zeta_1),\frac{\lambda}{2}\right)\diamond Y\left(\Delta^\alpha(y_k-\zeta_2),\frac{\lambda}{2}\right) < \rho\diamond\rho < \varepsilon.$$

Similarly we can show that $(G,B,Y)_\theta^\beta - \lim \Delta^\alpha cx = c\zeta$. □

Theorem 2.5. *If* $(G,B,Y)_\theta^\beta - \lim \Delta^\alpha x = \zeta$, *then there is a subsequence* $(\Delta^\alpha x_{\rho_k})$ *of* $\Delta^\alpha x$ *such that* $(G,B,Y)_\theta^\beta - \lim \Delta^\alpha x_{\rho_k} = \zeta$.

Proof. Take $(G,B,Y)_\theta^\beta - \lim \Delta^\alpha x = \zeta$. Then, for every $\lambda > 0$ and $\varepsilon \in (1,0)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r^\beta}\sum_{k\in I_r} G(\Delta^\alpha x_k-\zeta,\lambda) > 1-\varepsilon \text{ and } \frac{1}{h_r^\beta}\sum_{k\in I_r} B(\Delta^\alpha x_k-\zeta,\lambda) < \varepsilon, \frac{1}{h_r^\beta}\sum_{k\in I_r} Y(\Delta^\alpha x_k-\zeta,\lambda) < \varepsilon$$

for all $r \geq r_0$. Obviously, for each $r \geq r_0$, we choose $\rho_k \in I_r$ such that

$$G(\Delta^\alpha x_{\rho_k}-\zeta,\lambda) > \frac{1}{h_r^\beta}\sum_{k\in I_r} G(\Delta^\alpha x_k-\zeta,\lambda) > 1-\varepsilon$$

$$B(\Delta^\alpha x_{\rho_k}-\zeta,\lambda) < \frac{1}{h_r^\beta}\sum_{k\in I_r} B(\Delta^\alpha x_k-\zeta,\lambda) < \varepsilon$$

$$Y(\Delta^\alpha x_{\rho_k}-\zeta,\lambda) < \frac{1}{h_r^\beta}\sum_{k\in I_r} Y(\Delta^\alpha x_k-\zeta,\lambda) < \varepsilon.$$

It follows that $(G,B,Y)_\theta^\beta - \lim \Delta^\alpha x_{\rho_k} = \zeta$. □

Theorem 2.6. *Let* $0 < \alpha \leqslant \beta \leq 1$. *If* $(G,B,Y)_\theta^\beta - \lim \Delta^\alpha x = \zeta$, *then* $(G,B,Y)_\theta^\beta - \lim \Delta^\beta x = \zeta$.

Proof. Omitted. □

Theorem 2.7. *Let* $0 < \beta \leq 1$. *If* $\liminf_r q_r > 1$, *then* $(G,B,Y)^\beta \subset (G,B,Y)_\theta^\beta$.

Proof. Take $(G,B,Y)^\beta - \lim \Delta^\alpha x = \zeta$. Since $\frac{k_r^\beta}{h_r^\beta} > \frac{h_r^\beta}{h_r^\beta}$ for all $r \geq 1$, we can write

$$\begin{aligned}
\frac{1}{h_r^\beta}\sum_{k\in I_r} G(\Delta^\alpha x_k-\zeta,\lambda) &= \frac{1}{h_r^\beta}\sum_{k=1}^{k_r} G(\Delta^\alpha x_k-\zeta,\lambda) - \frac{1}{h_r^\beta}\sum_{k=1}^{k_{r-1}} G(\Delta^\alpha x_k-\zeta,\lambda)\\
&= \frac{k_r^\beta}{h_r^\beta}\left(\frac{1}{k_r^\beta}\sum_{k=1}^{k_r} G(\Delta^\alpha x_k-\zeta,\lambda)\right) - \frac{k_{r-1}^\beta}{h_r^\beta}\left(\frac{1}{k_{r-1}^\beta}\sum_{k=1}^{k_{r-1}} G(\Delta^\alpha x_k-\zeta,\lambda)\right)\\
&> \frac{k_r^\beta}{h_r^\beta}\left(\frac{1}{k_r^\beta}\sum_{k=1}^{k_r} G(\Delta^\alpha x_k-\zeta,\lambda)\right) > \left(\frac{1}{k_r^\beta}\sum_{k=1}^{k_r} G(\Delta^\alpha x_k-\zeta,\lambda)\right) > 1-\varepsilon.
\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r^\beta}{h_r^\beta} \leq \frac{(1+\delta)^\beta}{\delta^\beta} \text{ and } \frac{k_{r-1}^\beta}{h_r^\beta} \leq \frac{1}{\delta^\beta}.$$

From here, $\frac{1}{h_r^\beta} \sum_{k \in I_r} B(\Delta^\alpha x_k - \zeta, \lambda) < \varepsilon$ and $\frac{1}{h_r^\beta} \sum_{k \in I_r} Y(\Delta^\alpha x_k - \zeta, \lambda) < \varepsilon$ are obtained. Thus, $(G, B, Y)_\theta^\beta - \lim \Delta^\alpha x = \zeta$. □

Theorem 2.8. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, β and τ be fixed real numbers such that $0 < \beta \leq \tau \leq 1$. If*

$$\liminf_{r\to\infty} \frac{(h'_r)^\tau}{(h_r)^\beta} > 0 \text{ and } \lim_{r\to\infty} \frac{h'_r}{(h_r)^\tau} = 1 \tag{3}$$

holds and $A \subset F$ is neutrosophic-bounded (NB) in NNS V then $(G, B, Y)_\theta^\beta \subset (G, B, Y)_{\theta'}^\tau$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$, $h'_r = s_r - s_{r-1}$.

Proof. Let $\Delta^\alpha x \in (G, B, Y)_\theta^\beta$ and assume that (3) holds. Since $A \subset F$ is neutrosophic-bounded (NB) in $NNSV$, then there exists some $\lambda > 0$ such that $\frac{1}{h_r^\beta} \sum_{k \in I_r} G(\Delta^\alpha x_k - \zeta, \lambda) > 1 - \varepsilon$ and $\frac{1}{h_r^\beta} \sum_{k \in I_r} B(\Delta^\alpha x_k - \zeta, \lambda) < \varepsilon$, $\frac{1}{h_r^\beta} \sum_{k \in I_r} Y(\Delta^\alpha x_k - \zeta, \lambda) < \varepsilon$ for each $(\Delta^\alpha x_k - \zeta) \in A$. Now, since $I_r \subseteq J_r$ and $h_r \leq h'_r$ for all $r \in \mathbb{N}$, we may write

$$\begin{aligned}
\frac{1}{(h_r)^\beta} \sum_{k \in I_r} G(\Delta^\alpha x_k - \zeta, \lambda) &\leq \frac{1}{(h_r)^\beta} \sum_{k \in J_r} G(\Delta^\alpha x_k - \zeta, \lambda) \\
&= \frac{(h'_r)^\tau}{(h_r)^\beta} \frac{1}{(h'_r)^\tau} \sum_{k \in J_r} G(\Delta^\alpha x_k - \zeta, \lambda)
\end{aligned}$$

for all $r \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
\frac{1}{(h'_r)^\tau} \sum_{k \in J_r} B(\Delta^\alpha x_{p_k} - \zeta, \lambda) &= \frac{1}{(h'_r)^\tau} \sum_{k \in J_r - I_r} B(\Delta^\alpha x_k - \zeta, \lambda) + \frac{1}{(h'_r)^\tau} \sum_{k \in I_r} B(\Delta^\alpha x_k - \zeta, \lambda) \\
&\leq \frac{h'_r - h_r}{(h'_r)^\tau} \varepsilon + \frac{1}{(h'_r)^\tau} \sum_{k \in I_r} B(\Delta^\alpha x_k - \zeta, \lambda) \\
&\leq \frac{h'_r - (h_r)^\tau}{(h_r)^\tau} \varepsilon + \frac{1}{(h_r)^\beta} \sum_{k \in I_r} B(\Delta^\alpha x_k - \zeta, \lambda) \\
&\leq \left(\frac{h'_r}{(h_r)^\tau} - 1 \right) \varepsilon + \frac{1}{(h_r)^\beta} \sum_{k \in I_r} B(\Delta^\alpha x_k - \zeta, \lambda)
\end{aligned}$$

for every $r \in \mathbb{N}$. Therefore $\frac{1}{(h'_r)^\tau} \sum_{k \in J_r} G(\Delta^\alpha x_k - \zeta, \lambda) > 1 - \varepsilon$ and $\frac{1}{(h'_r)^\tau} \sum_{k \in J_r} B(\Delta^\alpha x_k - \zeta, \lambda) < \varepsilon$. It can be shown to be $\frac{1}{(h'_r)^\tau} \sum_{k \in J_r} Y(\Delta^\alpha x_k - \zeta, \lambda) < \varepsilon$ by similar operations. $(G, B, Y)_\theta^\beta \subset (G, B, Y)_{\theta'}^\tau$ is obtained as the result. □

Thus in the light of Theorem 2.8, we have the following result:

Corollary 2.9. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.*

If (1) *holds and NB then,*
(i) $(G, B, Y)_\theta^\beta \subset (G, B, Y)_{\theta'}$ *for* $0 < \beta \leq 1$,
(ii) $(G, B, Y)_\theta \subset (G, B, Y)_{\theta'}$.

Definition 2.10. *Take an NNS V. A sequence $\Delta^{\alpha}x = (\Delta^{\alpha}x_k)$ is named to be $\Delta^{\alpha}-$ strongly lacunary Cauchy of order β with regards to the NN N ($LCa - NN$) if, for every $\varepsilon \in (1,0)$ and $\lambda > 0$, there are $r_0, p \in \mathbb{N}$ satisfying*

$$\frac{1}{h_r^{\beta}}\sum_{k\in I_r} G(\Delta^{\alpha}x_k - \Delta^{\alpha}x_p, \lambda) > 1-\varepsilon \text{ and } \frac{1}{h_r^{\beta}}\sum_{k\in I_r} B(\Delta^{\alpha}x_k - \Delta^{\alpha}x_p, \lambda) < \varepsilon,$$

$$\frac{1}{h_r^{\beta}}\sum_{k\in I_r} Y(\Delta^{\alpha}x_k - \Delta^{\alpha}x_p, \lambda) < \varepsilon$$

for all $r \geq r_0$.

Theorem 2.11. *If a sequence $\Delta^{\alpha}x = (\Delta^{\alpha}x_k)$ in NNS is $\Delta^{\alpha}-$ strongly lacunary convergent of order β with regards to NN N , then it is strongly Cauchy of order β with regards to NN N .*

Proof. Let $(G,B,Y)_{\theta}^{\beta} - \lim \Delta^{\alpha}x = \zeta$. Select $\varepsilon > 0$. Then, for a given $\rho \in (0,1), (1-\rho) * (1-\rho) > 1-\varepsilon$ and $\rho\diamond\rho < \varepsilon$. Then, we have

$$\frac{1}{h_r^{\beta}}\sum_{k\in I_r} G\left(\Delta^{\alpha}x_k - \zeta, \frac{\lambda}{2}\right) > 1-\rho \text{ and } \frac{1}{h_r^{\beta}}\sum_{k\in I_r} B\left(\Delta^{\alpha}x_k - \zeta, \frac{\lambda}{2}\right) < \rho,$$

$$\frac{1}{h_r^{\beta}}\sum_{k\in I_r} Y\left(\Delta^{\alpha}x_k - \zeta, \frac{\lambda}{2}\right) < \rho.$$

We have to show that

$$\frac{1}{h_r^{\beta}}\sum_{k\in I_r} G(\Delta^{\alpha}x_k - \Delta^{\alpha}x_m, \lambda) > 1-\varepsilon \text{ and } \frac{1}{h_r^{\beta}}\sum_{k\in I_r} B(\Delta^{\alpha}x_k - \Delta^{\alpha}x_m, \lambda) < \varepsilon,$$

$$\frac{1}{h_r^{\beta}}\sum_{k\in I_r} Y(\Delta\alpha x_k - \Delta^{\alpha}x_m, \lambda) < \varepsilon.$$

We have three possible cases.

Case (i) we get for $\lambda > 0$

$$G(\Delta^{\alpha}x_k - \Delta^{\alpha}x_m, \lambda) \geq G\left(\Delta^{\alpha}x_k - \zeta, \frac{\lambda}{2}\right) * G\left(\Delta^{\alpha}x_m - \zeta, \frac{\lambda}{2}\right) > (1-\rho) * (1-\rho) > 1-\varepsilon.$$

Case (ii) we obtain

$$B(\Delta^{\alpha}x_k - \Delta^{\alpha}x_m, \lambda) \leq B\left(\Delta^{\alpha}x_k - \zeta, \frac{\lambda}{2}\right) \diamond B\left(\Delta^{\alpha}x_m - \zeta, \frac{\lambda}{2}\right) < \rho\diamond\rho < \varepsilon.$$

Case (iii) we have

$$Y(\Delta^{\alpha}x_k - \Delta^{\alpha}x_m, \lambda) \leq Y\left(\Delta^{\alpha}x_k - \zeta, \frac{\lambda}{2}\right) \diamond Y\left(\Delta^{\alpha}x_m - \zeta, \frac{\lambda}{2}\right) < \rho\diamond\rho < \varepsilon.$$

This shows that $(\Delta^{\alpha}x_k)$ is strongly Cauchy of order β with regards to *NN N* . □

Conclusion

Neutrosophic normed space has been an active field of research during the recent times. In the current studying, using the concept of lacunary sequence, we have introduced the new notation of strongly lacunary convergence of order β of difference sequences of fractional order in *NNS* and have given the an example for the new definition. Further investigate the uniqueness of the limit and the linearity of this new concept. Then, important coverage relations are given for the concept of $\Delta^{\alpha}-$ strongly lacunary convergent of order β. Finally strongly lacunary Cauchy of order β with regards to the *NN* have been introduced. We expect that the introduced notions and the results might be a reference for further studies in this field. For further studies one can investigate and generalize this results using multiplier sequences, sequence of modulus functions, etc.

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