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Existence and nonexistence results for fifth-order multi-point boundary value problems involving integral boundary condition

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Abstract. In this paper, by using the classical compression-expansion fixed point theorem of Krasnoselskii, we study the existence and nonexistence of monotone and convex positive solutions for a nonlinear fifth-order differential equation with multi-point and integral boundary condition. We establish some sufficient conditions for the existence of at least one or two monotone and convex positive solutions. Furthermore, the nonexistence results of positive solution are also considered. As applications, two examples are presented to illustrate the validity of our main results.

1. Introduction

Fifth order boundary value problems arise in a variety of different areas of mathematics and physics, they occur in the mathematical modelling of viscoelastic flows. Many papers, used different numerical methods to solve fifth order boundary value problems. In particular, in [32], the author used adomian decomposition method to construct the numerical solution of fifth-order boundary value problems with two-point boundary conditions. In [13], the author dealt with the Sinc-Galerkin method to solve a fifth order boundary value problem with two-point boundary conditions. For a detailed exposition see, for example, ([3], [6], [9], [12], [14], [15], [31]).

Many works, used different techniques to study higher order boundary value problems can be found in ([1], [2], [4], [5], [8], [10], [17], [19], [20], [22], [23], [24], [27], [29], [30], [33]), and the references therein.

Odda [26] studied the nonlinear fifth-order boundary value problem consisting of the equation

$$u^{(5)}(t) = f(t, u(t)), \ t \in (0, 1),$$

and the boundary conditions

$$u(0) = u'(0) = u'''(0) = u^{(4)}(1) = 0, \ \alpha u'(1) + \beta u''(1) = 0,$$

where $\alpha, \beta \geq 0, \alpha + \beta > 0$. By the use of the nonlinear alternative theorem, the author obtained existence results of at least one positive solution.

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Panos et al. [28] have discussed the following five-point fifth-order boundary value problem

$$\begin{cases} u^{(5)}(t) = c(t)f(t, u''(t), u'''(t), u^{(4)}(t)), \ t \in [0,1], \\ \alpha u(\xi_1) - \beta u'(\xi_1) = 0, \ \gamma u(\xi_2) + \delta u'(\xi_2) = 0, \\ u''(0) = u'''(\eta) = u^{(4)}(1) = 0, \end{cases}$$

where $f \in C([0,1] \times [0,\infty) \times \mathbb{R} \times \mathbb{R}, [0,\infty))$, $c \in C((0,1), [0,\infty))$, $\eta \in \left(\frac{1}{2}, 1\right)$, $\alpha, \beta, \gamma, \delta, \xi_1, \xi_2 \geq 0$, with $0 \leq \xi_1 < \xi_2 \leq 1$, and $p := \alpha\delta + \beta\gamma + \alpha\gamma(\xi_2 - \xi_1) \neq 0$. They obtained the existence of positive solutions for above problem by Kneser's Theorem, Sperner's Lemma.

Liu et al. [25] established the existence of monotone and convex positive solutions for the fourth-order differential equation

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \ t \in [0,1],$$

with the multi-point boundary conditions

$$u'''(1) = 0, \ u''(1) = 0, \ u'(0) = 0, \ u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i),$$

or

$$u'''(1) = 0, \ u''(1) = 0, \ u'(1) = 0, \ u(0) = \sum_{i=1}^{m-2} \beta_i u(\eta_i),$$

where $f \in C([0,1] \times \mathbb{R}^4, [0,\infty))$, $\beta_i > 0$, $i = 1, 2, ..., m-2$, $\sum_{i=1}^{m-2} \beta_i > 1$ and $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$.

In [11], the authors, by using lower and upper solution method and schauder fixed point theorem, studied the existence of solution for the following fifth-order boundary value problem

$$\begin{cases} u^{(5)}(t) + f(t, u(t), u'''(t)) = 0, \ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \ u'''(1) = u^{(4)}(1) = 0, \end{cases}$$

where $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$.

Bekri and Benaicha [7], interested in the existence of solution for the following fifth-order three-point boundary value problem

$$u^{(5)}(t) + f(t, u(t)) = 0, \ t \in (0,1),$$

$$u(0) = 0, \ u'(0) = u''(0) = u'''(0) = 0, \ u(1) = \alpha u(\eta),$$

where $0 < \eta < 1, \alpha \in \mathbb{R}, \alpha\eta^4 \neq 1, f \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

Motivated by the works above, in this paper, we study the existence and nonexistence of monotone and convex positive solutions for the following nonlinear fifth-order boundary value problem with multi-point and integral boundary condition

$$u^{(5)}(t) + f(t, u(t), u'(t), u''(t)) = 0, \ t \in (0,1), \tag{1}$$

$$u'(0) = u''(0) = u''(1) = 0, \ u'''(0) = \alpha, \ u(0) = \beta \int_\nu^\xi u(s)ds + \sum_{i=1}^{n} \gamma_i u(\eta_i), \tag{2}$$

where

H1) $f \in C([0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty), [0,\infty))$.

H2) $\alpha > 0, \beta \geq 0, \gamma_i \geq 0$ and $0 \leq \nu < \eta_1 < ... < \eta_n < \xi \leq 1, 1 \leq i \leq n$.

By imposing upon nonlinearity f some additional assumptions (in particular the non negativity of f if u, u', u'' are non negative) and using the fixed point theorem of cone expansion and compression of norm type, we prove the existence and nonexistence of monotone and convex positive solutions for the problem (1)-(2). Two examples illustrating the main results are given.

2. Definitions and Lemmas

Definition 2.1. *Let X be a real Banach space. By a cone we mean a nonempty, closed and convex set $P \subset X$ satisfying the following conditions*

1/ $\lambda P \subset P$, $\forall \lambda \geq 0$

2/ $P \cap (-P) = \{0\}$.

Definition 2.2. *An operator $A : X \to X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Definition 2.3. *A function $u(t)$ is called positive solution of* (1)-(2) *if $u \in C([0,1], \mathbb{R})$ and $u(t) \geq 0$ for all $t \in [0,1]$.*

We first consider the linear equation

$$u^{(5)}(t) + h(t) = 0, \ t \in [0,1], \tag{3}$$

subject to the boundary conditions (2).

For convenience, we denote $\mu = 1 - \Big(\beta(\xi - \nu) + \sum_{i=1}^{n} \gamma_i\Big)$.
Next, we will present the following auxiliary results.

Lemma 2.4. *If $\mu \neq 0$, then for $h \in C([0,1], \mathbb{R})$, the problem* (3)-(2) *has a unique solution*

$$\begin{aligned} u(t) &= \int_0^1 \Big[G(t,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n} \gamma_i G(\eta_i, s)\Big] h(s)ds \\ &\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i \varphi(\eta_i), \ \forall t \in [0,1], \end{aligned}$$

where

$$G(t,s) = \frac{1}{4!}\begin{cases} t^4(1-s)^2 - (t-s)^4, & 0 \leq s \leq t \leq 1, \\ t^4(1-s)^2, & 0 \leq t \leq s \leq 1, \end{cases} \tag{4}$$

$$H(t,s) = \frac{1}{5!}\begin{cases} t^5(1-s)^2 - (t-s)^5, & 0 \leq s \leq t \leq 1, \\ t^5(1-s)^2, & 0 \leq t \leq s \leq 1, \end{cases} \tag{5}$$

$$\varphi(t) = \frac{\alpha}{6}t^3\Big(1 - \frac{t}{2}\Big),$$

$$\begin{aligned} \psi(\xi,\nu) &= \int_\nu^\xi \varphi(\tau)d\tau \\ &= \frac{\alpha}{12}\Big(\frac{1}{2}(\xi^4 - \nu^4) - \frac{1}{5}(\xi^5 - \nu^5)\Big). \end{aligned}$$

Proof. Rewriting (3) as $u^{(5)}(t) = -h(t)$ and integrating five times over the interval $[0,t]$ for $t \in [0,1]$, we obtain

$$u(t) = -\frac{1}{4!}\int_0^t (t-s)^4 h(s)ds + \frac{1}{4!}c_1 t^4 + \frac{1}{6}c_2 t^3 + c_3 t^2 + c_4 t + c_5, \tag{6}$$

where $c_i \in \mathbb{R},\ i \in \{1, \ldots, 5\}$ are arbitrary real constants. By using the boundary conditions (2), we obtain

$$c_3 = c_4 = 0,\ c_2 = \alpha,\ c_1 = \int_0^1 (1-s)^2 h(s)ds - 2\alpha.$$

Further

$$\begin{aligned}
c_5 &= u(0)\\
&= \beta\int_\nu^\xi u(\tau)d\tau + \sum_{i=1}^n \gamma_i u(\eta_i)\\
&= \beta\int_\nu^\xi \Big[-\frac{1}{4!}\int_0^\tau (\tau-s)^4 h(s)ds + \frac{\tau^4}{4!}\Big(\int_0^1 (1-s)^2 h(s)ds - 2\alpha\Big) + \frac{\tau^3}{6}\alpha + c_5\Big]d\tau\\
&\quad + \sum_{i=1}^n \gamma_i\Big[-\frac{1}{4!}\int_0^{\eta_i} (\eta_i-s)^4 h(s)ds + \frac{\eta_i^4}{4!}\Big(\int_0^1 (1-s)^2 h(s)ds - 2\alpha\Big) + \frac{\eta_i^3}{6}\alpha + c_5\Big]\\
&= \beta\int_\nu^\xi \Big[-\frac{1}{4!}\int_0^\tau (\tau-s)^4 h(s)ds + \frac{\tau^4}{4!}\int_0^1 (1-s)^2 h(s)ds - \frac{\tau^4}{12}\alpha + \frac{\tau^3}{6}\alpha + c_5\Big]d\tau\\
&\quad + \sum_{i=1}^n \gamma_i\Big[-\frac{1}{4!}\int_0^{\eta_i} (\eta_i-s)^4 h(s)ds + \frac{\eta_i^4}{4!}\int_0^1 (1-s)^2 h(s)ds - \frac{\eta_i^4}{12}\alpha + \frac{\eta_i^3}{6}\alpha + c_5\Big].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
c_5 &= \frac{\beta}{\mu}\int_\nu^\xi \Big[-\frac{1}{4!}\int_0^\tau (\tau-s)^4 h(s)ds + \frac{\tau^4}{4!}\int_0^1 (1-s)^2 h(s)ds - \frac{\tau^4}{12}\alpha + \frac{\tau^3}{6}\alpha\Big]d\tau\\
&\quad + \frac{1}{\mu}\sum_{i=1}^n \gamma_i\Big[-\frac{1}{4!}\int_0^{\eta_i} (\eta_i-s)^4 h(s)ds + \frac{\eta_i^4}{4!}\int_0^1 (1-s)^2 h(s)ds - \frac{\eta_i^4}{12}\alpha + \frac{\eta_i^3}{6}\alpha\Big]\\
&= \frac{\beta}{\mu}\int_\nu^\xi \Big[\frac{1}{4!}\int_0^\tau [\tau^4(1-s)^2 - (\tau-s)^4]h(s)ds + \frac{1}{4!}\int_\tau^1 \tau^4(1-s)^2 h(s)ds\Big]d\tau\\
&\quad + \frac{\beta}{\mu}\int_\nu^\xi \frac{\alpha}{6}\tau^3\Big(1-\frac{\tau}{2}\Big)d\tau + \frac{1}{\mu}\sum_{i=1}^n \gamma_i\Big[\frac{1}{4!}\int_0^{\eta_i} [\eta_i^4(1-s)^2 - (\eta_i-s)^4]h(s)ds\\
&\quad + \frac{1}{4!}\int_{\eta_i}^1 \eta_i^4(1-s)^2 h(s)ds\Big] + \frac{1}{\mu}\sum_{i=1}^n \gamma_i\frac{\alpha}{6}\eta_i^3\Big(1-\frac{\eta_i}{2}\Big).
\end{aligned}$$

Putting the values of c_1, c_2 and c_5 in (6), we get

$$\begin{aligned}
u(t) &= -\frac{1}{4!}\int_0^t (t-s)^4 h(s)ds + \frac{1}{4!}\Big(\int_0^1 (1-s)^2 h(s)ds - 2\alpha\Big)t^4 + \frac{\tau^3}{6}\alpha\\
&\quad + \frac{\beta}{\mu}\int_\nu^\xi \Big[\frac{1}{4!}\int_0^\tau \Big[\tau^4(1-s)^2 - (\tau-s)^4\Big]h(s)ds + \frac{1}{4!}\int_\tau^1 \tau^4(1-s)^2 h(s)ds\Big]d\tau\\
&\quad + \frac{\beta}{\mu}\int_\nu^\xi \frac{\alpha}{6}\tau^3\Big(1-\frac{\tau}{2}\Big)d\tau + \frac{1}{\mu}\sum_{i=1}^n \gamma_i\Big[\frac{1}{4!}\int_0^{\eta_i} [\eta_i^4(1-s)^2 - (\eta_i-s)^4]h(s)ds\\
&\quad + \frac{1}{4!}\int_{\eta_i}^1 \eta_i^4(1-s)^2 h(s)ds\Big] + \frac{1}{\mu}\sum_{i=1}^n \gamma_i\frac{\alpha}{6}\eta_i^3\Big(1-\frac{\eta_i}{2}\Big)
\end{aligned}$$

$$
\begin{aligned}
= {} & \int_0^t \frac{1}{4!}[t^4(1-s)^2-(t-s)^4]h(s)ds+\int_t^1 \frac{1}{4!}t^4(1-s)^2h(s)ds+\frac{\alpha}{6}t^3\Big(1-\frac{t}{2}\Big)\\
& +\frac{\beta}{\mu}\int_\nu^\xi \Big[\frac{1}{4!}\int_0^\tau [\tau^4(1-s)^2-(\tau-s)^4]h(s)ds+\frac{1}{4!}\int_\tau^1 \tau^4(1-s)^2h(s)ds\Big]d\tau\\
& +\frac{\alpha\beta}{12\mu}\Big(\frac{1}{2}(\xi^4-\nu^4)-\frac{1}{5}(\xi^5-\nu^5)\Big)+\frac{1}{\mu}\sum_{i=1}^n \gamma_i\Big[\frac{1}{4!}\int_0^{\eta_i}[\eta_i^4(1-s)^2-(\eta_i-s)^4]h(s)ds\\
& +\frac{1}{4!}\int_{\eta_i}^1 \eta_i^4(1-s)^2h(s)ds\Big]+\frac{1}{\mu}\sum_{i=1}^n \gamma_i\frac{\alpha}{6}\eta_i^3\Big(1-\frac{\eta_i}{2}\Big)\\
= {} & \int_0^1 \Big[G(t,s)+\frac{\beta}{\mu}\int_\nu^\xi G(\tau,s)d\tau+\frac{1}{\mu}\sum_{i=1}^n \gamma_iG(\eta_i,s)\Big]h(s)ds\\
& +\varphi(t)+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^n \gamma_i\varphi(\eta_i)\\
= {} & \int_0^1 \Big[G(t,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^n \gamma_iG(\eta_i,s)\Big]h(s)ds\\
& +\varphi(t)+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^n \gamma_i\varphi(\eta_i).
\end{aligned}
$$

□

Now, we shall give some properties concerning the nonnegativity of ψ, and the boundedness of φ and its first and second derivatives.

Lemma 2.5. *The functions $\varphi, \varphi', \varphi''$ are nonnegative and increasing in $[0,1]$ and satisfy*

$$\frac{\alpha}{12}t^3 \le \varphi(t) \le \frac{\alpha}{12},\ 0 \le \varphi'(t) \le \frac{\alpha}{6},\ 0 \le \varphi''(t) \le \frac{\alpha}{4},\ \textit{for all } t \in [0,1],$$

and

$$\psi(\xi,\nu) \ge 0,\ \textit{for all } 0 \le \nu < \xi \le 1.$$

Next, we state some properties of the Green's function $G(t,s)$.

Lemma 2.6. *Let $G(t,s)$ be defined in Lemma 2.4 and fix $\sigma \in (0,1)$, then G satisfies the following properties:*

(C1) $G(t,s) \ge 0$, *for all* $t,s \in [0,1]$,

(C2) $0 \le \frac{\partial G}{\partial t}(t,s) \le \frac{1}{6}s(1-s)^2$, *for all* $t,s \in [0,1]$,

(C3) $0 \le \frac{\partial^2 G}{\partial t^2}(t,s) \le s(1-s)^2$, *for all* $t,s \in [0,1]$,

(C4) $t^4G(1,s) \le G(t,s) \le G(1,s)$, *for all* $t,s \in [0,1]$,

(C5) $\min_{\sigma\le t\le 1} G(t,s) \ge \sigma^4G(1,s)$, *for all* $s \in [0,1]$.

Proof. For the proofs of (C1), (C2), (C3), see *[18, Lemma 3]*, *[16, Lemma 2]* and *[8, Lemma 2.3]*.
(C4) For $0 \le s \le t \le 1$. If $s=1$, then $t=1$ and we have

$$G(t,s) = 0 = G(1,1) = t^4G(1,s).$$

If $s = 0$, we have

$$G(t,s) = \frac{1}{4!}(t^4 - t^4) = 0 = t^4 G(1,s).$$

If $s = t$,

$$G(t,s) = \frac{1}{4!}t^4(1-t)^2 \geq \frac{1}{4!}t^4(1-t)^2 - \frac{1}{4!}t^4(1-t)^4 = t^4 G(1,s).$$

For $0 < s < t \leq 1$, we have

$$\begin{aligned} \frac{G(t,s)}{G(1,s)} &= \frac{t^4(1-s)^2 - (t-s)^4}{(1-s)^2 - (1-s)^4} \\ &\geq \frac{t^4(1-s)^2 - (t-ts)^4}{(1-s)^2 - (1-s)^4} \\ &= \frac{t^4\Big((1-s)^2 - (1-s)^4\Big)}{(1-s)^2 - (1-s)^4} \\ &= t^4. \end{aligned}$$

Then

$$G(t,s) \geq t^4 G(1,s).$$

On the other hand, using (C2), we deduce that $G(t,s)$ is an increasing function with respect to t. As a consequence, we get

$$G(t,s) \leq G(1,s), \text{ for all } t,s \in [0,1].$$

For $t \leq s$, we have

$$\begin{aligned} G(t,s) = \frac{1}{4!}t^4(1-s)^2 &\leq \frac{1}{4!}s(1-s)^2 \\ &\leq \frac{1}{4!}s(1-s)^2(2-s) \\ &= G(1,s), \end{aligned}$$

and

$$\begin{aligned} G(t,s) = \frac{1}{4!}t^4(1-s)^2 &\geq \frac{1}{4!}t^4(1-s)^2 - \frac{1}{4!}t^4(1-s)^4 \\ &\geq \frac{1}{4!}t^4\Big((1-s)^2 - (1-s)^4\Big) \\ &= t^4 G(1,s). \end{aligned}$$

We conclude that

$$t^4 G(1,s) \leq G(t,s) \leq G(1,s), \ \forall t,s \in [0,1].$$

(C5) For $\sigma \leq t \leq 1$, by (C2), we have

$$\min_{\sigma \leq t \leq 1} G(t,s) = G(\sigma,s), \ s \in [0,1].$$

Thus, using (C4), we obtain

$$\min_{\sigma \leq t \leq 1} G(t,s) \geq \sigma^4 G(1,s), \ \forall s \in [0,1].$$

□

Lemma 2.7. *The Green's function $H(t,s)$ satisfies the following properties*

(i) $H(t,s) \geq 0$, *for all* $t,s \in [0,1]$,

(ii) $H(\xi,s) - H(\nu,s) \geq 0$, *for all* $s \in [0,1]$.

Proof. (i) It is obvious that $H(t,s) \geq 0$ for $t \leq s$. Assume that $0 \leq s \leq t \leq 1$, then

$$\begin{aligned} H(t,s) &= \frac{1}{5!}\Big[t^5(1-s)^2 - (t-s)^5\Big] \\ &\geq \frac{1}{5!}\Big[t^5(1-s)^2 - (t-ts)^5\Big] \\ &= \frac{1}{5!}t^5(1-s)^2(1-(1-s)^3) \geq 0. \end{aligned}$$

(ii) Using the first property of Lemma 2.6, we get

$$\frac{\partial H}{\partial t}(t,s) = G(t,s) \geq 0, \text{ for all } t,s \in [0,1],$$

which means that $H(t,s)$ is increasing with respect to t. Thus

$$H(\xi,s) \geq H(\nu,s), \text{ for all } s \in [0,1].$$

□

Lemma 2.8. *Let* $\sigma \in (0,1)$ *be fixed,* $\mu > 0$, *and* $h \in C([0,1],[0,\infty])$, *then the unique solution of problem* (3)-(2) *satisfies*

(1) $u(t) \geq 0,\ u'(t) \geq 0,\ u''(t) \geq 0$, *for all* $t \in [0,1]$,

(2) $\min_{\sigma\leq t\leq 1} u(t) \geq \frac{\sigma^4}{72}\|u\|$, *where* $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$, $\|u\|_\infty = \max_{0\leq t\leq 1}|u(t)|$, $\|u'\|_\infty = \max_{0\leq t\leq 1}|u'(t)|$, $\|u''\|_\infty = \max_{0\leq t\leq 1}|u''(t)|$.

Proof. (1) follows immediately from Lemmas 2.4-2.7.
(2) For all $t \in [0,1]$, we have

$$\begin{aligned} u(t) &= \int_0^1 \Big[G(t,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i\varphi(\eta_i) \\ &\leq \int_0^1 \Big[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad + \frac{\alpha}{12} + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i\varphi(\eta_i) \\ &\leq \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad + \frac{\alpha}{4} + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i\varphi(\eta_i) \end{aligned}$$

$$\leq 4!\Big(\int_0^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds + \frac{\alpha}{4}+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\Big).$$

Then we have

$$\|u\|_\infty \leq 4!\Big(\int_0^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds + \frac{\alpha}{4}+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\Big). \tag{7}$$

Also, from (C2) in Lemma 2.6 and Lemma 2.5, we have

$$\begin{aligned} u'(t) &= \int_0^1 \frac{\partial G}{\partial t}(t,s)h(s)ds+\varphi'(t) \\ &\leq \int_0^1 \frac{1}{6}s(1-s)^2h(s)ds+\frac{\alpha}{6} \\ &\leq \int_0^1 \Big[4!G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad +\frac{\alpha}{4}+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i) \\ &\leq 4!\Big(\int_0^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad +\frac{\alpha}{4}+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\Big). \end{aligned}$$

Then,

$$\|u'\|_\infty \leq 4!\Big(\int_0^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds + \frac{\alpha}{4}+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\Big). \tag{8}$$

Using the property (C3) of Lemma 2.6, we also get

$$\begin{aligned} u''(t) &= \int_0^1 \frac{\partial^2 G}{\partial t^2}(t,s)h(s)ds+\varphi''(t) \\ &\leq \int_0^1 s(1-s)^2h(s)ds+\frac{\alpha}{4} \\ &\leq \int_0^1 \Big[4!G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad +\frac{\alpha}{4}+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i) \end{aligned}$$

$$\leq \quad 4!\Big(\int_0^1 \Big[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds + \frac{\alpha}{4} + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\Big).$$

Then we have,

$$\|u''\|_\infty \leq 4!\Big(\int_0^1 \Big[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds + \frac{\alpha}{4} + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\Big). \tag{9}$$

So, from (7), (8), (9), we get

$$\|u\| \leq 4!\Big(\int_0^1 \Big[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds + \frac{\alpha}{4} + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\Big).$$

On the other hand, for $t \in [\sigma, 1]$, by (C5) of Lemma 2.6, we get

$$\begin{aligned} u(t) &= \int_0^1 \Big[G(t,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i) \\ &\geq \int_0^1 \Big[\sigma^4 G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad + \frac{\alpha}{12}\sigma^3 + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i) \\ &\geq \frac{\sigma^4}{3}\Big(\int_0^1 \Big[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]h(s)ds \\ &\quad + \frac{\alpha}{4} + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\Big) \\ &\geq \frac{\sigma^4}{72}\|u\|, \end{aligned}$$

Hence, we deduce that

$$\min_{\sigma\leq t\leq 1} u(t) \geq \frac{\sigma^4}{72}\|u\|.$$

□

In what follows, we shall consider the Banach space $E = C^2([0,1],\mathbb{R})$ equipped with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\},$$

and define the cone $P \subset E$ by

$$P = \left\{ u \in E : u(t) \geq 0,\ u'(t) \geq 0,\ u''(t) \geq 0,\ t \in [0,1], \min_{\sigma \leq t \leq 1} u(t) \geq \frac{\sigma^4}{72} \|u\| \right\}.$$

Let the operator $A : P \to E$ defined as follows

$$\begin{aligned} Au(t) &= \int_0^1 \left[G(t,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n} \gamma_i G(\eta_i,s) \right] f(s,u(s),u'(s),u''(s))ds \\ &\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i),\ t \in [0,1]. \end{aligned} \tag{10}$$

It is easy to see that the existence of a monotone and convex positive solution of the boundary value problem (1)-(2) is equivalent to the existence of a nontrivial fixed point of A on P.

Lemma 2.9. *A is a completely continuous operator and $A(P) \subset P$.*

Proof. By Lebesgue dominated convergence theorem along with Arzela-Ascoli theorem, it can be seen that A is completely continuous. Further, from Lemma 2.8 we obtain $A(P) \subset P$. □

To prove our results, we use the known Krasnoselskii fixed point theorem [21].

Theorem 2.10. *Let E be a Banach space, and let $P \subset E$, be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$$

be a completely continuous operator such that

(i) $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; or

(ii) $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence and nonexistence results

These notations are important considerations in our analysis hereinafter.

$$f_0 = \lim_{x+y+z \to 0} \left\{ \min_{t \in [0,1]} \frac{f(t,x,y,z)}{x+y+z} \right\}, \qquad f^0 = \lim_{x+y+z \to 0} \left\{ \max_{t \in [0,1]} \frac{f(t,x,y,z)}{x+y+z} \right\}$$

$$f_\infty = \lim_{x+y+z \to \infty} \left\{ \min_{t \in [0,1]} \frac{f(t,x,y,z)}{x+y+z} \right\}, \qquad f^\infty = \lim_{x+y+z \to \infty} \left\{ \max_{t \in [0,1]} \frac{f(t,x,y,z)}{x+y+z} \right\}.$$

$$\begin{aligned} \Lambda_1 &= 6\int_0^1 \left[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s) \right] ds, \\ \Lambda_2 &= \frac{\sigma^8}{72}\int_\sigma^1 \left[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s) \right] ds. \end{aligned}$$

Now, we show the existence and nonexistence of solutions for boundary value problem (1)-(2) by using the known Krasnoselskii fixed point theorem.

Theorem 3.1. *Suppose that $\Lambda_1 f^0 < 1 < \Lambda_2 f_\infty$, then the boundary value problem* (1)-(2) *has at least one monotone and convex positive solution for α small enough and has no monotone and convex positive solution for α large enough.*

Proof. We divide the proof into two steps.
Step 1. Since $\Lambda_1 f^0 < 1$, there exists $\varepsilon_1 > 0$ such that $\Lambda_1(f^0 + \varepsilon_1) \le 1$.
By the definition of f^0, there exists $\rho_1 > 0$ such that

$$f(t,x,y,z) \le (f^0 + \varepsilon_1)(x + y + z), \text{ for all } t \in [0,1],\ 0 < x + y + z \le \rho_1.$$

Set $\Omega_1 = \left\{u \in E, \|u\| < \frac{\rho_1}{3}\right\}$, and let α satisfy $0 < \alpha \le \frac{\mu}{6(2-\mu)}\rho_1$. Then for all $u \in P \cap \partial\Omega_1,\ t \in [0,1]$, we have

$$\begin{aligned}
Au(t) &= \int_0^1 \left[G(t,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\right] f(s,u(s),u'(s),u''(s))ds \\
&\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i \varphi(\eta_i) \\
&\le \int_0^1 \left[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\right] f(s,u(s),u'(s),u''(s))ds \\
&\quad + \frac{\alpha}{12} + \frac{\alpha\beta}{12\mu}(\xi - \nu) + \frac{\alpha}{12\mu}\sum_{i=1}^n \gamma_i \\
&\le \int_0^1 \left[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\right](f^0 + \varepsilon_1)(u(s) + u'(s) + u''(s))ds \\
&\quad + \frac{\alpha}{\mu}\Big(1 + \beta(\xi - \nu) + \sum_{i=1}^n \gamma_i\Big) \\
&\le 3(f^0 + \varepsilon_1)\|u\| \int_0^1 \left[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\right]ds + \frac{\alpha}{\mu}(2 - \mu) \\
&\le \frac{1}{2}\Lambda_1(f^0 + \varepsilon_1)\|u\| + \frac{\rho_1}{6} \\
&\le \frac{1}{2}\|u\| + \frac{1}{2}\|u\| = \|u\|.
\end{aligned}$$

Also, on the other hand, from Lemmas 2.5, 2.6, we have

$$\begin{aligned}
(Au)''(t) &= \int_0^1 \frac{\partial^2 G}{\partial t^2}(t,s) f(s,u(s),u'(s),u''(s))ds + \varphi''(t) \\
&\le \int_0^1 s(1-s)^2 f(s,u(s),u'(s),u''(s))ds + \frac{\alpha}{4} \\
&\le \int_0^1 4!G(1,s) f(s,u(s),u'(s),u''(s))ds + \frac{\alpha}{4} \\
&\le \int_0^1 \left[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\right] f(s,u(s),u'(s),u''(s))ds \\
&\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i \varphi(\eta_i)
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 \Big[4!G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big](f^0+\varepsilon_1)(u(s)+u'(s)+u''(s))ds \\
&\quad +\frac{\alpha}{\mu}\Big(1+\beta(\xi-\nu)+\sum_{i=1}^{n}\gamma_i\Big) \\
&\leq 3(f^0+\varepsilon_1)\|u\|\int_0^1 \Big[4!G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]ds+\frac{\alpha}{\mu}(2-\mu) \\
&\leq \frac{1}{2}\Lambda_1(f^0+\varepsilon_1)\|u\|+\frac{\rho_1}{6} \\
&\leq \frac{1}{2}\|u\|+\frac{1}{2}\|u\|=\|u\|.
\end{aligned}$$

Integrating the above inequality on $[0,t]$, we find

$$(Au)'(t)\leq\|u\|,\ \text{for all } t\in[0,1].$$

Then,

$$\|Au\|\leq\|u\|,\ \text{for all } u\in P\cap\partial\Omega_1.$$

On the other hand, since $\Lambda_2 f_\infty>1$, there exists $\varepsilon_2>0$ such that $\Lambda_2(f_\infty-\varepsilon_2)\geq 1$.

By the definitions of f_∞, there exists a constant $\widetilde{\rho_2}>0$ such that

$$f(t,x,y,z)\geq(f_\infty-\varepsilon_2)(x+y+z),\ t\in[0,1],\ x+y+z\geq\widetilde{\rho_2}.$$

Let $\Omega_2=\Big\{u\in E, \|u\|<\rho_2\Big\}$, where $\rho_2=2\max\{\frac{\rho_1}{3}, 36\sigma^{-4}\widetilde{\rho_2}\}$. Then for all $t\in[\sigma,1]$, by Lemma 2.7, we have

$$u(t)+u'(t)+u''(t)\geq u(t)\geq\frac{\sigma^4}{72}\|u\|\geq\frac{\sigma^4}{72}\rho_2\geq\widetilde{\rho_2},\ \text{for all } u\in P\cap\partial\Omega_2,$$

and

$$\begin{aligned}
(Au)(1) &= \int_0^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]f(s,u(s),u'(s),u''(s))ds \\
&\quad +\varphi(1)+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i) \\
&\geq \int_0^1 \Big[\sigma^4 G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]f(s,u(s),u'(s),u''(s))ds \\
&\quad +\frac{\alpha}{12}+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i) \\
&\geq \int_\sigma^1 \sigma^4\Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big](f_\infty-\varepsilon_2)(u(s)+u'(s)+u''(s))ds \\
&\geq \frac{\sigma^8}{72}\|u\|(f_\infty-\varepsilon_2)\int_\sigma^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]ds \\
&= \Lambda_2(f_\infty-\varepsilon_2)\|u\| \\
&\geq \|u\|.
\end{aligned}$$

Thus

$$\|Au\|\geq\|Au\|_\infty=\sup_{0\leq t\leq 1}|Au(t)|\geq(Au)(1)\geq\|u\|.$$

Therefore, by Theorem 2.10, we conclude that the operator A has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which is a monotone and convex positive solution of the boundary value problem (1)-(2). Clearly, if $\alpha = 0$, then the existence result for the problem (1)-(2) can be proved in an analogous way.

Step 2. We show that the boundary value problem (1)-(2) has no monotone and convex positive solution for α large enough. Suppose by contrary that there exist $0 < \alpha_1 < \alpha_2 < ... < \alpha_m < ...$, with $\lim_{m\to+\infty} \alpha_m = +\infty$ such that for any positive integer m, the boundary value problem

$$\begin{cases} u^{(5)}(t) + f(t, u(t), u'(t), u''(t)) = 0, \ t \in (0, 1), \\ u'(0) = u''(0) = u''(1) = 0, \ u'''(0) = \alpha_m, \\ u(0) = \beta \int_\nu^\xi u(s)ds + \sum_{i=1}^n \gamma_i u(\eta_i), \end{cases}$$

has a monotone and convex positive solution $u_m(t)$.
By (10), we have

$$\begin{aligned} u_m(1) &= \int_0^1 \Big[G(1, s) + \frac{\beta}{\mu}\Big(H(\xi, s) - H(\nu, s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i, s)\Big] f(s, u_m(s), u_m'(s), u_m''(s))ds \\ &\quad + \varphi(1) + \frac{\beta}{\mu}\psi(\xi, \nu) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i \varphi(\eta_i) \\ &\geq \varphi(1) + \frac{\beta}{\mu}\psi(\xi, \nu) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i \varphi(\eta_i) \\ &= \frac{\alpha_m}{12} + \frac{\alpha_m \beta}{6\mu}\int_\nu^\xi \tau^3\Big(1 - \frac{\tau}{2}\Big)d\tau + \frac{\alpha_m}{6\mu}\sum_{i=1}^n \gamma_i \eta_i^3\Big(1 - \frac{\eta_i}{2}\Big) \\ &= \alpha_m\Big(\frac{1}{12} + \frac{\beta}{6\mu}\int_\nu^\xi \tau^3\Big(1 - \frac{\tau}{2}\Big)d\tau + \frac{1}{6\mu}\sum_{i=1}^n \gamma_i \eta_i^3\Big(1 - \frac{\eta_i}{2}\Big)\Big) \to +\infty, \text{ when } m \to +\infty. \end{aligned}$$

Thus

$$\lim_{m\to+\infty} \|u_m\|_\infty = +\infty.$$

As a result we have

$$\lim_{m\to+\infty} \|u_m\| = +\infty.$$

Since $\Lambda_2 f_\infty > 1$, there exists $\widehat{\rho} > 0$ such that

$$f(t, x, y, z) \geq (f_\infty - \varepsilon')(x + y + z), \ t \in [0, 1], \ x + y + z \geq \frac{\sigma^4}{72}\widehat{\rho},$$

where $\varepsilon' > 0$ satisfies $\Lambda_2(f_\infty - \varepsilon') \geq 1$.
Let m be large enough, such that $\|u_m\| \geq \widehat{\rho}$. Since

$$u_m(s) + u_m'(s) + u_m''(s) \geq u_m(s) \geq \min_{s\in[\sigma,1]} u_m(s) \geq \frac{\sigma^4}{72}\|u_m\| \geq \frac{\sigma^4}{72}\widehat{\rho}, \ \forall s \in [\sigma, 1].$$

Then

$$\begin{aligned} \|u_m\| &\geq u_m(1) = \int_0^1 \Big[G(1, s) + \frac{\beta}{\mu}\Big(H(\xi, s) - H(\nu, s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i, s)\Big] f(s, u_m(s), u_m'(s), u_m''(s))ds \\ &\quad + \alpha_m\Big(\frac{1}{12} + \frac{\beta}{6\mu}\int_\nu^\xi \tau^3\Big(1 - \frac{\tau}{2}\Big)d\tau + \frac{1}{6\mu}\sum_{i=1}^n \gamma_i \eta_i^3\Big(1 - \frac{\eta_i}{2}\Big)\Big) \end{aligned}$$

$$
\begin{aligned}
&> \int_0^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]f(s,u_m(s),u_m'(s),u_m''(s))ds\\
&\geq \int_\sigma^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]f(s,u_m(s),u_m'(s),u_m''(s))ds\\
&\geq \int_\sigma^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big](f_\infty-\varepsilon')(u_m(s)+u_m'(s)+u_m''(s))ds\\
&\geq \frac{\sigma^4}{72}\|u_m\|(f_\infty-\varepsilon')\int_\sigma^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]ds\\
&\geq \frac{\sigma^8}{72}\|u_m\|(f_\infty-\varepsilon')\int_\sigma^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]ds\\
&\geq \Lambda_2(f_\infty-\varepsilon')\|u_m\|\geq\|u_m\|,
\end{aligned}
$$

which is a contradiction. □

As consequence of the above Theorem 3.1, we can easily state the following corollary.

Corollary 3.2. *Suppose that f is superlinear, i.e.,*

$$f^0=0,\ f_\infty=\infty.$$

Then the boundary value problem (1)-(2) *has at least one monotone and convex positive solution for α small enough and has no monotone and convex positive solution for α large enough.*

Theorem 3.3. *Suppose that $\Lambda_1 f^\infty < 1 < \Lambda_2 f_0$, then the boundary value problem* (1)-(2) *has at least one monotone and convex positive solution for any $\alpha\in[0,\infty)$.*

Proof. Since $\Lambda_2 f_0 > 1$, there exists $\rho_3 > 0$ such that

$$f(t,x,y,z)\geq(f_0-\varepsilon_3)(x+y+z),\ \text{for all } t\in[0,1],\ 0<x+y+z\leq\rho_3,$$

where $\varepsilon_3 > 0$ satisfies $\Lambda_2(f_0-\varepsilon_3)\geq 1$.
Let $\Omega_3=\{u\in E,\|u\|<\frac{\rho_3}{3}\}$. Then for all $u\in P\cap\partial\Omega_3$, by Lemmas 2.5, 2.6, we have

$$
\begin{aligned}
(Au)(1) &= \int_0^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]f(s,u(s),u'(s),u''(s))ds\\
&\quad+\varphi(1)+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\\
&\geq \int_0^1 \Big[\sigma^4 G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]f(s,u(s),u'(s),u''(s))ds\\
&\quad+\frac{\alpha}{12}+\frac{\beta}{\mu}\psi(\xi,\nu)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\\
&\geq \int_0^1 \sigma^4\Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big](f_0-\varepsilon_3)(u(s)+u'(s)+u''(s))ds\\
&\geq \frac{\sigma^8}{72}\|u\|(f_0-\varepsilon_3)\int_\sigma^1 \Big[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]ds\\
&= \Lambda_2(f_0-\varepsilon_3)\|u\|\\
&\geq \|u\|.
\end{aligned}
$$

Thus

$$\|Au\| \geq \|Au\|_\infty \geq \|u\|, \ \forall u \in P \cap \partial\Omega_3.$$

Next, in view of $\Lambda_1 f^\infty < 1$, there exists $\bar{\rho}_4 > 2\rho_3$ such that

$$f(t,x,y,z) \leq (f^\infty + \varepsilon_4)(x+y+z), \text{ for all } t \in [0,1], \ x+y+z \geq \bar{\rho}_4.$$

where $\varepsilon_4 > 0$ satisfies $\Lambda_1(f^\infty + \varepsilon_4) \leq 1$.

Let $\omega = \max\{f(t,x,y,z), \ (t,x,y,z) \in [0,1] \times [0,\bar{\rho}_4]^3\}$. Then

$$f(t,x,y,z) \leq (f^\infty + \varepsilon_4)(x+y+z) + \omega, \text{ for all } (t,x,y,z) \in [0,1] \times [0,\infty)^3.$$

Now, we choose

$$\rho_4 > \frac{1}{6}\max\left\{\bar{\rho}_4, \frac{2\omega\Lambda_1}{1-\Lambda_1(f^\infty+\varepsilon_4)}, \frac{12\alpha}{\mu}(2-\mu)\right\},$$

and let $\Omega_4 = \{u \in E, \|u\| < \rho_4\}$. Then for all $u \in P \cap \partial\Omega_4$, from Lemmas 2.4, 2.5, 2.6 and 2.7, we have

$$\begin{aligned}
Au(t) &= \int_0^1 \Big[G(t,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big] f(s,u(s),u'(s),u''(s))ds \\
&\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i \varphi(\eta_i) \\
&\leq \int_0^1 \Big[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big] f(s,u(s),u'(s),u''(s))ds \\
&\quad + \frac{\alpha}{12} + \frac{\alpha\beta}{12\mu}(\xi-\nu) + \frac{\alpha}{12\mu}\sum_{i=1}^n \gamma_i \\
&\leq \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big]\Big((f^\infty+\varepsilon_4)(u(s)+u'(s)+u''(s)) + \omega\Big)ds \\
&\quad + \frac{\alpha}{\mu}(2-\mu) \\
&\leq \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big](f^\infty+\varepsilon_4)(u(s)+u'(s)+u''(s))ds \\
&\quad + \omega\int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big]ds + \frac{\alpha}{\mu}(2-\mu) \\
&\leq 3(f^\infty+\varepsilon_4)\|u\|\int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big]ds \\
&\quad + \omega\int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^n \gamma_i G(\eta_i,s)\Big]ds + \frac{\alpha}{\mu}(2-\mu) \\
&\leq \frac{1}{2}\Lambda_1(f^\infty+\varepsilon_4)\|u\| + \frac{\omega\Lambda_1}{6} + \frac{\rho_4}{2} \\
&\leq \frac{1}{2}\Lambda_1(f^\infty+\varepsilon_4)\|u\| + \frac{1}{2}\Big(1-\Lambda_1(f^\infty+\varepsilon_4)\Big)\|u\| + \frac{\|u\|}{2} \\
&\leq \|u\|, \ t \in [0,1].
\end{aligned}$$

Then

$$\|Au\|_\infty \leq \|u\|, \ \forall u \in P \cap \partial\Omega_4.$$

Also, in view of Lemmas 2.5 and 2.6, it follows that

$$\begin{aligned}
(Au)''(t) &= \int_0^1 \frac{\partial^2 G}{\partial t^2}(t,s)f(s,u(s),u'(s),u''(s))ds + \varphi''(t)\\
&\le \int_0^1 s(1-s)^2 f(s,u(s),u'(s),u''(s))ds + \frac{\alpha}{4}\\
&\le \int_0^1 4!G(1,s)f(s,u(s),u'(s),u''(s))ds + \frac{\alpha}{4}\\
&\le \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big] f(s,u(s),u'(s),u''(s))ds\\
&\quad + \frac{\alpha}{4} + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i)\\
&\le \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]\Big((f^\infty+\varepsilon_4)(u(s)+u'(s)+u''(s))+\omega\Big)ds\\
&\quad + \frac{\alpha}{\mu}(2-\mu)\\
&\le 3(f^\infty+\varepsilon_4)\|u\| \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]ds\\
&\quad + \omega\int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big]ds + \frac{\alpha}{\mu}(2-\mu)\\
&\le \frac{1}{2}\Lambda_1(f^\infty+\varepsilon_4)\|u\| + \frac{\omega\Lambda_1}{6} + \frac{\rho_4}{2}\\
&\le \frac{1}{2}\Lambda_1(f^\infty+\varepsilon_4)\|u\| + \frac{1}{2}\Big(1-\Lambda_1(f^\infty+\varepsilon_4)\Big)\|u\| + \frac{\|u\|}{2}\\
&\le \|u\|,\ t\in[0,1],
\end{aligned}$$

then

$$(Au)''(t) \le \|u\|,\ t\in[0,1].$$

By integrating the above inequality on $[0,t]$, we get

$$(Au)'(t) \le \|u\|,\ t\in[0,1],$$

which implies that

$$\|Au\| \le \|u\|.$$

Therefore, by Theorem 2.10, it follows that the operator A has at least one fixed point $u \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$, which is a monotone and convex positive solution of the boundary value problem (1)-(2). □

From Theorem 3.3, we can easily derive the following corollary.

Corollary 3.4. *Suppose that f is sublinear , i.e.,*

$$f_0 = \infty,\ f^\infty = 0.$$

Then the boundary value problem (1)-(2) *has at least one monotone and convex positive solution for any $\alpha \in [0,\infty)$.*

Theorem 3.5. *Suppose that the following conditions hold*

(H1) $\Lambda_2 f_0 > 1$ *and* $\Lambda_2 f_\infty > 1$;

(H2) There exist $\delta > \frac{6\alpha}{\mu}(2-\mu)$, $M \in (0, \Lambda_1^{-1}]$ *such that*

$$\max_{(t,x+y+z)\in[0,1]\times[0,\delta]} f(t,x,y,z) < M\delta.$$

Then the boundary value problem (1)-(2) *has at least two monotone and convex positive solutions* $u_1(t)$, $u_2(t)$ *satisfying*

$$0 < \|u_1\| < \frac{\delta}{3} < \|u_2\|.$$

Proof. We choose r_1 and r_2 such that $0 < r_1 < \frac{\delta}{3} < r_2$.
If $\Lambda_2 f_0 > 1$, then by the proof of Theorem 3.3, we have

$$\|Au\| \geq \|u\|, \text{ for all } u \in P \cap \partial\Omega_{r_1} \tag{11}$$

where $\Omega_{r_1} = \{u \in E, \|u\| < r_1\}$.

If $\Lambda_2 f_\infty > 1$, then by the proof of Theorem 3.1, we have

$$\|Au\| \geq \|u\|, \text{ for all } u \in P \cap \partial\Omega_{r_2}, \tag{12}$$

where $\Omega_{r_2} = \{u \in E, \|u\| < r_2\}$.

Let $\Omega_\delta = \{u \in E, \|u\| < \frac{\delta}{3}\}$. Then for all $t \in [0,1], u \in u \in P \cap \partial\Omega_\delta$, by condition (H2), we get

$$\begin{aligned}
Au(t) &= \int_0^1 \Big[G(t,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big] f(s,u(s),u'(s),u''(s))ds \\
&\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i) \\
&< \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big] M\delta ds \\
&\quad + \frac{\alpha}{\mu}\Big(1 + \beta(\xi-\nu) + \sum_{i=1}^{n}\gamma_i\Big) \\
&\leq \frac{1}{6}M\delta\Lambda_1 + \frac{\alpha}{\mu}(2-\mu) \\
&\leq \frac{1}{6}\Lambda_1^{-1}\delta\Lambda_1 + \frac{\alpha}{\mu}(2-\mu) \\
&\leq \frac{\delta}{6} + \frac{\delta}{6} = \frac{\delta}{3} = \|u\|,
\end{aligned}$$

then, $\|Au\|_\infty < \|u\| = \frac{\delta}{3}$.

Similarly, from Lemmas 2.5, 2.6, we have

$$
\begin{aligned}
(Au)''(t) &= \int_0^1 \frac{\partial^2 G}{\partial t^2}(t,s)f(s,u(s),u'(s),u''(s))ds + \varphi''(t) \\
&\leq \int_0^1 4!G(1,s)f(s,u(s),u'(s),u''(s))ds + \frac{\alpha}{4} \\
&\leq \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big] f(s,u(s),u'(s),u''(s))ds \\
&\quad + \varphi(t) + \frac{\beta}{\mu}\psi(\xi,\nu) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i\varphi(\eta_i) \\
&< \int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{n}\gamma_i G(\eta_i,s)\Big] M\delta ds + \frac{\alpha}{\mu}(2-\mu) \\
&\leq \frac{1}{6}M\delta\Lambda_1 + \frac{\alpha}{\mu}(2-\mu) \\
&\leq \frac{1}{6}\Lambda_1^{-1}\delta\Lambda_1 + \frac{\alpha}{\mu}(2-\mu) \\
&\leq \frac{\delta}{6} + \frac{\delta}{6} = \frac{\delta}{3} = \|u\|,
\end{aligned}
$$

then, $(Au)''(t) < \|u\|$.
Integrating the above inequality on $[0,t]$, we find

$$(Au)'(t) < \|u\|, \text{ for all } t \in [0,1].$$

Consequently,

$$\|Au\| < \|u\|, \text{ for all } u \in P \cap \partial\Omega_\delta. \tag{13}$$

Therefore, by (11), (12) and (13), the boundary value problem (1)-(2) has at least two monotone and convex positive solutions u_1 and u_2 satisfying $0 < \|u_1\| < \frac{\delta}{3} < \|u_2\|$. □

Remark 3.6. *From the proof of Theorem 3.5 we obtain that if* (H2) *holds and* $\Lambda_2 f_0 > 1$ *(or* $\Lambda_2 f_\infty > 1$*), then problem* (1)-(2) *has at least one monotone and convex positive solution* u_1 *(or* u_2*) satisfying* $0 < \|u_1\| < \frac{\delta}{3}$ *(or* $\frac{\delta}{3} < \|u_2\|$*).*

Using the hypotheses that are given in Theorems 3.1 and 3.3 with an additional assumption on the nonlinear term f, we are able to deduce the following result which can be proved in an analogous way to the previous one.

Theorem 3.7. *Assume that the following conditions hold.*

(H3) $\Lambda_1 f^0 < 1$ *and* $\Lambda_1 f^\infty < 1$;

(H4) There exist $\theta > \frac{6\alpha}{\mu}(2-\mu)$, $N \in [\Lambda_2^{-1}, +\infty)$ *such that*

$$\min_{(t,x+y+z)\in[0,1]\times\left[\left(\frac{\sigma^2}{6}\sqrt{\frac{\theta}{6}}\right)^2,\theta\right]} f(t,x,y,z) > \left(\frac{\sigma^2}{6}\sqrt{\frac{\theta N}{6}}\right)^2.$$

Then the boundary value problem (1)-(2) *has at least two monotone and convex positive solutions* $u_1(t)$, $u_2(t)$ *satisfying*

$$0 < \|u_1\| < \frac{\theta}{3} < \|u_2\|.$$

Remark 3.8. *From the proof of Theorem 3.7 we obtain that if* (H4) *holds and* $\Lambda_1 f^0 < 1$ *(or* $\Lambda_1 f^\infty < 1$*), then problem* (1)-(2) *has at least one monotone and convex positive solution* u_1 *(or* u_2*) satisfying* $0 < \|u_1\| < \frac{\theta}{3}$ *(or* $\frac{\theta}{3} < \|u_2\|$*).*

4. Illustrative examples

To show the applicability and efficiency of the theoretical results, we give two examples.

Example 4.1. *Consider the following boundary value problem*

$$u^{(5)}(t) + f(t, u, u', u'') = 0,\ t \in (0, 1), \tag{14}$$

$$u'(0) = u''(0) = u''(1) = 0,\ u'''(0) = \alpha,\ u(0) = \beta \int_{\nu}^{\xi} u(s)ds + \sum_{i=1}^{2} \gamma_i u(\eta_i), \tag{15}$$

where
$f(t,x,y,z) = \frac{10^7(x+y+z)}{1+e^{-t}}\Big(\frac{1+\cos(x+y+z)}{1+x+y+z} + \frac{623(1+e^{-1})}{500\times 10^7}(x+y+z)\ln\Big(\frac{3+x+y+z}{2+x+y+z}\Big)\Big) \in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$,
$\alpha = 2, \xi = \frac{3\sigma}{4}, \nu = \frac{\sigma}{4}, \beta = \frac{1}{2}, \eta_1 = \frac{\sigma}{2}, \eta_2 = \frac{3\sigma}{5}, \gamma_1 = \frac{1}{4}, \gamma_2 = \frac{1}{5}$, *for* $\sigma \in (0,1)$.
Obviously, $0 \le \nu < \eta_1 < \eta_2 < \xi \le 1$, *and* $\mu = \frac{1}{4}(\frac{11}{5} - \sigma) > 0$.
We have

$$\begin{aligned} f_0 &= \lim_{x+y+z\to 0} \left\{ \min_{t\in[0,1]} \frac{f(t,x,y,z)}{x+y+z} \right\} = 10^7, \\ f^\infty &= \lim_{x+y+z\to\infty} \left\{ \max_{t\in[0,1]} \frac{f(t,x,y,z)}{x+y+z} \right\} = \frac{623}{500}. \end{aligned}$$

By simple calculations, we find that

$$\begin{aligned} \Lambda_1 &= 6\int_0^1 \Big[4!G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{2}\gamma_i G(\eta_i,s)\Big]ds \\ &= 6\Big[\frac{2}{15} + \frac{\beta}{\mu 5!}\Big(\frac{\xi^5 - \nu^5}{3} + \frac{\nu^6 - \xi^6}{6}\Big) + \frac{1}{\mu 4!}\Big(\gamma_1\eta_1^4\Big(\frac{1}{3} - \frac{\eta_1}{5}\Big) + \gamma_2\eta_2^4\Big(\frac{1}{3} - \frac{\eta_2}{5}\Big)\Big)\Big] \\ &= 6\Big(\frac{2}{15} + \frac{(242 - 91\sigma)\sigma^5}{36864(11 - 5\sigma)} + \frac{(415450 - 140187\sigma)\sigma^4}{36\times 10^6(11 - 5\sigma)}\Big). \end{aligned}$$

In addition, we have

$$\begin{aligned} \Lambda_2 &= \frac{\sigma^8}{72}\int_\sigma^1 \Big[G(1,s) + \frac{\beta}{\mu}\Big(H(\xi,s) - H(\nu,s)\Big) + \frac{1}{\mu}\sum_{i=1}^{2}\gamma_i G(\eta_i,s)\Big]ds \\ &= \frac{\sigma^8}{72}\Big[\frac{1}{72}(-1+\sigma)^3\Big(-\sigma(2-\sigma) - \frac{2}{5}(1-\sigma)^2\Big) - \frac{\beta}{360\mu}(\xi^5 - \nu^5)(-1+\sigma)^3 - \frac{(-1+\sigma)^3}{72\mu}(\gamma_1\eta_1^4 + \gamma_2\eta_2^4)\Big] \\ &= \frac{\sigma^8(1-\sigma)^3}{72}\Big(\frac{1}{360}(2 + 6\sigma - 3\sigma^2) + \frac{121\sigma^5}{18432(11 - 5\sigma)} + \frac{8309\sigma^4}{72\times 10^4(11 - 5\sigma)}\Big). \end{aligned}$$

Putting

$$\Theta_1(\sigma) = \Lambda_2 f_0 - 1,\ \Theta_2(\sigma) = \Lambda_1 f^\infty - 1.$$

We easily check that

$$\Theta_1(\sigma) > 0 \text{ and } \Theta_2(\sigma) < 0, \text{ for all } \sigma \in \Big[\frac{11}{20}, \frac{18}{25}\Big].$$

Therefore, all conditions of Theorem 3.3 are fulfilled. Hence, the boundary value problem (14)-(15) *has at least one monotone and convex positive solution. The plot of the functions* $\Theta_1(\sigma)$, $\Theta_2(\sigma)$ *for different values of* σ *ranging from* 0 *to* 1 *are depicted in Figs.1, 2.*

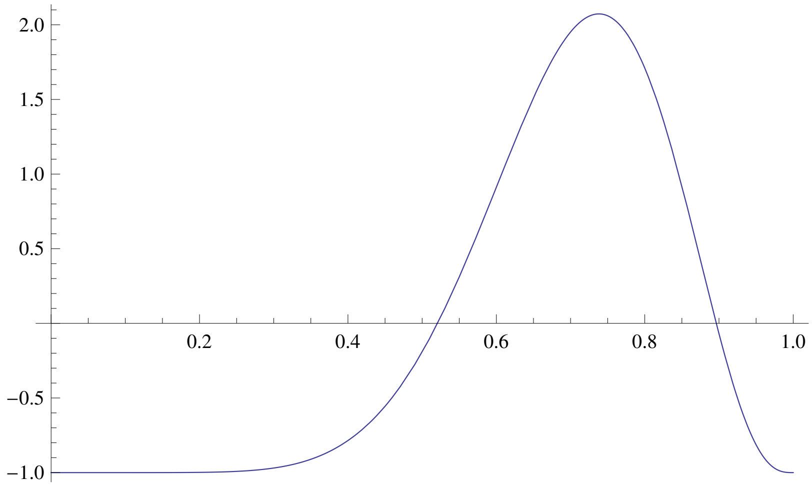

Figure 1: Graph of the function $\Theta_1(\sigma)$, $\sigma \in \left[0,1\right]$.

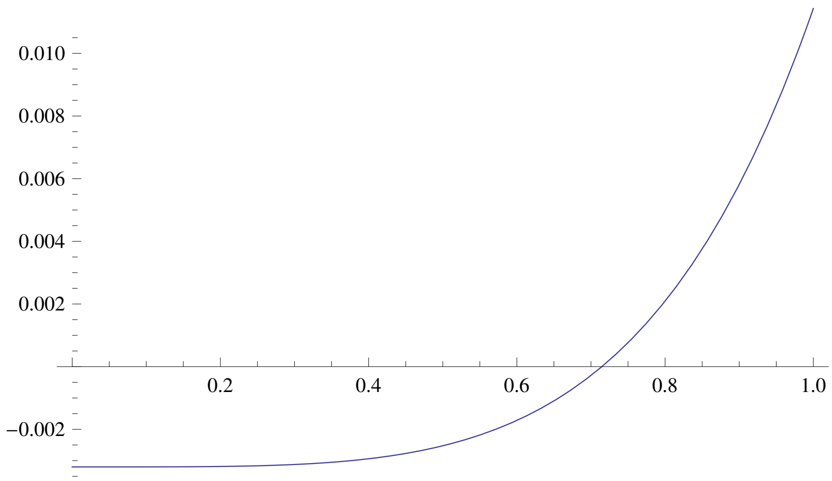

Figure 2: Graph of the function $\Theta_2(\sigma)$, $\sigma \in \left[0,1\right]$.

Example 4.2. *Consider the following boundary value problem*

$$u^{(5)}(t) + f(t, u, u', u'') = 0, \ t \in (0,1), \tag{16}$$

$$u'(0) = u''(0) = u''(1) = 0, \ u'''(0) = \alpha, \ u(0) = \beta \int_{\nu}^{\xi} u(s)ds + \sum_{i=1}^{2} \gamma_i u(\eta_i), \tag{17}$$

where

$$f(t,x,y,z) = \begin{cases} (6+t)|\sin 6\times 10^7(x+y+z)|, & x+y+z \leq 7, \\ (6+t)|\sin 6\times 10^7(x+y+z)| + 37\times 10^7\sqrt{t+1} & \\ \times(x+y+z)^2 \arctan\left(\frac{1}{x+y+z}\right), & x+y+z > 7, \end{cases}$$

$f(t,x,y,z) \in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, $\alpha = \frac{5}{19}$, $\xi = 1$, $\nu = 0$, $\beta = \frac{1}{4}$, $\eta_1 = \frac{\sigma}{2}$, $\eta_2 = 0$, $\gamma_1 = \frac{1}{3}$, $\gamma_2 = 0$, *for* $\sigma \in (0,1)$. *Obviously,* $0 \leq \nu < \eta_1 < \eta_2 < \xi \leq 1$, *and* $\mu = \frac{5}{12} > 0$.

We have

$$f_0 = \lim_{x+y+z\to 0}\left\{\min_{t\in[0,1]}\frac{f(t,x,y,z)}{x+y+z}\right\} = 36\times 10^7,$$
$$f_\infty = \lim_{x+y+z\to \infty}\left\{\min_{t\in[0,1]}\frac{f(t,x,y,z)}{x+y+z}\right\} = 37\times 10^7.$$

By simple calculations, we find

$$\begin{aligned}\Lambda_1 &= 6\int_0^1\left[4!G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{2}\gamma_i G(\eta_i,s)\right]ds\\ &= 6\Big[\frac{2}{15}+\frac{\beta}{\mu 5!}\Big(\frac{\xi^5}{3}-\frac{\xi^6}{6}\Big)+\frac{1}{\mu 4!}\Big(\gamma_1\eta_1^4\Big(\frac{1}{3}-\frac{\eta_1}{5}\Big)\Big)\Big]\\ &= \frac{161}{200}+\frac{\sigma^4}{80}\Big(\frac{1}{3}-\frac{\sigma}{10}\Big)\\ &= \frac{1932+10\sigma^4-3\sigma^5}{2400}.\end{aligned}$$

And

$$\begin{aligned}\Lambda_2 &= \frac{\sigma^8}{72}\int_\sigma^1\left[G(1,s)+\frac{\beta}{\mu}\Big(H(\xi,s)-H(\nu,s)\Big)+\frac{1}{\mu}\sum_{i=1}^{2}\gamma_i G(\eta_i,s)\right]ds\\ &= \frac{\sigma^8}{72}\Big[\frac{1}{72}(-1+\sigma)^3\Big(-\sigma(2-\sigma)-\frac{2}{5}(1-\sigma)^2\Big)-\frac{\beta}{360\mu}\Big(\xi^5(-1+\sigma)^3+\frac{(\xi-\sigma)^6}{2}\Big)-\frac{(-1+\sigma)^3}{72\mu}\gamma_1\eta_1^4\Big]\\ &= \frac{\sigma^8(1-\sigma)^3}{72}\Big[\frac{1}{360}(2+6\sigma-3\sigma^2)+\frac{1}{1200}\Big(2-(1-\sigma)^3\Big)+\frac{\sigma^4}{1440}\Big]\\ &= \frac{\sigma^8(1-\sigma)^3}{5184}\Big[\frac{1}{5}(2+6\sigma-3\sigma^2)+\frac{3}{50}\Big(1+3\sigma-3\sigma^2+\sigma^3\Big)+\frac{\sigma^4}{20}\Big]\\ &= \frac{\sigma^8(1-\sigma)^3\Big(46+138\sigma-78\sigma^2+6\sigma^3+5\sigma^4\Big)}{5184\times 10^2}.\end{aligned}$$

Let $M=\frac{61}{50}$, $\delta = 7$, *then*

$$\max_{(t,x+y+z)\in[0,1]\times[0,7]} f(t,x,y,z)\le 7<\frac{427}{50}=M\delta.$$

Putting

$$\Phi_1(\sigma)=\Lambda_1^{-1}-M,\ \Phi_2(\sigma)=\Lambda_2 f_0-1,\ \Phi_3(\sigma)=\Lambda_2 f_\infty-1.$$

We easily check that

$$\Phi_1(\sigma)>0,\ \Phi_2(\sigma)>0,\ \Phi_3(\sigma)>0,\ \textit{for all}\ \sigma\in\Big[\frac{7}{20},\frac{19}{20}\Big].$$

All conditions of Theorem 3.5 are fulfilled, then, the boundary value problem (16)-(17) *has at least two monotone and convex positive solutions. The graphical representation of* $\Phi_1(\sigma)$, $\Phi_2(\sigma)$ *and* $\Phi_3(\sigma)$ *are illustrated in Fig.3, Fig.4 and Fig.5 respectively.*

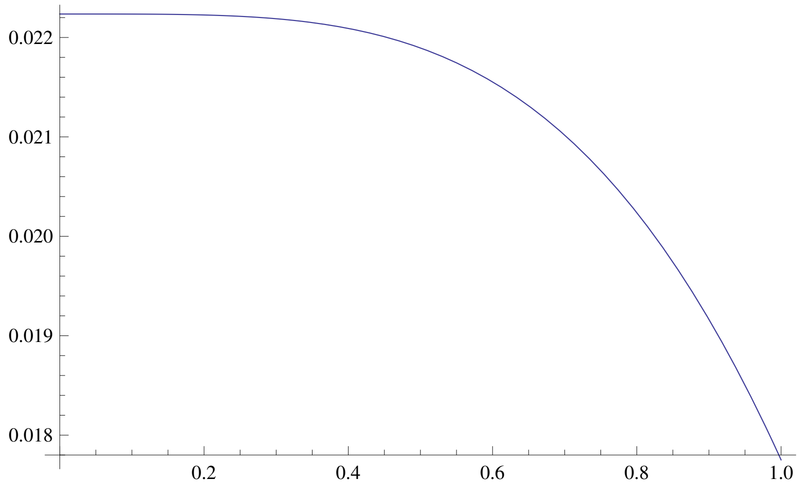

Figure 3: Graph of the function $\Phi_1(\sigma)$, $\sigma \in \left[0, 1\right]$.

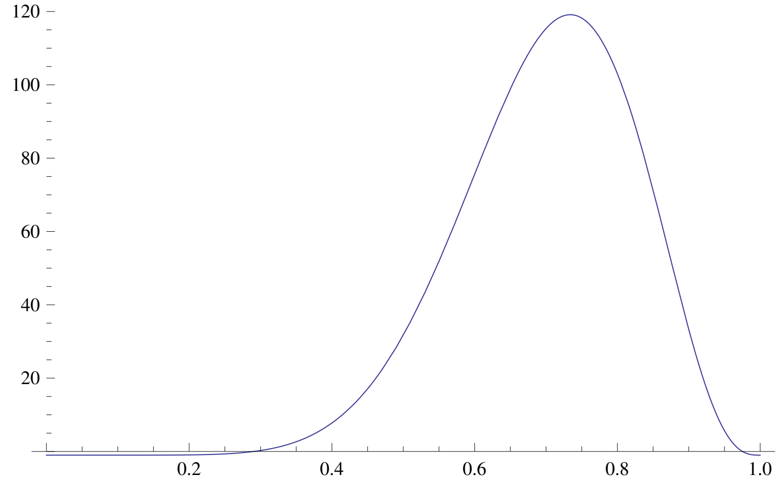

Figure 4: Graph of the function $\Phi_2(\sigma)$, $\sigma \in \left[0, 1\right]$.

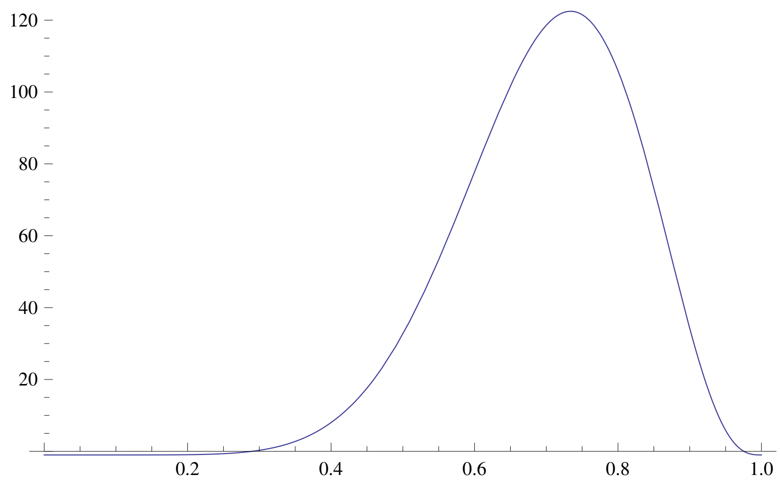

Figure 5: Graph of the function $\Phi_3(\sigma)$, $\sigma \in \left[0, 1\right]$.

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