# The extremal $t$-uniform unicyclic hypergraph on Estrada index

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**Abstract.** For a $t$-uniform hypergraph $H = (V(H), E(H))$, the Estrada index $EE(H)$ of $H$ is defined as $\sum_{i \in [n]} e^{\theta_i}$, where $\theta_1, \ldots, \theta_n$ are the eigenvalues of the adjacency matrix of $H$. In this paper, the extremal $t$-uniform unicyclic hypergraph which has maximum Estrada index are characterized.

## 1. Introduction

Let $H = (V(H), E(H))$ be a hypergraph with vertex set $V(H)(= [n] = \{1, 2, \ldots, n\})$ and edge set $E(H)$, where $E(H) \subseteq 2^{V(H)}$ and $2^{V(H)}$ stands for the power set of $V(H)$. A hypergraph $H$ is $t$-uniform if $|e| = t$ for any $e \in E(H)$, and 2-uniform hypergraphs are well-known ordinary graphs. For any nonempty subset $S$ of $V(H)$, the sub-hypergraph $H[S]$ induced by $S$ is $(S, E(H[S]))$, where $S$ is its vertex set and $E(H[S]) = \{V(e) \cap S : e \in E(H)\}$. Let $E(v) = \{e | v \in e \in E(H)\}$ and let $d(v)(= |E(v)|)$ be the degree of $v$ in $H$. For any $e \in E(v) \subseteq E(H)$, if $|e| \geq 3$, let $E(v) \setminus \{v\} = \{e \setminus \{v\} : e \in E(v)\}$ be obtained from $E(v)$ by $v$-shrinking on $E(v)$. Especially, we say that $e \setminus \{v\}$ is obtained from $e$ by $v$-shrinking on $e$. Let $N_H(v) = \{u | u, v \in e \in E(H)\}$. A vertex with degree 1 is called pendent vertex and an edge $e$ is called a pendent edge if $e$ contains exactly $|e| - 1$ pendent vertices.

A walk $W$ in $H$ is a sequence of alternating vertices and edges $v_0 e_1 v_1 e_2 \cdots e_p v_p$, where $v_{i-1}, v_i \in e_i$ for $i \in [p]$. If $v_i$ (resp. $e_i$) are all distinct for $i \in [p] \cup \{0\}$ (resp. $i \in [p]$), then $W$ is called a path. If $|e_i \cap e_j| \leq 1$ for $i \neq j, i, j \in [p]$, $W$ is also called a loose path. A cycle is a loose path satisfying $v_0 = v_p$. For any $u, v \in V(H)$, if is at least one path connecting $u$ with $v$, we say that $H$ is connected. An edge $e \in E(H)$ is a cut edge if $H - e$ is disconnected, where $H - e = (V(H), E(H) \setminus \{e\})$. For more detailed notations and terminologies related to walks, please see Table 1.

For a connected $t$-uniform hypergraph $H$ with order $n$ and size $\varepsilon$, its cyclomatic number $c(H)$ is defined as $\epsilon(t - 1) - n + 1$. In particular, 0-cyclic hypergraph and 1-cyclic hypergraph are called supertree and $t$-uniform unicyclic hypergraph, respectively. Let $\mathcal{U}(n)$ be the class of $t$-uniform unicyclic hypergraphs of order $n$. For $m, t \geq 3$, let $C_{m,t} = v_1 e_1 v_2 e_2 v_3 \cdots v_m e_m v_1$ be a $t$-uniform cycle of length $m$, for $i \in [m]$,

$$e_i = \{v_i, v_{i+1}, u_{i1}, u_{i2}, \ldots, u_{i,t-2}\}, \quad v_{m+1} = v_1,$$
$$V_0 = \{v_i : i \in [m]\}. \tag{1}$$

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Table 1: **Nomenclature**

| | |
|---|---|
| $W_k(H; u, v)$ | the set of $(u, v)$-walks of length $k$ in $H$ |
| $M_k(H; u, v)$ | $|W_k(H; u, v)|$, that is, the order of $W_k(H; u, v)$ |
| $W_k(H; u)$ | the set of $(u, u)$-walks of length $k$ in $H$ |
| $W_k(H)$ | the set of $\bigcup_{u \in V(H)} W_k(H; u)$ |
| $M_k(H)$ | $|W_k(H)|$ |
| $W_k(H; u, [v])$ | the set of $(u, u)$-walks of length $k$ which pass through $v$ in $H$ |
| $(H_1; u_1, v_1) \preceq (H_2; u_2, v_2)$ | $M_k(H_1; u_1, v_1) \leq M_k(H_2; u_2, v_2)$ for any positive integer $k$ |
| $(H_1; u_1, v_1) \prec (H_2; u_2, v_2)$ | $(H_1; u_1, v_1) \preceq (H_2; u_2, v_2)$ and $M_{k_0}(H_1; u_1, v_1) < M_{k_0}(H_2; u_2, v_2)$ for some positive integer $k_0$ |
| $(H_1; u_1) \preceq (H_2; u_2)$ | $M_k(H_1; u_1) \leq M_k(H_2; u_2)$ for any positive integer $k$ |
| $(H_1; u_1) \prec (H_2; u_2)$ | $(H_1; u_1) \preceq (H_2; u_2)$ and $M_{k_0}(H_1; u_1) < M_{k_0}(H_2; u_2)$ for some positive integer $k_0$ |

Let $\mathcal{U}(m, n)$ be the class of $t$-uniform unicyclic hypergraph of order $n$ obtained from $C_{m,t}$ (which is described as in (1.1)) by attaching $\frac{n}{t-1} - m$ pendent edges to some vertices in $V_0$. Note that $\epsilon(t - 1) - n + 1 = 1$, it has $\epsilon = \frac{n}{t-1}$ a nonnegative integer. For some other notations and terminologies which are not given in this paper, please refer to [1].

The adjacency matrix $A(H) = (a_{ij})$ of $H$ is defined as

$$a_{ij} = \begin{cases} |\{e \in E(H) : i, j \in e \subseteq V(H)|, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

The characteristic polynomial of $H$ is the determinant $|xI - A(H)|$, where $I$ is the identity matrix of order $n$. The eigenvalues $\theta_1, \ldots, \theta_n$ of $H$ are the zeroes of $|xI - A(H)|$.

In [5], Estrada and Rodrgíuea-Veláquez proposed a measure $\frac{1}{n} \sum_{i \in [n]} e^{\theta_i}$ to describe the sub-hypergraph centrality of $H$, which has found extensively applications such as in chemistry, complex networks and other interdisciplinary fields [5, 8]. In [11], the authors named directly $\sum_{i \in [n]} e^{\theta_i}$ as Estrada index of $H$, denoted it by $EE(H)$. More applications on Estrada index for ordinary graphs have been proposed, for examples, see [6, 7, 9, 10]. At the same time, many mathematical properties about the Estrada index of ordinary graphs have been obtained, such as [2–4, 13, 14] and references therein. Note that graphs are limited to representing pairwise relationships between entities. However, in real-world there are more multi-way relationships, which can be modeled by hypergraphs. So the applications of theory of hypergraphs have attracted the interest of many researchers, but this research is still in its beginning stage. Since Li et al. [11] generalized the notation of Estrada index from graphs to hypergraphs, few results in this field have been obtained. Recently, some extremal properties of Estrada index on $t$-uniform hypergraphs are obtained[12, 15, 16]. In this paper, we will continue to study the properties on Estrada index for hypergraphs.

The following lemmas are useful in our main results.

**Lemma 1.1.** [12] For a t-uniform hypergraph $H = (V(H), E(H))$ with a cut edge $e = \{u_1, u_2, \ldots, u_t\}$, $t \geq 3$, if $d_H(u_1) = 1$ and $d_H(u_2) \geq 2$, then $(H; u_1) \prec (H; u_2)$.

**Lemma 1.2.** [12] For a hypergraph $H = (V(H), E(H))$ with $u, v \in V(H)$ and $E_0 = \{e_1, e_2, \ldots, e_s\}$, where $e_h \in 2^{V(H)}, e_h \cap \{u, v\} = \emptyset$ for $h \in [s]$. Let

$$E_u = \{e'_h = e_h \cup \{u\} : e_h \in E_0, h \in [s]\},$$
$$E_v = \{e'_h = e_h \cup \{v\} : e_h \in E_0, h \in [s]\};$$

$H_u = H - E_0 + E_u$ and $H_v = H - E_0 + E_v$. If for any $w \in \bigcup_{h \in [s]} e_h$, $(H; u) \prec (H; v)$ and $(H; w, u) \preceq (H; w, v)$, then $EE(H_u) < EE(H_v)$.

## 2. Extremal *t*-uniform unicyclic hypergraph with maximum Estrada index

In this section, we consider the extremal structure of *t*-uniform unicyclic hypergraph with maximum Estrada index.

**Lemma 2.1.** *For a t-uniform unicyclic hypergraph H with unique cycle $C_{m,t}$ (which is described as in (1.1)), let $u \in V_0$ and $u_1$ be a pendent neighbor of u in H, then $(H; u_1) \prec (H; u)$.*

*Proof.* Let $u_1 \in f = \{u, u_1, u_2, \ldots, u_{t-1}\}$.

**Case 1.** $f \notin E(C_{m,t})$, then $f$ must be a cut edge of $H$, by Lemma 1.1, we have $(H; u_1) \prec (H; u)$.

**Case 2.** $f \in E(C_{m,t})$.

Let $A_1$ be the hypergraph obtained from $H$ by $u$-shrinking on $E(u)$, and $A_2$ be the hypergraph obtained from $H$ by $u_1$-shrinking on $E(u_1)$. Then

$$
\begin{aligned}
W_k(H; u_1) &= W_k(A_1; u_1) \cup W_k(H; u_1, [u]), \\
W_k(H; u) &= W_k(A_2; u) \cup W_k(H; u, [u_1]).
\end{aligned}
$$

Further we have

$$
\begin{aligned}
M_k(H; u_1) &= M_k(A_1; u_1) + M_k(H; u_1, [u]), \\
M_k(H; u) &= M_k(A_2; u) + M_k(H; u, [u_1]).
\end{aligned}
$$

It is easy to see that

$$
M_k(A_1; u_1) \le M_k(A_2; u) \ (resp., M_k(A_1; u_1, u_i)) \le M_k(A_2; u, u_i))
$$

for any nonegative integer $k$ and $i \in [t-2] \setminus \{1\}$, and for $k = 2$, by direct calculation, we have

$$
M_2(A_1; u_1) = t - 2 < t - 2 + (d_H(u) - 1)(t - 1) = M_2(A_2; u).
$$

For any $W \in W_k(H; u_1, [u])$, let $W = W_1 W_2$, where $W_1$ is either a walk which consists of $(u_1, u_1)$-section of length $l_1 - 1$ in $A_1$ and a $(u_1, f, u)$-section of length 1 or $(u_1, u_i)$-section of length $l_1 - 1$ in $A_1$ and a $(u_i, f, u)$-section of length 1 for $i \in \{2, \ldots, t-1\}$; and $W_2$ is a $(u, u_1)$-section with length $l_2$ in $H$. Then we have

$$
\begin{aligned}
M_k(H; u_1) &= M_k(A_1; u_1) + \sum_{l_1+l_2=k, l_1, l_2 \ge 1} M_{l_1-1}(A_1; u_1) M_{l_2}(H; u, u_1) + \\
&\quad \sum_{i=2}^{t-1} \sum_{l_1+l_2=k, l_1, l_2 \ge 1} M_{l_1-1}(A_1; u_1, u_i) M_{l_2}(H; u, u_1).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
M_k(H; u) &= M_k(A_2; u) + \sum_{l_1+l_2=k, l_1, l_2 \ge 1} M_{l_1-1}(A_2; u) M_{l_2}(H; u_1, u) + \\
&\quad \sum_{i=2}^{t-1} \sum_{l_1+l_2=k, l_1, l_2 \ge 1} M_{l_1-1}(A_2; u, u_i) M_{l_2}(H; u_1, u).
\end{aligned}
$$

Hence $M_k(H; u_1) \le M_k(H; u)$ for any nonegative integer $k$ and $M_2(H; u_1) < M_2(H; u)$. So $(H; u_1) \prec (H; u)$. $\square$

**Lemma 2.2.** *For $m \ge 2, t \ge 3$, let $H_0$ be a t-uniform unicyclic hypergraph with maximum Estrada index among t-uniform unicyclic hypergraphs of order n with unique cycle $C_{m,t}$ (which is described as in (1)), then $H_0 \in \mathcal{U}(m, n)$.*

*Proof.* Let $u$ be a vertex in $V_0$, if there exists a non-pendent neighbor $u_1$ of $u$ outside $C_{m,t}$ or in $V(C_{m,t}) \setminus V_0$, for $i \in [d_{H_0}(u_1) - 1]$, let $\tilde{e}_i$ be the edge which is incident to $u_1$, $e_i = \tilde{e}_i \setminus \{u_1\}$ but $u \notin e_i$. For convenience, let

$$
\begin{aligned}
E_0 &= \{e_1, e_2, \dots, e_{d_{H_0}(u_1)-1}\}, \\
E_u &= \{e_i \cup \{u\}, e_i \in E_0, i \in [d_{H_0}(u_1) - 1]\}, \\
E_{u_1} &= \{e_i \cup \{u_1\}, e_i \in E_0, i \in [d_{H_0}(u_1) - 1]\}.
\end{aligned}
$$

Further let $H = H_0 - E_0$. Then we have the following results:

(i) $d_H(u_1) = 1$ and $d_H(u) \geq 2$. By Lemma 2.1, $(H; u_1) \prec (H; u)$;

(ii) For any $w \in V(E_0)$, $M_k(H; w, u_1) = M_k(H; w, u) = 0$, then $(H; w, u_1) \preceq (H; w, u)$;

(iii) $H + E_u$ is also a $t$-uniform unicyclic hypergraphs of order $n$ with unique cycle $C_{m,t}$.
By Lemma 1.2, we have $EE(H_0) < EE(H + E_u)$, a contradiction. Thus $H_0 \in \mathcal{U}(m, n)$. $\square$

From Lemma 2.2, in order to obtain $t$-uniform unicyclic hypergraph with maximum Estrada index in $\mathcal{U}(n)$, we only need to consider the hypergraphs in $\mathcal{U}(m, n)$.

**Lemma 2.3.** *For $U \in \mathcal{U}(m, n)$ with unique cycle $C_{m,t}$ (which is described as in (1)), there exist $n_i(n_i \geq 0)$ pendent edges attaching at $v_i$ for $i \in [m]$. Let $H$ be the hypergraph obtained from $U$ by $v_1$-shrinking on $e_m$. If $m \geq 4$ and $n_1 \leq n_3$, then*

(i) $(H; v_1) \prec (H; v_3)$;

(ii) $(H; w, v_1) \preceq (H; w, v_3)$ for $w \in \{v_m, u_{m1}, u_{m2}, \dots, u_{m,t-2}\}$.

*Proof.* (i) Let $\hat{H}$ be the hypergraph obtained from $H$ by $v_3$-shrinking on $e_2$ and $H_1$ be the component of $\hat{H}$ containing $v_2$, and $\tilde{H}$ be the hypergraph obtained from $H$ by $v_1$-shrinking on $e_1$ and $H_2$ be the component of $\tilde{H}$ containing $v_2$. Obviously, $H_1$ is a proper sub-hypergraph of $H_2$. Then

(1) $(H_1; v_1) \prec (H_2; v_3)$, $(H_1; v_1, v_2) \prec (H_2; v_3, v_2)$ and $(H_1; v_1, u_{1i}) \preceq (H_2; v_3, u_{2i})$ for $i \in [t - 2]$.

(2) For any positive integer $k$,

$$
\begin{aligned}
M_k(H; v_1) &= M_k(H_1; v_1) + M_k(H; v_1, [v_3]), \\
M_k(H; v_3) &= M_k(H_2; v_3) + M_k(H; v_3, [v_1]).
\end{aligned}
$$

For any $W \in W_k(H; v_1, [v_3])$, let $W = W_1 W_2$, where $W_1$ is the shortest $(v_1, v_3)$-section, which consists of $(v_1, v_2)$-section (or $(v_1, u_{2i})$-section, $i \in [t - 2]$) of length $l_1 - 1$ in $H_1$ and $(v_2, e_2, v_3)$(or $(u_{2i}, e_2, v_3)$)-section of length 1, and $W_2$ is the remaining $(v_3, v_1)$-section of length $l_2$ in $H$. Then we have

$$
\begin{aligned}
M_k(H; v_1, [v_3]) = &\sum_{l_1 + l_2 = k, l_1, l_2 \geq 2} M_{l_1 - 1}(H_1; v_1, v_2) M_{l_2}(H; v_3, v_1) + \\
&\sum_{i=1}^{t-2} \sum_{l_1 + l_2 = k, l_1, l_2 \geq 2} M_{l_1 - 1}(H_1; v_1, u_{2i}) M_{l_2}(H; v_3, v_1).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
M_k(H; v_3, [v_1]) = &\sum_{l_1 + l_2 = k, l_1, l_2 \geq 2} M_{l_1 - 1}(H_2; v_3, v_2) M_{l_2}(H; v_1, v_3) + \\
&\sum_{i=1}^{t-2} \sum_{l_1 + l_2 = k, l_1, l_2 \geq 2} M_{l_1 - 1}(H_2; v_3, u_{1i}) M_{l_2}(H; v_1, v_3).
\end{aligned}
$$

Then $M_k(H; v_1, [v_3]) < M_k(H; v_3, [v_1])$. Hence $(H; v_1) \prec (H; v_3)$.

(ii) For $h \in [m]$ and $j \in [n_h]$, let $e_{hj} = \{v_h, v^1_{hj}, \ldots, v^{t-1}_{hj}\}$ be the pendent edges attaching at $v_h$, and

$$V_{v_h} = \bigcup_{j \in [n_h]} (e_{hj} \setminus \{v_h\}).$$

For a $W \in W_k(H; w, v_1)$ and $w \in \{v_m, u_{m1}, u_{m2}, \ldots, u_{m,t-2}\}$, let $W = W_1 W_2$, where $W_1$ is the longest $(w, v_2)$-section of $W$, and $W_2$ is the the remaining $(v_2, v_1)$-section of $W$ which the internal vertices (if exist) are only possible to belong to $V_{v_1} \cup \{u_{11}, u_{12}, \ldots, u_{1,t-2}\}$. Let $W'_2$ be obtained from $W_2$ by replacing $v_1$ by $v_3$, $u_{1k}$ by $u_{2k}$ for $k \in [t-2]$, and $v^s_{1j}$ by $v^s_{3j}$ for $j \in [n_1]$ and $s \in [t-1]$. Then $W' = W_1 W'_2 \in W_k(H; w, v_3)$. Thus $(H; w, v_1) \preceq (H; w, v_3)$. $\square$

By Lemma 2.3 and Lemma 1.2, we have the following result.

**Lemma 2.4.** *For a $U \in \mathcal{U}(m, n)$ with unique cycle $C_{m,t}$ (which is described as in (1)), there exist $n_i(n_i \geq 0)$ pendent edges attaching at $v_i$ for $i \in [m]$. Let $H$ be the hypergraph obtained from $U$ by $v_1$-shrinking on $e_m$ and $e_0 = e_m \setminus \{v_1\}$. If $m \geq 4$ and $n_3 = \max_{i \in [m]} n_i$, then $EE(U) < EE(H - e_0 + (e_0 \cup \{v_3\}))$.*

**Remark 1.** $H - e_0 + (e_0 \cup \{v_3\})$ is a $t$-uniform unicyclic hypergraphs of order $n$ and the length of its unique cycle is $m - 2$.

Let $C_3(n_1, n_2, n_3)$ be a $t$-uniform unicyclic hypergraphs obtained from $C_{3,t}$ by attaching $n_i$ pendent edges at $v_i$ for $i \in [3]$. Without loss of generality, let $n_1 \geq n_2 \geq n_3$ in the following.

**Lemma 2.5.** *If $n_1 \geq 1$ and $n_2 \geq 0$, then $(C_3(n_1, n_2, 0); v_3) \prec (C_3(n_1, n_2, 0); v_1)$.*

*Proof.* Let $\tilde{H}$ (resp. $\check{H}$) be the hypergraph obtained from $C_3(n_1, n_2, 0)$ by $v_1$-shrinking on $e_3$ and $v_2$-shrinking on $e_1$ (resp. $v_3$-shrinking on $e_3$ and $v_2$-shrinking on $e_2$) at the same time, and $H_1$ (resp. $H_2$) be the component of $\tilde{H}$ (resp. $\check{H}$) containing $v_2$. Obviously, $H_1$ is a proper sub-hypergraph of $H_2$. Then

(1) $(H_1; v_3) \prec (H_2; v_1)$;

(2) $(H_1; w, v_3) \prec (H_2; w, v_1)$ for $w \in \{u_{31}, u_{32}, \ldots, u_{3,t-2}\}$;

(3) For any positive integer $k$,

$$M_k(C_3(n_1, n_2, 0); v_3) = M_k(H_1; v_3) + M_k(C_3(n_1, n_2, 0); v_3, [v_1]),$$
$$M_k(C_3(n_1, n_2, 0); v_1) = M_k(H_2; v_1) + M_k(C_3(n_1, n_2, 0); v_1, [v_3]).$$

By (1), we know that $M_k(H_1; v_3) \leq M_k(H_2; v_1)$, and there exists at least one $k_0$ satisfying

$$M_{k_0}(H_1; v_3) \leq M_{k_0}(H_2; v_1).$$

Now we only need to prove $M_k(C_3(n_1, n_2, 0); v_3, [v_1]) \leq M_k(C_3(n_1, n_2, 0); v_1, [v_3])$.

In fact, for any $W \in W_k(C_3(n_1, n_2, 0); v_3, [v_1])$, let $W = W_1 W_2$, where $W_1$ is the shortest $(v_3, v_1)$-section, which consists of $(v_3, v_3)$-section (or $(v_3, u_{3i})$-section, $i \in [t-2]$) of length $l_1 - 1$ in $H_1$ and $(v_3, e_3, v_1)$(or $(u_{3i}, e_3, v_1)$)-section of length 1, and $W_2$ is the remaining $(v_1, v_3)$-section of length $l_2$ in $C_3(n_1, n_2, 0)$. Then we have

$$M_k(C_3(n_1, n_2, 0); v_3, [v_1]) = \sum_{l_1 + l_2 = k, l_1, l_2 \geq 1} M_{l_1 - 1}(H_1; v_3) M_{l_2}(C_3(n_1, n_2, 0); v_1, v_3) +$$
$$\sum_{i=1}^{t-2} \sum_{l_1 + l_2 = k, l_1, l_2 \geq 1} M_{l_1 - 1}(H_1; v_3, u_{3i}) M_{l_2}(C_3(n_1, n_2, 0); v_1, v_3).$$

Similarly,

$$M_k(C_3(n_1, n_2, 0); v_1, [v_3]) = \sum_{l_1 + l_2 = k, l_1, l_2 \geq 1} M_{l_1 - 1}(H_2; v_1) M_{l_2}(C_3(n_1, n_2, 0); v_3, v_1) +$$
$$\sum_{i=1}^{t-2} \sum_{l_1 + l_2 = k, l_1, l_2 \geq 1} M_{l_1 - 1}(H_2; v_1, u_{3i}) M_{l_2}(C_3(n_1, n_2, 0); v_3, v_1).$$

Then by (1) and (2), we have $M_k(C_3(n_1, n_2, 0); v_3, [v_1]) \le M_k(C_3(n_1, n_2, 0); v_1, [v_3])$. This completes the proof. □

**Lemma 2.6.** *If $n_1 \ge n_2 \ge n_3 \ge 1$, then $EE(C_3(n_1, n_2, n_3)) < EE(C_3(n_1 + n_3, n_2, 0))$.*

*Proof.* Let $E_1 = \{e_{31}, \ldots, e_{3,n_3}\}$ be set of the pendent edges attaching at $v_3$ and

$$E_0 = \{e_{31} \setminus \{v_3\}, \ldots, e_{3,n_3} \setminus \{v_3\}\}.$$

Denote $H = C_3(n_1, n_2, n_3) - E_1 + E_0$, now we have the following results.

(1) The component of $H$ containing $v_3$ is $C_3(n_1, n_2, 0)$. By Lemma 2.5, we know that

$$(C_3(n_1, n_2, 0); v_3) \prec (C_3(n_1, n_2, 0); v_1);$$

(2) For any vertex $w \in V(E_0)$, $M_k(C_3(n_1, n_2, 0); w, v_3) = M_k(C_3(n_1, n_2, 0); w, v_1) = 0$, then

$$(C_3(n_1, n_2, 0); w, v_3) \preceq (C_3(n_1, n_2, 0); w, v_1).$$

Further by Lemma 1.2, we obtain $EE(C_3(n_1, n_2, n_3)) < EE(C_3(n_1 + n_3, n_2, 0))$. □

**Remark 2.** For $H \in \mathcal{U}(3, n)$, repeated by lemmas 2.5 and 2.6, we have $EE(H) \le EE(C_3(n_1 + n_2 + n_3, 0))$ with equality if and only if $H \cong C_3(n_1 + n_2 + n_3, 0)$, where $n = (t - 1)(n_1 + n_2 + n_3 + 3)$.

Let $C_2(n_1, n_2)$ be a $t$-uniform unicyclic hypergraphs obtained from $C_{2,t}$ by attaching $n_i$ pendent edges at $v_i$ for $i \in [2]$.

**Lemma 2.7.** *If $n_1 \ge n_2 \ge 1$, then $EE(C_2(n_1, n_2)) < EE(C_2(n_1 + n_2, 0))$.*

*Proof.* Let $E_1 = \{e_{21}, \ldots, e_{2,n_2}\}$ be set of the pendent edges attaching at $v_2$ and

$$E_0 = \{e_{21} \setminus \{v_2\}, \ldots, e_{2,n_2} \setminus \{v_2\}\}.$$

Denote $H = C_2(n_1, n_2) - E_1 + E_0$, now we have the following results.

(1) The component of $H$ containing $v_2$ is $C_2(n_1, 0)$ and $(C_2(n_1, 0); v_2) \prec (C_2(n_1, 0); v_1)$;

(2) For any vertex $w \in V(E_0)$, $M_k(C_2(n_1, 0); w, v_2) = M_k(C_2(n_1, 0); w, v_1) = 0$, then

$$(C_2(n_1, 0); w, v_2) \preceq (C_2(n_1, 0); w, v_1).$$

Further by Lemma 1.2, we obtain $EE(C_2(n_1, n_2)) < EE(C_2(n_1 + n_2, 0))$. □
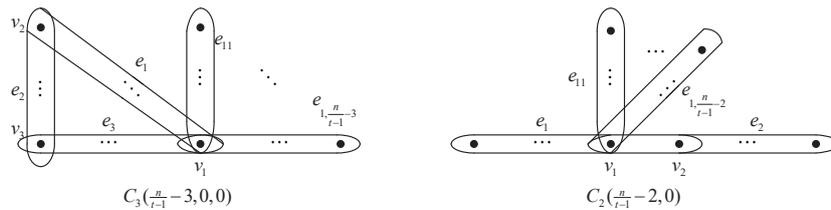


Figure 1: The hypergraphs $C_3(\frac{n}{t-1} - 3, 0, 0)$ and $C_2(\frac{n}{t-1} - 2, 0)$.

Let $G[C_3(\frac{n}{t-1} - 3, 0, 0)]$ (resp. $G[C_2(\frac{n}{t-1} - 2, 0)]$) be the graph obtained from $C_3(\frac{n}{t-1} - 3, 0, 0)$ (resp. $C_2(\frac{n}{t-1} - 2, 0)$) by replacing each edge of $C_3(\frac{n}{t-1} - 3, 0, 0)$ (resp. $C_2(\frac{n}{t-1} - 2, 0)$) by a complete graph $K_t$, as shown in Figure 2. Then $G[C_3(\frac{n}{t-1} - 3, 0, 0)]$ (resp. $G[C_2(\frac{n}{t-1} - 2, 0)]$) has the same adjacent matrix as $C_3(\frac{n}{t-1} - 3, 0, 0)$ (resp. $C_2(\frac{n}{t-1} - 2, 0)$). Hence $G[C_3(\frac{n}{t-1} - 3, 0, 0)]$ (resp. $G[C_2(\frac{n}{t-1} - 2, 0)]$) has the same characteristic polynomial as $C_3(\frac{n}{t-1} - 3, 0, 0)$ (resp. $C_2(\frac{n}{t-1} - 2, 0)$). Then

$$EE(G[C_3(\frac{n}{t-1} - 3, 0, 0)]) = EE(C_3(\frac{n}{t-1} - 3, 0, 0)), \tag{2}$$

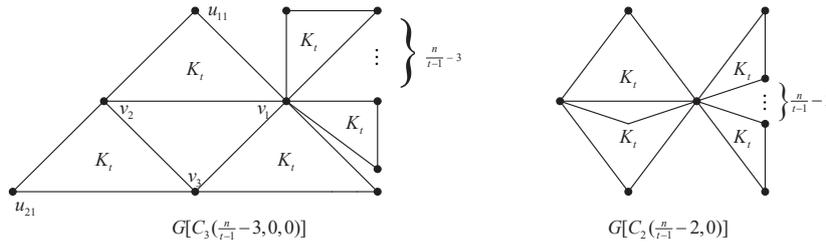$$EE(G[C_2(\frac{n}{t-1} - 2, 0)]) = EE(C_2(\frac{n}{t-1} - 2, 0)). \tag{3}$$

Figure 2: The graphs $G[C_3(\frac{n}{t-1} - 3, 0, 0)]$ and $G[C_2(\frac{n}{t-1} - 2, 0)]$.

**Lemma 2.8.** $EE(C_3(\frac{n}{t-1} - 3, 0, 0)) < EE(C_2(\frac{n}{t-1} - 2, 0))$.
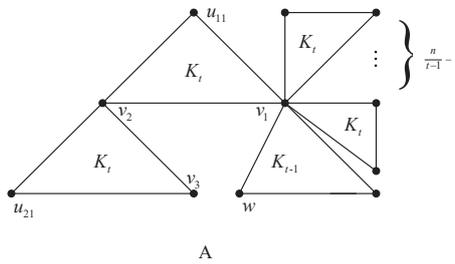


Figure 3: The graph $A$.

*Proof.* Let $A_1$ be the hypergraph obtained from $C_3(\frac{n}{t-1} - 3, 0, 0)$ (as shown in Figure 1) by $v_3$-shrinking on $e_3$, and $A$ (as shown in Figure 3) be the graph obtained from $A_1$ by replacing each edge $e$ of $A_1$ by a complete graph $K_{|e|}$.

Let $V_1 = \{v_3, u_{21}, u_{22}, \ldots, u_{2,t-2}\}$. Since $v_2$ is a cut-vertex of $A$, then for any $W \in W_k(A; v_3)$, it can be decomposed into two closed sections: $(v_3, v_3)$-walk $W_1$ whose internal vertices (if exist) are only possible to belong to $V_1 \cup \{v_2\}$; $(v_2, v_2)$-walk $W_2$ whose internal vertices (if exist) are only possible to belong to $V(A) - V_1$. Note that if the length of $W_2$ is at least 1, then $v_2$ must be an internal vertex of $W_1$.

Let $W_1'$ be a walk which is obtained from $W_1$ by replacing $v_3$ by $u_{11}$, $u_{21}$ by $v_1$ and $u_{2i}$ by $u_{1i}$ for $i \in \{2, 3, \ldots, t - 2\}$. Obviously, $W' = W_1' W_2 \in W_k(A; u_{11})$. It is easy to see that $M_4(A; v_3) < M_4(A; u_{11})$. Thus $(A; v_3) \prec (A; u_{11})$.

Let $V_2 = \{v_1, u_{31}, u_{32}, \ldots, u_{3,t-2}\}$. For any $W \in W_k(A; w, v_3)$ and $w \in V_2$, let $W = W_1 W_2$, where $W_1$ is the longest $(w, v_2)$-section of $W$ which the internal vertices (if exist) are only possible to belong to $V(A) - V_1$, and $W_2$ is the the remaining $(v_2, v_3)$-section of $W$ which the internal vertices (if exist) are only possible to belong to $V_1 \cup \{v_2\}$. $W_2'$ is obtained from $W_2$ by replacing $v_3$ by $u_{11}$, $u_{21}$ by $v_1$ and $u_{2i}$ by $u_{1i}$ for $i \in \{2, 3, \ldots, t - 2\}$. Obviously, $W' = W_1 W_2' \in W_k(A; w, u_{11})$. Thus $(A; w, v_3) \prec (A; w, u_{11})$.

Let $E_{v_3} = \{wv_3 : w \in \{v_1, u_{31}, \ldots, u_{3,t-2}\}\}$ and $E_{u_{11}} = \{wu_{11} : w \in \{v_1, u_{31}, \ldots, u_{3,t-2}\}\}$. By Lemma 1.2, we have $EE(A + E_{v_3}) < EE(A + E_{u_{11}})$.

Further by lemmas 2.2, 2.7 and (2), (3), we have $EE(C_3(\frac{n}{t-1} - 3, 0, 0)) < EE(C_2(\frac{n}{t-1} - 2, 0))$. $\square$

From lemmas 2.2, 2.7 and 2.8, Remark 2, we have our main result.

**Theorem 2.9.** *Let $H$ be a $t$-uniform unicyclic hypergraph, $EE(H) \leq EE(C_2(\frac{n}{t-1} - 2, 0))$ and the equality if and only if $H \cong C_2(\frac{n}{t-1} - 2, 0)$.*

## 3. Statements and Declarations

**Author Contributions:** The authors made equal contributions to this paper.
**Conflicts of Interest:** The authors declare no conflict of interest.

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