Filomat 37:19 (2023), 6277–6288
https://doi.org/10.2298/FIL2319277C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Inequalities on a class of analytic functions defined by generalized Mittag-Leffler function

Murat Çağlar[a], **K. R. Karthikeyan**[b], **G. Murugusundaramoorthy**[c]

[a]*Department of Mathematics, Faculty of Science, Erzurum Technical University, Erzurum, Türkiye*
[b]*Department of Applied Mathematics and Science, National University of Science & Technology, Muscat, Oman*
[c]*Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Deemed to be University, Vellore, Tamilnadu, India*

Abstract. By making use of the generalized difference operator, we have defined a new class of λ-pseudo Pascu type functions of complex order using subordination. Interesting results such as subordination results, inequalities for the initial Taylor-Maclaurin coefficients and unified solution of Fekete-Szegő problem have been obtained. Also, the study has been extended to quantum calculus by replacing the ordinary derivative with a q-derivative in the defined function class. Several applications, known or new of the main results are also presented.

1. Introduction and Preliminaries

Current developments and numerous extensions of well-known special transcendental functions, we referred to expository articles by Srivastava [1–5]. In this vein, Srivastava in [1, 2] has allied methodological principles in expository writing so as to convey such a representative insight into the diversity of the *Special Function Theory*. The outcome is a unique and masterly primer of articles are very much comprehensive and self-contained pertaining to the study of higher transcendental functions. Further, Srivastava in [3] detailed some recent developments and potential directions for further researches which can be based on a non-trivial family of the Riemann-Liouville type fractional integrals and fractional derivatives. The main highlight of the article is that, providing extensions and generalizations of known and readily accessible definitions and results by introducing some obviously redundant and seemingly inconsequential parameters or by changing the variable of integration in an integral definition. As long ago as 1940, Wright [6] investigated a rather general form of the various multi-parameter extensions of the Mittag-Leffler functions. in which he introduced and systematically studied the asymptotic expansion of the following Taylor-Maclaurin series (see [6, p. 424]):

$$E_{\theta,\vartheta}(\chi,z)=\sum_{n=0}^{\infty}\frac{\chi(n)z^n}{\Gamma(\theta n+\vartheta)}\quad (z,\ \theta,\ \vartheta,\in\mathbb{C}, Re(\theta)>0),$$

2020 *Mathematics Subject Classification*. 30C45; 30C80.

Keywords. Analytic functions; Bazilevič functions; Starlike and convex functions; Subordination; Fekete-Szegő problem; Coefficient inequalities; q-calculus; Jackson's q-derivative operator.

Received: 04 February 2022; Accepted: 14 February 2023

Communicated by Hari M. Srivastava

Email addresses: mcaglar25@gmail.com (Murat Çağlar), karthikeyan@nu.edu.com (K. R. Karthikeyan), gmsmoorthy@yahoo.com (G. Murugusundaramoorthy)

where $\chi(n)$ is a function satisfying suitable conditions. In his recent survey-cum-expository review articles, Srivastava [1-3] introduced and investigated a hybrid unification of various multiparameter generalizations of the Mittag-Leffler function and the Hurwitz-Lerch zeta function in the following form (see [1, p. 9, Eq. (25)],[2, p. 140, Eq. (15)], and [3, p. 1516, Eq. (8.4)])

$$E^{a}_{\theta,\vartheta}(\tau;z) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^{\tau}\Gamma(\theta n+\vartheta)}, \quad (z,\ \theta,\ \vartheta, \in \mathbb{C}, Re(\theta)>0). \tag{1}$$

Srivastava [1–3] also used the hybrid unification in (1) as the kernel of a general form of the operators of the Riemann-Liouville type fractional calculus. For comprehensive study on the extensions of the Mittag-Leffler functions and its relationship with higher transcendental functions, we refer to survey cum expository articles by Srivastava [1–5] .Srivastava et al. [11, Eq. 8] considered the following family of the multi-index Mittag-Leffler functions as a kernel of some fractional-calculus operators

$$E^{\gamma,\omega,\delta,\epsilon}_{(\theta_j,\vartheta_j)_m}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\omega n}\,(\delta)_{\epsilon n}}{\prod_{j=1}^{m}\Gamma\left(\theta_j n+\vartheta_j\right)}\frac{z^n}{n!}, \tag{2}$$

$$\left(\theta_j,\ \vartheta_j,\ \gamma,\ \omega,\ \delta,\ \epsilon \in \mathbb{C};\ \mathrm{Re}(\theta_j)>0;\ \mathrm{Re}\left(\sum_{j=1}^{m}\theta_j\right) > \mathrm{Re}(\omega+\epsilon)-1\right).$$

Many researchers studied well-known families of *Univalent Function Theory* involving the *Fox H-function* [12] (also see [13, p. 271]) defined by a Mellin–Barnes integral, which is a generalization of the *Meijer G-function* (see [14, p. 45]) and the *Fox–Wright function* (see [2, Definition 2]) has almost all the special functions as its special cases(see[15–18]) and Mittag-Leffler functions by Attiya [19], Srivastava and El-Deeb [20], Srivastava et al. [21–24] , Tomovski et al. [25] (also see [26], Hurwitz-Lerch Zeta functions [27–30], also by Sălăgean-difference operator [31, 32] . Motivated by aforementioned studies on *Univalent Function Theory* in this article we defined a new generalized operator given in (4) denoted by $D^m_k(\theta,\ \vartheta,\ \rho)$ and a new class of λ-pseudo Pascu type functions of complex order using subordination. Interesting results such as subordination results, inequalities for the initial Taylor-Maclaurin coefficients and unified solution of Fekete-Szegő problem have been obtained.

2. λ-pseudo Pascu type functions of complex order

Let $\mathcal{H}$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z : |\ z\ |< 1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots$. Let

$$\mathcal{A}_n = \{f \in \mathcal{H},\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots\}$$

and let $\mathcal{A} = \mathcal{A}_1$. Ibrahim and Darus [31] introduced the Sălăgean-difference operator $D^m_k f(z) : \mathcal{A} \to \mathcal{A}$ which is defined by

$$D^m_k f(z) = z + \sum_{n=2}^{\infty}\left[n + \frac{k}{2}(1+(-1)^{n+1})\right]^m a_n z^n, \qquad m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}. \tag{3}$$

If we let $k = 0$, then $D^m_k f(z)$ reduces to the well-known Sălăgean differential operator. $D^m_k f(z)$ is a modified Dunkl operator of complex variables which has many applications in the field of algebra and complex analysis, for details refer to [32] and references provided therein.

Using Hadamard product, we now define the following operator $D^m_k(\theta,\ \vartheta,\ \rho)f : \mathbb{U} \longrightarrow \mathbb{U}$ by

$$D^m_k(\theta,\ \vartheta,\ \rho)f(z) = \left[D^m_k f(z) * \mathcal{R}^{\rho}_{\theta,\vartheta}(z)\right] = z + \sum_{n=2}^{\infty}\left[n+\frac{k}{2}(1+(-1)^{n+1})\right]^m \frac{\Gamma(\vartheta)(\rho)_{n-1}}{\Gamma(\vartheta+\theta(n-1))\,(n-1)!}a_n z^n. \tag{4}$$

Remark 2.1. *Here we list only few special cases of the operator $D_k^m(\theta,\ \vartheta,\ \rho)f$, for details refer to [33] and references provided therein.*

1. *The operator $D_k^m(\theta,\ \vartheta,\ \rho)f$ is closely related to the operator recently used by Mashwan et. al. [33].*
2. *If we let $\theta = 0$ and $\rho = 1$ in (4), then $D_k^m(\theta,\ \vartheta,\ \rho)f$ reduces to $D_k^m f(z)$ defined by Ibrahim and Darus [31, 32].*

Let $\mathcal{P}$ denote the class of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ that are analytic in $\mathbb{U}$ and $Re\{p(z)\} > 0$ for all z in $\mathbb{U}$. Throughout this paper, we shall assume that $\Phi(z)$ is an analytic function in $\mathbb{U}$ such that $Re\{\Phi(z)\} > 0$ $(z \in \mathbb{U})$. Further, we assume that $\Phi(z)$ has a power series expansion of the form

$$\Phi(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 + \cdots,\ z \in \mathbb{U},\ L_1 \neq 0. \tag{5}$$

Using the operator $D_k^m(\theta,\ \vartheta,\ \rho)f$, we now introduce the following the class of functions using the principle of subordination:

Definition 2.2. *Let the class $\mathcal{NB}_k^{\lambda,m}(\alpha;\ \theta,\ \vartheta,\ \rho;\ b;\ \Phi)$ consist of function in $\mathcal{A}$ satisfying the subordination condition*

$$1 + \frac{1}{b}\left\{\frac{2z^{1-\lambda}\left[(1-\alpha)D_k^{m+1}(\theta,\ \vartheta,\ \rho)f(z) + \alpha D_k^{m+2}(\theta,\ \vartheta,\ \rho)f(z)\right]^{\lambda}}{(1-\alpha)H_k^m(\theta,\ \vartheta,\ \rho)f(z) + \alpha H_k^{m+1}(\theta,\ \vartheta,\ \rho)f(z)} - 1\right\} \prec \Phi(z), \tag{6}$$

where $H_k^m(\theta,\ \vartheta,\ \rho)f = \left[D_k^m(\theta,\ \vartheta,\ \rho)f(z) - D_k^m(\theta,\ \vartheta,\ \rho)f(-z)\right]$, $0 \leq \alpha \leq 1$, $\lambda \geq 1$, $b \in \mathbb{C}\setminus\{0\}$, $\theta,\ \vartheta,\ \rho \in \mathbb{C}$, $Re(\theta) > 0$.

Remark 2.3. *Here we will point out some special cases of our class $\mathcal{NB}_k^{\lambda,m}(\alpha;\ \theta,\ \vartheta,\ \rho;\ b;\ \Phi)$.*

1. *If we let $\theta = k = \alpha = 0$, $\lambda = \rho = 1$ and $\phi(z) = (1 + Xz)/(1 + Yz)$, the class $\mathcal{NB}_k^{\lambda,m}(\alpha;\ \theta,\ \vartheta,\ \rho;\ b;\ \Phi)$ will reduce to the class of functions*

$$\mathbb{J}^b(X, Y, m) = \left\{f \in \mathcal{A} :\ 1 + \frac{1}{b}\left(\frac{2D_0^{m+1}f(z)}{D_0^m f(z) - D_0^m f(-z)} - 1\right) \prec \frac{1 + Xz}{1 + Yz}\right\}.$$

The class $\mathbb{J}^b(X, Y, m)$ was recently introduced and studied by Arif et al. in [34].

2. *If we let $\theta = \alpha = 0$, $\lambda = \rho = 1$ and $\phi(z) = (1 + Xz)/(1 + Yz)$, the class $\mathcal{NB}_k^{\lambda,m}(\alpha;\ \theta,\ \vartheta,\ \rho;\ b;\ \Phi)$ will reduce to the class of functions $\mathbb{J}_k^b(X, Y, m)$ recently introduced and studied by Ibrahim in [32].*
3. *If we let $b = 1$ and $\phi(z) = (1 + Xz)/(1 + Yz)$, the class $\mathcal{NB}_k^{\lambda,m}(\alpha;\ \theta,\ \vartheta,\ \rho;\ b;\ \Phi)$ will reduce to the class of functions closely related to the class recently introduced and studied by Mashwan et al. [33].*

3. Conditions For Starlikeness and Coefficient Inequalities of $\mathcal{NB}_k^{\lambda,m}(\alpha;\ \theta,\ \vartheta,\ \rho;\ b;\ \Phi)$

Throughout this paper, we let

$$D_k^m(\theta,\ \vartheta,\ \rho)f'(z) = [D_k^m(\theta,\ \vartheta,\ \rho)f(z)]' \quad \text{and} \quad H_k^m(\theta,\ \vartheta,\ \rho)f'(z) = [H_k^m(\theta,\ \vartheta,\ \rho)f(z)]'.$$

By Definition 2.2, the superordinate function Φ is assumed to be in $\mathcal{P}$, it is well-known that class of function $\mathcal{P}$ need not be univalent. Throughout this section, we assume $\Phi \in \mathcal{P}$ to be convex univalent in $\mathbb{U}$.

If $f \in \mathcal{NB}_k^{1,m}(\alpha;\ \theta,\ \vartheta,\ \rho;\ b;\ \Phi)$, then by Definition 2.2 there exist a function $p(z) \in \mathcal{P}$ with $p(z) \prec \Phi(z)$ such that

$$\frac{2\left[(1-\alpha)D_k^{m+1}(\theta,\ \vartheta,\ \rho)f(z) + \alpha D_k^{m+2}(\theta,\ \vartheta,\ \rho)f(z)\right]}{(1-\alpha)H_k^m(\theta,\ \vartheta,\ \rho)f(z) + \alpha H_k^{m+1}(\theta,\ \vartheta,\ \rho)f(z)} = 1 + b\left[p(z) - 1\right]. \tag{7}$$

Replacing z by $-z$ in (7)

$$\frac{-2\left[(1-\alpha)D_k^{m+1}(\theta,\vartheta,\rho)f(-z)+\alpha D_k^{m+2}(\theta,\vartheta,\rho)f(-z)\right]}{(1-\alpha)H_k^{m}(\theta,\vartheta,\rho)f(z)+\alpha H_k^{m+1}(\theta,\vartheta,\rho)f(z)}=1+b\left[p(-z)-1\right]. \tag{8}$$

Adding (7) and (8), we have the following after simplification

$$1+\frac{1}{b}\left\{\frac{(1-\alpha)D_k^{m+1}(\theta,\vartheta,\rho)h(z)+\alpha D_k^{m+2}(\theta,\vartheta,\rho)h(z)}{(1-\alpha)D_k^{m}(\theta,\vartheta,\rho)h(z)+\alpha D_k^{m+1}(\theta,\vartheta,\rho)h(z)}-1\right\}\prec\frac{p(z)+p(-z)}{2}, \tag{9}$$

with $D_k^m(\theta,\vartheta,\rho)h(z)=\frac{D_k^m(\theta,\vartheta,\rho)f(z)-D_k^m(\theta,\vartheta,\rho)f(-z)}{2}$. If we assume $\Phi(z)$ to be univalent, then from (9) follows that $\frac{p(z)+p(-z)}{2}\prec\Phi(z)$.

On summarising the above discussion, we have the following.

Theorem 3.1. *Let the function $\Phi\in\mathcal{P}$ be convex univalent in $\mathbb{U}$. If $f\in\mathcal{NB}_k^{1,m}(\alpha;\theta,\vartheta,\rho;b;\Phi)$, then the odd function*

$$h(z)=\frac{1}{2}[f(z)-f(-z)]$$

satisfies

$$1+\frac{1}{b}\left\{\frac{(1-\alpha)D_k^{m+1}(\theta,\vartheta,\rho)h(z)+\alpha D_k^{m+2}(\theta,\vartheta,\rho)h(z)}{(1-\alpha)D_k^{m}(\theta,\vartheta,\rho)h(z)+\alpha D_k^{m+1}(\theta,\vartheta,\rho)h(z)}-1\right\}\prec\Phi(z).$$

Remark 3.2. *For appropriate choice of the parameters involved, we can deduce the results obtained by Arif et al. [34, Theorem 4] and Ibrahim [32, Theorem 2.1.].*

3.1. Conditions For Starlikeness

Motivated by the results presented in Chapter 4 of [35], here we obtain some conditions for starlikeness. We now state the following result which will be used in the sequel.

Lemma 3.3. *[35, Theorem 3.6.1.] Let the function q be univalent in the open unit disc $\mathbb{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$ with $\phi(w)\neq 0$ when $w\in q(\mathbb{U})$. set $Q(z)=zq'(z)\phi(q(z))$, $h(z)=\theta(q(z))+Q(z)$. Suppose that*

1. *Q is starlike univalent in $\mathbb{U}$, and*
2. *$Re(\frac{zh'(z)}{Q(z)})>0$ for $z\in\mathbb{U}$.*

If $\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z))$, then $p(z)\prec q(z)$ and q is the best dominant.

For conivenience, we denote $\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)=(1-\alpha)D_k^{m+1}(\theta,\vartheta,\rho)f(z)+\alpha D_k^{m+2}(\theta,\vartheta,\rho)f(z)$ and $\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)=(1-\alpha)H_k^m(\theta,\vartheta,\rho)f(z)+\alpha H_k^{m+1}(\theta,\vartheta,\rho)f(z)$.

Theorem 3.4. *Let the function $\Phi\in\mathcal{P}$ be convex univalent in $\mathbb{U}$ and let $\kappa(z):=t\Phi^2(z)+tz\Phi'(z)+(s-t)\Phi(z)$ with $t>0$ and $s>t$. If the function $f\in\mathcal{A}$ satisfies the conditions*

$$\frac{H_k^m(\theta,\vartheta,\rho)f(z)}{H_k^{m+1}(\theta,\vartheta,\rho)f(z)}\neq-\frac{\alpha}{1-\alpha}\quad \textit{and}\quad \frac{D_k^{m+1}(\theta,\vartheta,\rho)f(z)}{D_k^{m+2}(\theta,\vartheta,\rho)f(z)}\neq-\frac{\alpha}{1-\alpha},$$

then

$$\frac{1}{b}\left\{\frac{2z^{1-\lambda}\left[\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)\right]^{\lambda}}{\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)}-1\right\}\times\left[(s+(b-1)t-b\lambda t)+bt\lambda\frac{z\left[\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)\right]'}{\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)}-b\frac{tz\left[\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)\right]'}{\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)}\right.$$
$$\left.+\frac{t}{b}\left\{\frac{2z^{1-\lambda}\left[\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)\right]^{\lambda}}{\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)}-1\right\}\right]+t\left(\lambda\frac{z\left[\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)\right]'}{\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)}-\frac{z\left[\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)\right]'}{\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)}-\lambda+1\right)+s\prec\kappa(z), \tag{10}$$

implies $f\in\mathcal{NB}_k^{\lambda,m}(\alpha;\theta,\vartheta,\rho;b;\Phi)$. *Moreover, the function* Φ *is the best dominant of the left-hand side of* (6).

Proof. If we define the function p by

$$p(z):=1+\frac{1}{b}\left\{\frac{2z^{1-\lambda}\left[\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)\right]^{\lambda}}{\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)}-1\right\},\ z\in\mathbb{U},$$

then from the hypothesis, it follows that p is analytic in $\mathbb{U}$. By a straight forward computation, we have

$$bzp'(z)=1+b\left[p(z)-1\right]\left[\lambda\frac{z\left[\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)\right]'}{\mathcal{T}_k^m(\theta,\vartheta,\rho)f(z)}-\frac{z\left[\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)\right]'}{\mathcal{R}_k^m(\theta,\vartheta,\rho)f(z)}-\lambda+1\right],$$

and thus, the subordination (10) is equivalent to

$$tp^2(z)+tzp'(z)+(s-t)p(z)\prec\kappa(z). \tag{11}$$

Setting $\Omega(w):=tw^2+(s-t)w$ and $\Upsilon(w):=t$, then Ω and Υ are analytic functions in $\mathbb{C}$, with $\Upsilon(0)\neq 0$. Therefore

$$Q(z)=z\Phi'(z)\Upsilon(\Phi(z))=tz\Phi'(z)\quad\text{and}\quad\kappa(z)=\Omega(\Phi(z))+Q(z)=t\Phi^2(z)+tz\Phi'(z)+(s-t)\Phi(z),$$

and using the fact that Φ is a convex univalent function in $\mathbb{U}$, it follows that

$$Re\frac{zQ'(z)}{Q(z)}=Ret\left(1+\frac{z\Phi''(z)}{\Phi'(z)}\right)>0,\ z\in\mathbb{U},\quad(Q'(0)=tg'(0)\neq 0),$$

hence Q is a starlike univalent function in $\mathbb{U}$. Further, the convexity of Φ together with $\Re[\Phi(z)]>0$ implies

$$Re\frac{z\kappa'(z)}{Q(z)}=Re\left\{2\Phi(z)+\frac{z\Phi''(z)}{\Phi'(z)}+\frac{\beta}{\alpha}\right\}>0,\ z\in\mathbb{U}.$$

Since both of the conditions of Lemma 3.3 are satisfied it follows that (11) implies $p(z)\prec\Phi(z)$, and Φ is the best dominant of p, which prove our conclusions. □

Theorem 3.5. *If the function* $f\in\mathcal{A}$ *satisfies the conditions* $\frac{\left[D_k^m f(z)-D_k^m f(-z)\right]}{z}\neq 0$ *and let* $\kappa(z)=\frac{X^2z^2+(3X-Y)z+1}{(1+Yz)^2}$, $-1\leq Y<X\leq 1$.*Then*

$$\frac{1}{b}\left(\frac{2D_k^{m+1}f(z)}{D_k^m f(z)-D_k^m f(-z)}-1\right)\left[2+\frac{z\left(D_k^{m+1}f(z)\right)'}{D_k^{m+1}f(z)}-\frac{z\left[D_k^m f(z)-D_k^m f(-z)\right]'}{\left[D_k^m f(z)-D_k^m f(-z)\right]}+\frac{1}{b}\left(\frac{2D_k^{m+1}f(z)}{D_k^m f(z)-D_k^m f(-z)}-1\right)\right]+$$
$$\frac{\lambda}{b}\frac{z\left(D_k^{m+1}f(z)\right)'}{D_k^{m+1}f(z)}-\frac{z\left[D_k^m f(z)-D_k^m f(-z)\right]'}{b\left[D_k^m f(z)-D_k^m f(-z)\right]}+1\prec\kappa(z),$$

implies $f\in\mathbb{J}_k^b(X,Y,m)$ *[see Remark 2.3].*

Proof. If we define the functions

$$\Phi(z) = \frac{1+Xz}{1+Yz} \quad \text{and} \quad p(z) = 1 + \frac{1}{b}\left(\frac{2D_k^{m+1}f(z)}{D_k^m f(z) - D_k^m f(-z)} - 1\right),$$

then p is analytic in $\mathbb{U}$, and Φ is a convex univalent function in $\mathbb{U}$ with $Re\{\Phi(z)\} > 0$, $z \in \mathbb{U}$. Proceeding as in the proof of Theorem 3.4 with $\alpha = 0$ and $\lambda = s = t = 1$, we can establish the assertion of the Theorem 3.5. □

As a consequence of Theorem 3.5, we get the following result.

Corollary 3.6. *If the function $f \in \mathcal{A}$ satisfies the conditions $\frac{[f(z)-f(-z)]}{z} \neq 0$ and let $\kappa(z) = \frac{X^2z^2+(3X-Y)z+1}{(1+Yz)^2}$, $-1 \le Y < X \le 1$. Then*

$$\frac{2zf'(z)}{f(z)-f(-z)}\left[1 + \frac{zf''(z)}{f'(z)} - \frac{z\left[f(z)-f(-z)\right]'}{f(z)-f(-z)} + \left(\frac{2zf'(z)}{f(z)-f(-z)}\right)\right] \prec \kappa(z) \quad \Longrightarrow \quad \frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+Xz}{1+Yz}.$$

Remark 3.7. *If we let $\alpha = 1$ and by choosing appropriate values to the other parameters involved, we can obtain the conditions for convexity.*

3.2. Solution to Fekete-Szegő Problem for the Functions of $\mathcal{NB}_k^{\lambda,m}(\alpha;\theta,\vartheta,\rho;b;\Phi)$

We will give the solution of the Fekete-Szegő problem for the functions that belong to the classes we defined in the first section.

Lemma 3.8. *[36] If $p(z) = 1 + \sum\limits_{k=1}^{\infty} p_k z^k \in \mathcal{P}$, and v is complex number, then $\left|p_2 - vp_1^2\right| \le 2\max\{1; |2v-1|\}$ and the result is sharp for the functions $p_1(z) = \frac{1+z}{1-z}$ and $p_2(z) = \frac{1+z^2}{1-z^2}$.*

Throughout this subsection, we denote Π_n to be of the form

$$\Pi_n = \frac{\Gamma(\vartheta)(\rho)_n}{\Gamma(\vartheta + \theta(n-1))(n-1)!}.$$

Theorem 3.9. *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{NB}_k^{\lambda,m}(\alpha;\theta,\vartheta,\rho;b;\Phi)$, then for all $\mu \in \mathbb{C}$ we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|bL_1|}{(2+k)(3+k)^m[1+\alpha(2+k)]|\Pi_3|}\max\{1; |2\Upsilon - 1|\},$$

where Υ is given by

$$\Upsilon := \frac{1}{4}\left(2 - \frac{2L_2}{L_1} + \frac{bL_1(\lambda-1)}{\lambda} + \frac{\mu bL_1(2+k)(3+k)^m[1+\alpha(2+k)]\Pi_3}{\lambda^2(1+\alpha)^2 2^{2m+2}\Pi_2^2}\right).$$

The inequality is sharp for each $\mu \in \mathbb{C}$.

Proof. As $f \in \mathcal{NB}_k^{\lambda,m}(\alpha;\theta,\vartheta,\rho;b;\Phi)$, by (6) we have

$$\frac{2z^{1-\lambda}\left[(1-\alpha)D_k^{m+1}(\theta,\vartheta,\rho)f(z) + \alpha D_k^{m+2}(\theta,\vartheta,\rho)f(z)\right]^{\lambda}}{(1-\alpha)H_k^m(\theta,\vartheta,\rho)f(z) + \alpha H_k^{m+1}(\theta,\vartheta,\rho)f(z)} = 1 + b\{\Phi[w(z)] - 1\} \tag{12}$$

where $w(z) = \frac{p(z)-1}{p(z)+1}$ is a Schwartz function. The left hand side of (12) is given by

$$\frac{2z^{1-\lambda}\left[(1-\alpha)D_k^{m+1}(\theta,\vartheta,\rho)f(z) + \alpha D_k^{m+2}(\theta,\vartheta,\rho)f(z)\right]^{\lambda}}{(1-\alpha)H_k^m(\theta,\vartheta,\rho)f(z) + \alpha H_k^{m+1}(\theta,\vartheta,\rho)f(z)} = 1 + \left[\lambda(1+\alpha)2^{m+1}\Pi_2\right]a_2z +$$

$$\left[(\alpha(2+k)+1)(2+k)(3+k)^m\Pi_3a_3 + 2^{2m+1}\lambda(\lambda-1)(1+2\alpha)^2\Pi_2^2a_2^2\right]z^2 + \cdots. \tag{13}$$

From [37, Theorem 4], it can be easily seen that the right hand side of (12)

$$1+b\left\{\Phi[w(z)]-1\right\}=1+b\left[\frac{p_1L_1}{2}z+\frac{L_1}{2}\left[p_2-\frac{p_1^2}{2}\left(1-\frac{L_2}{L_1}\right)\right]z^2+\cdots\right]. \tag{14}$$

From (13) and (14), we obtain

$$a_2=\frac{bp_1L_1}{2\left[\lambda(1+\alpha)2^{m+1}\Pi_2\right]} \tag{15}$$

and

$$a_3=\frac{bL_1}{2(2+k)(3+k)^m[1+\alpha(2+k)]\Pi_3}\left[p_2-\frac{1}{4}\left(2-\frac{2L_2}{L_1}+\frac{bL_1(\lambda-1)}{\lambda}\right)p_1^2\right]. \tag{16}$$

To prove the Fekete-Szegő inequality for the class $\mathcal{NB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,b;\,\Phi)$, we consider

$$\left|a_3-\mu a_2^2\right|\le\frac{|bL_1|}{2(2+k)(3+k)^m[1+\alpha(2+k)]|\Pi_3|}\left[2+\frac{|p_1|^2}{4}\left(\left|\frac{2L_2}{L_1}-\frac{bL_1(\lambda-1)}{\lambda}\right.\right.\right.$$
$$\left.\left.\left.-\frac{\mu bL_1(2+k)(3+k)^m[1+\alpha(2+k)]\Pi_3}{\lambda^2(1+\alpha)^2 2^{2m+2}\Pi_2^2}\right|-2\right)\right]. \tag{17}$$

Denoting

$$\Upsilon:=\left|\frac{2L_2}{L_1}-\frac{bL_1(\lambda-1)}{\lambda}-\frac{\mu bL_1(2+k)(3+k)^m[1+\alpha(2+k)]\Pi_3}{\lambda^2(1+\alpha)^2 2^{2m+2}\Pi_2^2}\right|,$$

if $\Upsilon\le 2$, from (17) we obtain

$$\left|a_3-\mu a_2^2\right|\le\frac{|bL_1|}{(2+k)(3+k)^m[1+\alpha(2+k)]|\Pi_3|}. \tag{18}$$

Further, if $\Upsilon\ge 2$ from (17) we deduce

$$\left|a_3-\mu a_2^2\right|\le\frac{|bL_1|}{(2+k)(3+k)^m[1+\alpha(2+k)]|\Pi_3|}\left(\left|\frac{2L_2}{L_1}-\frac{bL_1(\lambda-1)}{\lambda}\right.\right.$$
$$\left.\left.-\frac{\mu bL_1(2+k)(3+k)^m[1+\alpha(2+k)]\Pi_3}{\lambda^2(1+\alpha)^2 2^{2m+2}\Pi_2^2}\right|\right). \tag{19}$$

An examination of the proof shows that the equality for (18) holds if $p_1=0$, $p_2=2$. Equivalently, by Lemma 3.8 we have $p(z^2)=p_2(z)=\dfrac{1+z^2}{1-z^2}$. Therefore, the extremal function of the class $\mathcal{NB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,b;\,\Phi)$ is given by

$$1+\frac{1}{b}\left\{\frac{2z^{1-\lambda}\left[(1-\alpha)D_k^{m+1}(\theta,\,\vartheta,\,\rho)f(z)+\alpha D_k^{m+2}(\theta,\,\vartheta,\,\rho)f(z)\right]^{\lambda}}{(1-\alpha)H_k^m(\theta,\,\vartheta,\,\rho)f(z)+\alpha H_k^{m+1}(\theta,\,\vartheta,\,\rho)f(z)}-1\right\}=\Phi\left[p(z^2)\right].$$

Similarly, the equality for (18) holds if $p_2=2$. Equivalently, by Lemma 3.8 we have $p(z)=p_1(z)=\dfrac{1+z}{1-z}$. Therefore, the extremal function in $\mathcal{NB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,b;\,\Phi)$ is given by

$$1+\frac{1}{b}\left\{\frac{2z^{1-\lambda}\left[(1-\alpha)D_k^{m+1}(\theta,\,\vartheta,\,\rho)f(z)+\alpha D_k^{m+2}(\theta,\,\vartheta,\,\rho)f(z)\right]^{\lambda}}{(1-\alpha)H_k^m(\theta,\,\vartheta,\,\rho)f(z)+\alpha H_k^{m+1}(\theta,\,\vartheta,\,\rho)f(z)}-1\right\}=\Phi\left[p_1(z)\right],$$

and the proof of the theorem is complete. □

If we let $\theta = \alpha = 0$, $\rho = 1$ and $\phi(z) = (1 + Xz)/(1 + Yz)$ in Theorem 3.9, then we have the following result.

Corollary 3.10. *If $f(z) \in \mathcal{A}$ satisfies the condition*

$$1 + \frac{1}{b}\left(\frac{2z^{1-\lambda}\left[D_k^{m+1}f(z)\right]^{\lambda}}{D_k^m f(z) - D_k^m f(-z)} - 1\right) \prec \frac{1 + Xz}{1 + Yz}, \quad (-1 \leq Y < X \leq 1),$$

then for all $\mu \in \mathbb{C}$ we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(X - Y)|b|}{(2 + k)(3 + k)^m} \max\{1; |2\Upsilon - 1|\},$$

where Υ is given by

$$\Upsilon := \frac{1}{4}\left(2(1 + Y) + \frac{(X - Y)b(\lambda - 1)}{\lambda} + \frac{\mu b(X - Y)(2 + k)(3 + k)^m}{2^{2m+2}\lambda^2}\right).$$

The inequality is sharp for each $\mu \in \mathbb{C}$.

If we let $m = k = \theta = \alpha = 0$, $\rho = 1$ and $\phi(z) = (1 + Xz)/(1 + Yz)$ in Corollary 3.10, then we have the following result.

Corollary 3.11. *If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Xz}{1 + Yz}, \quad (-1 \leq Y < X \leq 1),$$

then for all $\mu \in \mathbb{C}$ we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(X - Y)}{2} \max\left\{1; \left|Y + \frac{\mu(X - Y)}{4}\right|\right\}.$$

The inequality is sharp for each $\mu \in \mathbb{C}$.

4. λ-Pseudo Pascu Functions Involving Quantum Calculus

In this section, we will present a q-differential symmetric operator analogous to the operator $D_k^m(\theta, \vartheta, \rho)f(z)$ defined in Section 1. The study of *Geometric Function Theory* in dual with quantum calculus was initiated by Srivastava [38]. For recent developments and applications of quantum calculus in *Geometric Function Theory*, refer to the recent survey-cum-expository article of Srivastava [39] and references provided therein.

Here we will restrict ourselves to give just a very brief introduction of the q-calculus. For $f \in \mathcal{A}$, the *Jackson's q-derivative operator* or *q-difference operator* for a function $f \in \mathcal{A}$ is defined by

$$\Delta_q f(z) := \begin{cases} f'(0), & \text{if } z = 0, \\ \dfrac{f(z) - f(qz)}{(1 - q)z}, & \text{if } z \neq 0. \end{cases} \tag{20}$$

From (20), if $f \in \mathcal{A}_1$ we can easily see that $\Delta_q f(z) = 1 + \sum\limits_{n=2}^{\infty} [n]_q a_n z^{n-1}$, for $z \neq 0$ and note that $\lim\limits_{q \to 1^-} \Delta_q f(z) = f'(z)$. For our study ,we let $[n]_q = \sum_{k=1}^{n} q^{k-1}$, $[0]_q = 0$, $(q \in \mathbb{C})$ and the q-shifted factorial by

$$(a; q)_n = \begin{cases} 1, & n = 0 \\ (1 - a)(1 - aq)\ldots\left(1 - aq^{n-1}\right), & n \in \mathbb{N}. \end{cases}$$

Srivastava *et al*. [40–45] introduced function classes of q-starlike functions related with conic region and also studied the impact of Janowski functions on those conic regions. Inspired by aforementioned works on $q-$ *calculus*, we now define the q-analogue of the operator $D_k^m(\theta,\,\vartheta,\,\rho)f(z)$ as follows.

$$\mathcal{M}_q^m(k;\,\theta,\,\vartheta,\,\rho)f(z) = z + \sum_{n=2}^{\infty}\left([n]_q + \frac{k}{2}(1+(-1)^{n+1})\right)^m \frac{(q^\rho;q)_{n-1}\Gamma_q(\vartheta)}{\Gamma_q\left(\vartheta+\theta(n-1)\right)(q;q)_{n-1}} z^n. \tag{21}$$

Remark 4.1. *The q-analogue of the three-parameter Mittag-Leffler function was provided by Purohit and Kalla in [47, p. 18]. If we let $\theta = 0$ and $\rho = 1$ in* (21)*, then $\mathcal{M}_q^m(k;\,\theta,\,\vartheta,\,\rho)f$ reduces to $\mathcal{S}_q^{k,m}f$ defined by Ibrahim [46].*

We now define the q-analogue of the function class $\mathcal{NB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,b;\,\Phi)$ (see Definition 6).

Definition 4.2. *For $u,\,v \in \mathbb{C}$, with $u \neq v$, $|v| \leq 1$, let the class $q-\mathcal{KB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,\Phi)$ consist of function in $\mathcal{A}$ satisfying the subordination condition*

$$\frac{(u-v)z^{1-\lambda}\left[(1-\alpha)\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha\mathcal{M}_q^{m+2}(k;\,\theta,\,\vartheta,\,\rho)f(z)\right]^\lambda}{(1-\alpha)L_q^m(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha L_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)} \prec \Phi(z), \tag{22}$$

where $L_q^m(k;\,\theta,\,\vartheta,\,\rho)f = \left[\mathcal{M}_q^m(k;\,\theta,\,\vartheta,\,\rho)f(uz) - \mathcal{M}_q^m(k;\,\theta,\,\vartheta,\,\rho)f(vz)\right]$, $0 \leq \alpha \leq 1$, $\lambda \geq 1$, $b \in \mathbb{C}\setminus\{0\}$, $\theta,\,\vartheta,\,\rho \in \mathbb{C}$, $Re(\theta) > 0$ and $\Phi \in \mathcal{P}$ is defined as in (5)*.*

4.1. Conditions For Starlikeness and Solution to Fekete-Szegő Problem of $q-\mathcal{KB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,\Phi)$

Here we establish the conditions for starlikeness analogous to Theorem 3.4. Throughout this subsection, we let Ω_n to denote $\Omega_n = \frac{(q^\rho;q)_{n-1}\Gamma_q(\vartheta)}{\Gamma_q(\vartheta+\theta(n-1))(q;q)_{n-1}}$.

Theorem 4.3. *Let the function $\Phi \in \mathcal{P}$ be convex univalent in $\mathbb{U}$ with $Re\{\Phi(z)\} > 0$, $(z \in \mathbb{U})$. If the function $f \in \mathcal{A}$ satisfies the conditions*

$$\frac{L_q^m(k;\,\theta,\,\vartheta,\,\rho)f(z)}{L_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)} \neq -\frac{\alpha}{1-\alpha} \qquad \textit{and} \qquad \frac{\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)}{\mathcal{M}_q^{m+2}(k;\,\theta,\,\vartheta,\,\rho)f(z)} \neq -\frac{\alpha}{1-\alpha},$$

then

$$\begin{aligned}
&\frac{2[(u-v)z]^{1-\lambda}\left[(1-\alpha)\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha\mathcal{M}_q^{m+2}(k;\,\theta,\,\vartheta,\,\rho)f(z)\right]^\lambda}{(1-\alpha)L_q^m(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha L_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)}\\
&\left[(1-\lambda)(u-v)+\lambda\frac{(1-\alpha)z[\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)]'+\alpha z[\mathcal{M}_q^{m+2}(k;\,\theta,\,\vartheta,\,\rho)f(z)]'}{\left[(1-\alpha)\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha\mathcal{M}_q^{m+2}(k;\,\theta,\,\vartheta,\,\rho)f(z)\right]}\right.\\
&+\frac{2[(u-v)z]^{1-\lambda}\left[(1-\alpha)\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha\mathcal{M}_q^{m+2}(k;\,\theta,\,\vartheta,\,\rho)f(z)\right]^\lambda}{(1-\alpha)L_q^m(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha L_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)}\\
&-\frac{(1-\alpha)z\left[u(\mathcal{M}_q^m(k;\,\theta,\,\vartheta,\,\rho)f(uz))'-v(\mathcal{M}_q^m(k;\,\theta,\,\vartheta,\,\rho)f(vz))'\right]}{(1-\alpha)L_q^m(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha L_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)}\\
&\left.-\alpha\frac{z\left[u(\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(uz))'-v(\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(vz))'\right]}{(1-\alpha)L_q^m(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha L_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)}\right] \prec \kappa(z),
\end{aligned} \tag{23}$$

with $\kappa(z) = \Phi^2(z) + z\Phi'(z)$ implies $f \in q-\mathcal{KB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,\Phi)$. Moreover, the function Φ is the best dominant of the left-hand side of (22)*.*

Proof. If we define the function $\ell(z)$ by

$$\ell(z) := \frac{2[(u-v)z]^{1-\lambda}\left[(1-\alpha)\mathcal{M}_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha\mathcal{M}_q^{m+2}(k;\,\theta,\,\vartheta,\,\rho)f(z)\right]^{\lambda}}{(1-\alpha)L_q^{m}(k;\,\theta,\,\vartheta,\,\rho)f(z)+\alpha L_q^{m+1}(k;\,\theta,\,\vartheta,\,\rho)f(z)},$$

then from the hypothesis, it follows that $\ell(z)$ is analytic in $\mathbb{U}$. Now retracing the steps as in Theorem 3.4, we can establish the assertion of the Theorem. □

Remark 4.4. *As $q \to 1^-$, Theorem 4.3 reduces to the results listed in Section 3.*

For completeness, we just state the coefficient estimate and the Fekete-Szegő inequality for functions belonging to $q-\mathcal{KB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,\Phi)$.

Theorem 4.5. *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in q-\mathcal{KB}_k^{\lambda,m}(\alpha;\,\theta,\,\vartheta,\,\rho;\,\Phi)$, then for all $\mu \in \mathbb{C}$ we have*

$$\left|a_3-\mu a_2^2\right| \le \frac{|L_1|\max\{1;\,|2\mathcal{F}_1-1|\}}{[1+\alpha q(1+q)+k]\,[1+q+q^2+k]^m\left[\lambda(1+q+q^2)+u^2+v^2+uv\right]|\Omega_3|},$$

where $\mathcal{F}_1$ is given by

$$\mathcal{F}_1 = \frac{1}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left[2\lambda(1+q)(u+v)-2(u+v)^2+\lambda(1-\lambda)(1+q)^2\right]}{2\left[\lambda(1+q)-(u+v)\right]^2}\right.$$
$$\left.+\frac{\mu L_1\,[1+\alpha q(1+q)+k]\,[1+q+q^2+k]^m\left[\lambda(1+q+q^2)+u^2+v^2+uv\right]\Omega_3}{\left[\lambda(1+q)-(u+v)\right]^2(1+\alpha q)^2(1+q)^{2m}\Omega_2^2}\right).$$

The inequality is sharp for each $\mu \in \mathbb{C}$.

Remark 4.6. *If we let $u = 1$, $v = -1$, $q \to 1^-$ in Theorem 4.5 and let $b = 1$ in Theorem 3.9, then the results coincide.*

5. Conclusion

Using the newly defined operator, λ-pseudo Pascu type functions of complex order was defined to unify the study of various classes of analytic function. Further keeping with the latest trend of research we have extended the study using Quantum calculus. Srivastava in [3, 39] showed that, all the results investigated using quantum derivative (q-derivative) can be translated into the corresponding so called post-quantum analogues ($(p,\,q)$-derivative) using a straightforward parametric and argument variation of the following types $D_{p,q}f(z) = D_{\frac{q}{p}}f(pz)$ and $D_qf(z) = D_{p,pq}f\left(\frac{z}{p}\right)$, $(0 < q < p \le 1)$. Hence the additional parameter p is unnecessary, so here we have restricted our study with q-derivative rather than extending to $(p,\,q)$-derivative. We also point out relevant connections of the various q-results, which we investigate here, with those in several related earlier works on this subject.

References

[1] H. M. Srivastava, A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. Symmetry **2021**, 13, 2294.

[2] H. M. Srivastava, An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. J. Adv. Engrg. Comput. **2021**, 5 , 135–166.

[3] H. M. Srivastava, Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. J. Nonlinear Convex Anal. **2021**, 22, 1501–1520.

[4] H. M. Srivastava, On an extension of the Mittag-Leffler function, Yokohama Math. J. **16** (1968), 77–88.

[5] H. M. Srivastava, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions. SpringerPlus 2, 67 (2013). https://doi.org/10.1186/2193-1801-2-67.

[6] E. M. Wright, The asymptotic expansion of integral functions defined by Taylor series, Philos. Trans. Roy. Soc. London Ser. A **238** (1940), 423–451.
[7] E. W. Barnes,The asymptotic expansion of integral functions defined by Taylor's series. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci. **206** (1906), 249–297.
[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.
[9] H. M. Srivastava and J. Choi, *Series associated with the zeta and related functions*, Kluwer Academic Publishers, Dordrecht, 2001.
[10] S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch zeta functions and associated fractional derivative and other integral representations, Appl. Math. Comput. **154** (2004), no. 3, 725–733.
[11] H. M. Srivastava, M. Bansal and P. Harjule, A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function, Math. Methods Appl. Sci. **41** (2018), no. 16, 6108–6121.
[12] H. M. Srivastava, K. C. Gupta and S. P. Goyal, *The H-functions of one and two variables*, South Asian Publishers Pvt. Ltd., New Delhi, 1982.
[13] H. M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. **283(284)** (1976), 265–274.
[14] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1984.
[15] H. M. Srivastava, Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators, Appl. Anal. Discrete Math. **1** (2007), no. 1, 56–71.
[16] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric series*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1985.
[17] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. **103** (1999), no. 1, 1–13.
[18] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct. **14** (2003), no. 1, 7–18.
[19] A. A. Attiya, Some applications of Mittag-Leffler function in the unit disk, Filomat **30** (2016), no. 7, 2075–2081.
[20] H. M. Srivastava and S. M. El-Deeb, Fuzzy differential subordinations based upon the Mittag-Leffler type Borel distribution, Symmetry (2021), no. 13, 1023.
[21] H. M. Srivastava, A. Kumar, S. Das and K. Mehrez, Geometric properties of a certain class of Mittag–Leffler-type functions, Fractal Fract. (2022), no. 6, 54.
[22] H. M. Srivastava, A.-K. N. Alomari, K. M. Saad and W. M. Hamanah, Some dynamical models involving fractional-order derivatives with the Mittag-Leffler type kernels and their applications based upon the Legendre spectral collocation method, Fractal Fract. (2021), no. 5, 131.
[23] H. M. Srivastava, A. Kashuri, P. O. Mohammed, A. M. Alsharif and J. L. G. Guirao, New Chebyshev type inequalities via a general family of fractional integral operators with a modified Mittag-Leffler kernel, AIMS Math. **6** (2021), no. 10, 11167–11186.
[24] H. M. Srivastava, A. Fernandez and D. Baleanu, Some new fractional-calculus connections between Mittag–Leffler functions, Mathematics (2019), no 7, 485.
[25] Ž. Tomovski, R. Hilfer and H. M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, Integral Transforms Spec. Funct. **21** (2010), no. 11, 797–814.
[26] H. M. Srivastava and Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, Appl. Math. Comput. **211** (2009), no. 1, 198–210.
[27] H. M. Srivastava, R. K. Saxena, T. K. Pogány and R. Saxena, Integral and computational representations of the extended Hurwitz-Lerch zeta function, Integral Transforms Spec. Funct. **22** (2011), no. 7, 487–506.
[28] H. M. Srivastava, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, SpringerPlus **2** (2013), Article ID 2:67, 1-14.
[29] H. M. Srivastava, D. Jankov, T. K. Pogány and R. K. Saxena, Two-sided inequalities for the extended Hurwitz-Lerch zeta function, Comput. Math. Appl. **62** (2011), no. 1, 516–522.
[30] D. Răducanu and H. M. Srivastava, A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch zeta function. Integral Transforms Spec. Funct. **2007**, 18, no. 11-12, 933–943.
[31] R. W. Ibrahim and M. Darus, Subordination inequalities of a new Salagean-difference operator, Int. J. Math. Comput. Sci. **14** (2019), no. 3, 573–582.
[32] R. W. Ibrahim, On a Janowski formula based on a generalized differential operator, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **69** (2020), no. 2, 1320–1328.
[33] W. K. Mashwan, B. Ahmad, M. G. Khan,S. Mustafa, S. Arjika and B. Khan, Pascu-Type analytic functions by using Mittag-Leffler functions in Janowski domain, Mathematical Problems in Engineering, Volume 2021, Article ID 1209871, 7 pages.
[34] M. Arif, K. Ahmad, J.-L. Liu and J. Sokół, A new class of analytic functions associated with Sălăgean operator, J. Funct. Spaces **2019**, Art. ID 6157394, 8 pp.
[35] T. Bulboacă, *Differential subordinations and superordinations. Recent results*, House of Science Book Publ., Cluj-Napoca, 2005.
[36] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA.
[37] K. R. Karthikeyan, G. Murugusundaramoorthy and T.Bulboacă, Properties of λ-pseudo-starlike functions of complex order defined by subordination. Axioms. **2021**, 10, Art. ID: 86.
[38] H. M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in *Univalent functions, fractional calculus, and their applications (Kōriyama, 1988)*, 329–354, Ellis Horwood Ser. Math. Appl, Horwood, Chichester.
[39] H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory

of complex analysis, Iran. J. Sci. Technol. Trans. A Sci. **44** (2020), no. 1, 327–344.

[40] H. M. Srivastava, Q. Z. Ahmad, N. Khan, N. Khan, B. Khan, Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, Mathematics, **7** (2019), 181, 1-15.

[41] H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, Coefficient inequalities for q-starlike functions associated with the Janowski functions, Hokkaido Math. J., **48** (2019), 407-425.

[42] H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, M. Tahir, A generalized conic domain and its applications to certain subclasses of analytic functions, Rocky Mountain J. Math., **49** (2019), 2325-2346.

[43] H. M. Srivastava, N. Khan, M. Darus, M. T. Rahim, Q. Z. Ahmad, Y. Zeb, Properties of spiral-like close-to-convex functions associated with conic domains, Mathematics, **7** (2019), Article ID 706, 1-12.

[44] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general classes of q-starlike functions associated with the Janowski functions, Symmetry, **11** (2019), 1-14.

[45] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general families of q-starlike functions associated with the Janowski functions, Filomat, **33** (2019), 2613-2626.

[46] R. W. Ibrahim, Geometric process solving a class of analytic functions using q-convolution differential operator, J. Taibah Univ. Sci. **14** (2020), no. 1, 670–677.

[47] S. D. Purohit and S. L. Kalla, A generalization of q-Mittag-Leffler function, Mat. Bilten No. 35 (2011), 15–26.