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Weakly n-hyponormal weighted shifts: a sufficient condition and their examples

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Abstract. The n-hyponormal and weakly n-hyponormal weighted shifts were developed to study bridges of operators between the subnormal and hyponormal operators on an infinite dimensional complex Hilbert space about 30 years ago. In this paper we discuss the distinction between the classes of n-hyponormal and weakly n-hyponormal weighted shifts. For such a purpose we consider an arbitrary contractive hyponormal weighted shift W_α and find a sufficient condition for the weak n-hyponormality of W_α. We provide a general technique for distinction between the n-hyponormality and the weak n-hyponormality of W_α, and investigate the distinction between the classes of n-hyponormal and weakly n-hyponormal weighted shifts with Bergman shift and some other examples.

1. Introduction and preliminaries

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is *subnormal* if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace. For a positive integer $n \in \mathbb{N}$, an operator T is (*strongly*) *n-hyponormal* if the $(n+1)\times(n+1)$ operator matrix $\left[T^{*j}T^i\right]_{i,j=0}^n$ is positive. It is well-known that T is subnormal if and only if T is n-hyponormal for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, an operator T is *weakly n-hyponormal* if $p(T)$ is hyponormal for every polynomial p of degree n or less ([5],[6]). In particular, the weak 2-hyponormality [weak 3-hyponormality, or weak 4-hyponormality, resp.] is referred to as quadratic hyponormality [cubic hyponormality, or quartic hyponormality, resp.]. An operator $T \in B(\mathcal{H})$ is said to be *polynomially hyponormal* if T is weakly n-hyponormal for all $n \in \mathbb{N}$. Obviously, 1-hyponormal [or weakly 1-hyponormal] operator $T \in B(\mathcal{H})$ is hyponormal, i.e., $T^*T \geq TT^*$. It is known that every subnormal operator is polynomially hyponormal and every n-hyponormal operator is weakly n-hyponormal, namely we get

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"subnormal ⇒ n-hyponormal ⇒ weakly n-hyponormal ⇒ hyponormal ($n \in \mathbb{N}$)."

Many operator theorists have studied the converse implications; for example, see [5],[6],[10],[18],[21],[24], etc. In [12, Theorem 2.1], Curto-Putinar proved theoretically that there exists a polynomially hyponormal operator which is not 2-hyponormal. One can confirm the existence of a weighted shift that is polynomially hyponormal but not subnormal ([26, Theorem 3.4]). But one does not know any concrete example of a weighted shift that is polynomially hyponormal but not subnormal yet. Also it is not known whether a polynomially hyponormal weighted shift but not 2-hyponormal exists ([12, Remark 2.9]). Thus many operator theorists have studied the structure of n-hyponormal and weakly n-hyponormal weighted shifts for more than 30 years. The flatness is important to detect the structure of such weighted shifts (cf. [3],[5],[6],[23]). The flatness of subnormal weighted shifts was begun by J. Stampfli ([27]); he proved that if W_α is a subnormal weighted shift with a weight sequence $\alpha = \{\alpha_k\}_{k=0}^\infty$ in $\mathbb{R}_+ \setminus \{0\}$ and $\alpha_0 = \alpha_1$, then $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$, where $\mathbb{R}_+$ is the set of nonnegative real numbers. In [6] R. Curto improved Stampfli's result as that if W_α is a 2-hyponormal weighted shift with first two equal weights, then $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$. And he also proved that a weighted shift W_α is quadratically hyponormal, where

$$\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots, \tag{1.1}$$

in [6, Proposition 7]. This means that the quadratic hyponormality of a weighted shift W_α does not preserve the flatness property, which motivated the following problem.

Problem 1.1 ([7, Problem 4]). Describe all quadratically hyponormal weighted shifts W_α with $\alpha_0 = \alpha_1$.

Since R. Curto introduced Problem 1.1 in 1991, several operator theorists have studied this problem for more than 30 years (cf. [3],[5],[6],[9],[14],[15],[16],[17],[22],[23], etc.). Some of them are closely related to the Bergman shift. In particular, Exner-Jung-Park generalized Curto's example with weights in (1.1), namely, in [17, Theorem 2.2], they proved that if $\alpha = \{\alpha_i\}_{i=0}^\infty$ is given by

$$\alpha : \sqrt{x}, \sqrt{x}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots,$$

where x is a positive real number, then the associate weighted shift W_α is quadratically hyponormal if and only if $\delta_1 \le x \le \delta_2$, where $|\delta_1 - 0.1673| < \frac{1}{1000}$ and $|\delta_2 - 0.7439| < \frac{1}{1000}$. In [23], Li-Cho-Lee proved that

if W_α is a cubically hyponormal weight shift with first two equal weights, then $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$.

This means that every weakly n-hyponormal weighted shift W_α with first two equal weights satisfies the flatness property for $n \ge 3$. Hence we can see that Problem 1.1 does not extend to the weak n-hyponormality of weighted shifts for $n \ge 3$. However, the following problem is interesting to us still.

Problem 1.2. Let $\alpha(x, y)$ be a weight sequence defined by

$$\alpha(x, y) : x, y, \alpha_0, \alpha_1, \ldots,$$

where x and y are positive real variables and let $W_{\alpha(x,y)}$ be the associate weighted shift. Denote the regions in $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$ by

$$\mathcal{WH}_{\alpha(x,y)}^{\langle n \rangle} = \{(x, y) : W_{\alpha(x,y)} \text{ is weakly } n\text{-hyponormal}\}, \quad n \ge 2;$$
$$\mathcal{SH}_{\alpha(x,y)}^{\langle n \rangle} = \{(x, y) : W_{\alpha(x,y)} \text{ is } n\text{-hyponormal}\}, \quad n \ge 2.$$

Describe the region $\mathcal{WH}_{\alpha(x,y)}^{\langle n \rangle} \setminus \mathcal{SH}_{\alpha(x,y)}^{\langle l \rangle}$ for $n \ge 3$ and $2 \le l \le n$.

In terms of Problem 1.2, we recall some known results as following.

- If $\alpha(x,y): \sqrt{x}, \sqrt{y}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$, then $\left(\frac{141}{250}, \frac{2}{3}\right) \in \mathcal{WH}^{\langle 3\rangle}_{\alpha(x,y)} \backslash \mathcal{SH}^{\langle 2\rangle}_{\alpha(x,y)}$, which means that $\mathcal{WH}^{\langle 3\rangle}_{\alpha(x,y)} \backslash \mathcal{SH}^{\langle 2\rangle}_{\alpha(x,y)} \neq \emptyset$ ([21, Corollary 3.5]).

- If $\alpha(x,y): \sqrt{x}, \sqrt{y}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \ldots$, then $\{(x, \frac{3}{4}) : \frac{200}{297} < x \leq \frac{667}{990}\} \subset \mathcal{WH}^{\langle 4\rangle}_{\alpha(x,y)} \backslash \mathcal{SH}^{\langle 3\rangle}_{\alpha(x,y)}$ ([10, Corollary 5]).

- If $\alpha(x,y): \sqrt{x}, \sqrt{y}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$, then $\mathcal{SH}^{\langle\infty\rangle}_{\alpha(x,y)} = \cap_{n=1}^{\infty} \mathcal{SH}^{\langle n\rangle}_{\alpha(x,y)} = \emptyset$ ([20]).

Concerning Problem 1.2, we recall that the following question as a general version of [17, Theorem 2.2] is natural.

For $\alpha(x,y): \sqrt{x}, \sqrt{y}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$, describe the full range of the set $\{(x,y) : W_{\alpha(x,y)}$ is quadratically hyponormal$\}$.

This is an open problem arising from the authors of [17]. In this paper we discuss a sufficient condition for a nonempty region in $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)}$ which satisfies $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)} \backslash \mathcal{SH}^{\langle n\rangle}_{\alpha(x,y)} \neq \emptyset$ for $n \geq 3$.

This paper consists of four sections. In Section 2, we construct a subregion of $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)}$ for $n \geq 3$, which will be denoted by $\mathcal{CWH}^{\langle n\rangle}_{\alpha(x,y)}$ (see Algorithm 2.3). And we see that the associated weighted shifts $W_{\alpha(x,y)}$ to pair $(x,y) \in \mathcal{CWH}^{\langle n\rangle}_{\alpha(x,y)}$ have the weak n-hyponormality of $W_{\alpha(x,y)}$ (see Lemma 2.4). In Section 3, we apply the Bergman shift as an example to find the subregion $\mathcal{CWH}^{\langle n\rangle}_{\alpha(x,y)}$ satisfying $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)} \backslash \mathcal{SH}^{\langle n\rangle}_{\alpha(x,y)} \neq \emptyset$ for $n \geq 3$ via Lemma 2.4. The techniques of Sections 2 and 3 via Algorithm 2.3 provide an idea to find examples of a weighted shift W_α satisfying $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)} \backslash \mathcal{SH}^{\langle n\rangle}_{\alpha(x,y)} \neq \emptyset$ for $n \geq 3$. In Section 4, we will discuss the subregion of $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)}$ satisfying $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)} \backslash \mathcal{SH}^{\langle n\rangle}_{\alpha(x,y)} \neq \emptyset$ for $n \geq 3$ with an example of a weighted shift which is not Bergman shift.

Some of the calculations in this paper were aided by using the software tool *Mathematica* ([29]).

2. Description of a subregion $\mathcal{CWH}^{\langle n\rangle}_{\alpha(x,y)}$ of $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)}$

For a sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ of positive real numbers and $n, k \geq 0$, denote the Hankel matrix of α by

$$H_{n,k}(\alpha) := \begin{bmatrix} \alpha_k & \alpha_{k+1} & \cdots & \alpha_{k+n} \\ \alpha_{k+1} & \alpha_{k+2} & \cdots & \alpha_{k+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k+n} & \alpha_{k+n+1} & \cdots & \alpha_{k+2n} \end{bmatrix}. \tag{2.1}$$

We consider $\gamma := \{\gamma_i\}_{i=0}^{\infty}$ defined by

$$\gamma_0 := 1 \text{ and } \gamma_i := \alpha_{i-1}^2 \gamma_{i-1}, \ i \geq 1, \tag{2.2}$$

which are sometimes referred to as *moments* of α.

We begin this section with an equivalent condition for the weak n-hyponormality for a contractive hyponormal weighted shift, which is revised slightly from [18, Theorem 2.3].

Lemma 2.1. *Suppose $n \geq 2$. Let W_α be a contractive hyponormal weighted shift with $\alpha := \{\alpha_i\}_{i=0}^{\infty}$ and let $\gamma := \{\gamma_i\}_{i=0}^{\infty}$ be as in* (2.2)*. For any finite sequences $\{\epsilon_i\}_{i=1}^{n-1}$ and $\{\delta_i\}_{i=1}^{n-1}$ in $\mathbb{R}_+$, it holds that W_α is weakly n-hyponormal if and only* *if the following condition holds:*

$$
\begin{aligned}
\Delta_n^{\alpha}(\phi,p,q) = {} & \gamma_n \left|\phi_n p_0\right|^2 + \left(\begin{bmatrix} \gamma_{n-1} & \gamma_n \\ \gamma_n & \gamma_{n+1}-\epsilon_1 \end{bmatrix} \begin{bmatrix} \phi_{n-1}p_0 \\ \phi_n p_1 \end{bmatrix}, \begin{bmatrix} \phi_{n-1}p_0 \\ \phi_n p_1 \end{bmatrix} \right) \\
& + \sum_{k=2}^{n-1} \left(\begin{bmatrix} \gamma_{n-k} & \gamma_{n-k+1} & \cdots & \gamma_n \\ \gamma_{n-k+1} & \gamma_{n-k+2}-\epsilon_k & \cdots & \gamma_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \end{bmatrix} \begin{bmatrix} \phi_{n-k}p_0 \\ \phi_{n-k+1}p_1 \\ \vdots \\ \phi_n p_k \end{bmatrix}, \begin{bmatrix} \phi_{n-k}p_0 \\ \phi_{n-k+1}p_1 \\ \vdots \\ \phi_n p_k \end{bmatrix} \right) \\
& + \left(\begin{bmatrix} \gamma_0 & \gamma_1\phi_1 & \gamma_2 & \cdots & \gamma_n\phi_n \\ \gamma_1\overline{\phi_1} & \gamma_2|\phi_1|^2+\epsilon & \gamma_3\overline{\phi_1} & \cdots & \gamma_{n+1}\overline{\phi_1}\phi_n \\ \gamma_2 & \gamma_3\phi_1 & \gamma_4 & \cdots & \gamma_{n+2}\phi_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_n\overline{\phi_n} & \gamma_{n+1}\phi_1\overline{\phi_n} & \gamma_{n+2}\overline{\phi_n} & \cdots & \gamma_{2n}|\phi_n|^2+\delta \end{bmatrix} \begin{bmatrix} q_0 \\ p_1 \\ \phi_2 p_2 \\ \vdots \\ \phi_{n-1}p_{n-1} \\ p_n \end{bmatrix}, \begin{bmatrix} q_0 \\ p_1 \\ \phi_2 p_2 \\ \vdots \\ \phi_{n-1}p_{n-1} \\ p_n \end{bmatrix} \right) \\
& + \sum_{k=1}^{n-1} \left(\begin{bmatrix} \gamma_k & \cdots & \gamma_n & \cdots & \gamma_{k+n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \gamma_n & \cdots & \gamma_{2n-k}-\delta_k & \cdots & \gamma_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \gamma_{k+n} & \cdots & \gamma_{2n} & \cdots & \gamma_{k+2n} \end{bmatrix} \begin{bmatrix} q_k \\ \phi_1 p_{k+1} \\ \vdots \\ \phi_{n-k}p_n \\ \vdots \\ \phi_n p_{k+n} \end{bmatrix}, \begin{bmatrix} q_k \\ \phi_1 p_{k+1} \\ \vdots \\ \phi_{n-k}p_n \\ \vdots \\ \phi_n p_{k+n} \end{bmatrix} \right) \\
& + \sum_{k=n}^{\infty} \left(H_{n,k}(\gamma) \begin{bmatrix} q_k \\ \phi_1 p_{k+1} \\ \phi_2 p_{k+2} \\ \vdots \\ \phi_n p_{k+n} \end{bmatrix}, \begin{bmatrix} q_k \\ \phi_1 p_{k+1} \\ \phi_2 p_{k+2} \\ \vdots \\ \phi_n p_{k+n} \end{bmatrix} \right)
\end{aligned}
\tag{2.3}
$$

is positive for any $\phi = \left\{\phi_i\right\}_{i=1}^{n}$, $p = \{p_i\}_{i=0}^{\infty}$ *and* $q = \{q_i\}_{i=0}^{\infty}$ *in* $\mathbb{C}$, *where*

$$\epsilon = \sum_{l=1}^{n-1} \epsilon_l \left|\phi_{n-l+1}\right|^2 \quad \text{and} \quad \delta = \sum_{l=1}^{n-1} \delta_l \left|\phi_{n-l}\right|^2 . \tag{2.4}$$

Proof. Observe that the expressions of the right sides of (2.3) above and (2.8) in [18, Theorem 2.3] coincide exactly. □

Let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a weight sequence of positive real numbers and let $\gamma := \{\gamma_i\}_{i=0}^{\infty}$ be as in (2.2). We consider the matrix-valued functions F_k and G_k on $[0,\infty)$ defined by

$$F_1(h) = \begin{bmatrix} \gamma_{n-1} & \gamma_n \\ \gamma_n & \gamma_{n+1}-h \end{bmatrix}, \tag{2.5}$$

$$F_k(h) = \begin{bmatrix} \gamma_{n-k} & \gamma_{n-k+1} & \cdots & \gamma_n \\ \gamma_{n-k+1} & \gamma_{n-k+2}-h & \cdots & \gamma_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \end{bmatrix}, \quad 2 \le k \le n-1. \tag{2.6}$$

and

$$G_k(h)=\begin{bmatrix}\gamma_k & \cdots & \gamma_n & \cdots & \gamma_{k+n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \gamma_n & \cdots & \gamma_{2n-k}-h & \cdots & \gamma_{2n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \gamma_{k+n} & \cdots & \gamma_{2n} & \cdots & \gamma_{k+2n}\end{bmatrix},\ 1\le k\le n-1, \tag{2.7}$$

respectively.

The following lemma comes immediately from Lemma 2.1.

Lemma 2.2. *Let W_α be a contractive hyponormal weighted shift with $\alpha:=\{\alpha_i\}_{i=0}^\infty$ and let $\gamma:=\{\gamma_i\}_{i=0}^\infty$ be as in* (2.2)*. Suppose $F_k(\epsilon_k)\ge 0$, $G_k(\delta_k)\ge 0$ for some ϵ_k and δ_k in $\mathbb{R}_+$ for $1\le k\le n-1$, and $H_{n,k}(\gamma)\ge 0$ for all $k\ge n$. Assume that, for any $\phi:=\left\{\phi_i\right\}_{i=1}^n$ in $\mathbb{C}$,*

$$\Phi_n(\epsilon,\delta):=\begin{bmatrix}\gamma_0 & \gamma_1\phi_1 & \gamma_2 & \cdots & \gamma_n\phi_n\\ \gamma_1\overline{\phi_1} & \gamma_2|\phi_1|^2+\epsilon & \gamma_3\overline{\phi_1} & \cdots & \gamma_{n+1}\overline{\phi_1}\phi_n\\ \gamma_2 & \gamma_3\phi_1 & \gamma_4 & \cdots & \gamma_{n+2}\phi_n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \gamma_n\overline{\phi_n} & \gamma_{n+1}\phi_1\overline{\phi_n} & \gamma_{n+2}\overline{\phi_n} & \cdots & \gamma_{2n}|\phi_n|^2+\delta\end{bmatrix}\ge 0,$$

where ϵ and δ are as in (2.4)*. Then W_α is weakly n-hyponormal.*

We now give the algorithm to construct the subregion $\mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle}$ of $\mathcal{WH}_{\alpha(x,y)}^{\langle n\rangle}$ for weak n-hyponormality.

Algorithm 2.3. Suppose $n\ge 2$. Let $\alpha=\{\alpha_i\}_{i=0}^\infty$ be a weight sequence of positive real numbers and let $\gamma=\{\gamma_i\}_{i=0}^\infty$ be moments of α. Suppose $\alpha(x,y)$ is the 2-step backward extension weight sequence of α, namely,

$$\alpha(x,y): x, y, \alpha_0, \alpha_1, \ldots, \tag{2.8}$$

where x and y are positive real variables. Let $W_{\alpha(x,y)}$ be the associated weighted shift to $\alpha(x,y)$. To construct the subregion $\mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle}$, we provide steps as following.

- **I.** Take the largest possible ϵ_k so that $F_k(\epsilon_k)\ge 0$ for $1\le k\le n-2$.
- **II.** Take the largest possible δ_k so that $G_k(\delta_k)\ge 0$ for $2\le k\le n-1$.
- **III.** For ϵ_k and δ_k in Steps I and II, find the range of (x,y) satisfying $G_1(0)\ge 0$, $\Delta_n(x,y)\ge 0$ for any $\phi:=\left\{\phi_i\right\}_{i=1}^n$ in $\mathbb{C}$ with $\phi_1=1$, where

$$\Delta_n(x,y):=\left[\begin{array}{cc|ccc|c}\frac{1}{(xy)^2} & \frac{1}{y^2} & \gamma_0 & \cdots & \gamma_{n-3} & \gamma_{n-2}\phi_n\\ \frac{1}{y^2} & \gamma_0+\frac{\epsilon}{(xy)^2} & \gamma_1 & \cdots & \gamma_{n-2} & \gamma_{n-1}\phi_n\\ \hline \gamma_0 & \gamma_1 & & & & \gamma_n\phi_n\\ \vdots & \vdots & & H_{n-3,2}(\gamma) & & \vdots\\ \gamma_{n-3} & \gamma_{n-2} & & & & \gamma_{n+1}\phi_n\\ \hline \gamma_{n-2}\overline{\phi_n} & \gamma_{n-1}\overline{\phi_n} & \gamma_n\overline{\phi_n} & \cdots & \gamma_{2n-1}\overline{\phi_n} & \gamma_{2n-2}|\phi_n|^2+\frac{\delta}{(xy)^2}\end{array}\right], \tag{2.9}$$

  where ϵ and δ are as in (2.4).
- **IV.** Denote the set $\mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle}$ consisting of pair (x,y) obtained from Step III.

The following lemma follows from Lemma 2.2 immediately.

Lemma 2.4. *Suppose $n \geq 2$ and W_α is a contractive n-hyponormal weighted shift with $\alpha = \{\alpha_i\}_{i=0}^\infty$. Let $\gamma = \{\gamma_i\}_{i=0}^\infty$ be a moment sequence of α and $\alpha(x, y)$ be a weight sequence as in (2.8).If $(x, y) \in \mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle}$, then $W_{\alpha(x,y)}$ is weakly n-hyponormal.*

Proof. Since W_α is n-hyponormal, obviously $H_{n,k}(\gamma) \geq 0$ for all $k \geq n$. Hence, according to Algorithm 2.3, the proof is complete. □

Before closing this section, we note that the set $\mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle}$ can be empty possibly, namely we can find an example satisfying $\mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle} = \emptyset$ for $n \geq 3$; indeed, consider a sequence $\alpha : a, a, a, b, b, ...$ with $0 < a < b$ for such an example.

3. Bergman weighted shift and description of $\mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle}$

Let $W_{\alpha(x,y)}$ be a contractive hyponormal weighted shift with a weight sequence $\alpha(x, y)$ as in (2.8). In this section, we will discuss the range of $\mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle}$ with the Bergman weighted shift W_α which is one of the typical models to study the weak n-hyponormality of weighted shifts (cf. [6],[9],[10],[13],[17],[18],[20],[21],[22], [25]). Recall that if $\alpha(x, y) : \sqrt{x}, \sqrt{y}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$, then $\mathcal{SH}_{\alpha(x,y)}^{\langle\infty\rangle} = \cap_{n=1}^\infty \mathcal{SH}_{\alpha(x,y)}^{\langle n\rangle} = \emptyset$. To avoid this case, we consider the 2-step backward extension $\alpha(x, y)$ of $\left\{\sqrt{\frac{i+1}{i+2}}\right\}_{i=2}^\infty$ which is given by

$$\alpha(x, y) : \sqrt{x}, \sqrt{y}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots. \tag{3.1}$$

In this case, we know that $\mathcal{SH}_{\alpha(x,y)}^{\langle\infty\rangle} \neq \emptyset$, and $\mathcal{CWH}_{\alpha(x,y)}^{\langle n\rangle}$ can be compared possibly to the known results in Section 1.

Consider the associated moment sequence $\gamma = \{\gamma_j\}_{j=0}^\infty$ of $\alpha(x, y)$ as in (3.1) and the Hankel matrix $H_{n,k}(\gamma)$ as in (2.1). Then it follows that for $k \geq 2$ and $n \geq 0$,

$$\det \frac{1}{3xy} H_{n,k}(\gamma) = \det \left[\frac{1}{k+i+j+1}\right]_{i,j=0}^n = \frac{G(n+2)^2 G(k+n+2)^2}{G(k+1)G(k+2n+3)}, \tag{3.2}$$

where $G(\cdot)$ is *Barnes G-function*[1)]. (cf. [10, p.460],[13, Lemma 2.1],[25, Lemma 2.2]). Now consider the sequence $\zeta = \left\{\frac{1}{k+1}\right\}_{k=0}^\infty$. By (2.1) and (3.2), we can see easily that $H_{n,k}(\zeta)$ is the Cauchy matrix as following

$$H_{n,k} := H_{n,k}(\zeta) = \left[\frac{1}{k+i+j+1}\right]_{i,j=0}^n, \quad k, n \geq 0. \tag{3.3}$$

We start our work with an elementary lemma which can be proved by a direct computation.

Lemma 3.1. *Let $M_{n,k,l}$ be the submatrix obtained by deleting the l-th row and column of $H_{n,k}$. Then*

$$\det M_{n,k,l} = \frac{[(n+k+l)!]^2}{(k+2l-1)[(n-l+1)!]^2[(k+l-1)!]^2[(l-1)!]^2} \det H_{n,k}.$$

Consider a matrix $H_{n,k,l}(s)$ whose entries h_{ij} are defined by

$$h_{ij} = \begin{cases} \frac{1}{k+2l-3} - s & \text{if } i = j = l-1, \\ \frac{1}{k+i+j-1} & \text{otherwise.} \end{cases}$$

[1)]The *Barnes G-function* is presented by $G(n) = 1!2!\cdots(n-2)!$ ([1]).

Obviously we get $\det H_{n,k,l}(s) = \det H_{n,k} - s \det M_{n,k,l}$. For brevity, we denote by

$$\Omega_n := \frac{G(n+1)^3\, G(n+5)}{G(2n+3)} \quad (n \geq 3) \text{ and } \Omega_1 = \Omega_2 = 1.$$

By using (3.2) and Lemma 3.1, we obtain two elementary formulas of the Hankel matrices below.

Lemma 3.2. *Suppose that $x, y > 0$ and $n \geq 2$. Then we have the following statements.*

(i) *Let $Q_n(y) := \left[q_{i+j}\right]_{i,j=0}^{n}$ be an $(n+1)\times(n+1)$ matrix with*

$$q_0 := 1, \quad q_1 := \frac{1}{3y}, \quad \text{and} \quad q_k := \frac{1}{k+1}, \quad k \geq 2.$$

Then

$$\det Q_n(y) = \frac{\Omega_n \tau_n(y)}{n+3},$$

where

$$\tau_n(y) = \frac{1}{(n+2)^2\,(n+1)^3} - \frac{n}{(n+2)\,(n+1)}\left(\frac{1}{3y} - \frac{1}{2}\right) - \frac{n^2\,(n+1)}{12}\left(\frac{1}{3y} - \frac{1}{2}\right)^2.$$

(ii) *Let $A_n(x,y) := \frac{1}{3xy} H_{n,0}(\gamma)$ and $B_n(x,y)$ be the submatrix of $A_n(x,y)$ obtained by deleting the second row and column of $A_n(x,y)$. Then*

$$\det A_n(x,y) = \frac{\Omega_n}{n+3}\left(\frac{\frac{1}{3xy} - 1}{(n+1)(n+2)^2} + \tau_n(y)\right)$$

and

$$\det B_n(x,y) = \frac{n^2 \Omega_n}{12\,(n+3)\,(n+1)}\left(\left(\frac{1}{3xy} - 1\right)(n+1)^2 + 4\right).$$

Proof. (i) Use (3.2) and Lemma 3.1.

(ii) It follows from a simple computation that

$$\det A_n(x,y) = \left(\frac{1}{3xy} - 1\right)\det H_{n-1,2} + \det Q_n(y).$$

According to the definition of the matrix $H_{n,k}$ in (3.3), it holds that

$$\det H_{n-1,2} = \frac{\Omega_n}{(n+1)(n+3)(n+2)^2} \quad \text{and} \quad \det Q_n(y) = \frac{\Omega_n \tau_n(y)}{(n+3)},$$

which proves this lemma. □

If we apply the weight sequence $\alpha(x,y)$ to (2.5)-(2.7), the functions $F_k(s)$ and $G_k(t)$ are represented by

$$F_1(h) = \begin{bmatrix} \frac{3xy}{n} & \frac{3xy}{n+1} \\ \frac{3xy}{n+1} & \frac{3xy}{n+2} - h \end{bmatrix},$$

$$F_k(h) = \begin{bmatrix} \frac{3xy}{n-k+1} & \frac{3xy}{n-k+2} & \cdots & \frac{3xy}{n+1} \\ \frac{3xy}{n-k+2} & \frac{3xy}{n-k+3} - h & \cdots & \frac{3xy}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{3xy}{n+1} & \frac{3xy}{n+2} & \cdots & \frac{3xy}{n+k+1} \end{bmatrix}, \quad 2 \leq k \leq n-2,$$

and

$$G_k(h) = \begin{bmatrix} \frac{3xy}{k+1} & \cdots & \frac{3xy}{n+1} & \cdots & \frac{3xy}{n+k+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{3xy}{n+1} & \cdots & \frac{3xy}{2n-k+1} - h & \cdots & \frac{3xy}{2n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{3xy}{n+k+1} & \cdots & \frac{3xy}{2n+1} & \cdots & \frac{3xy}{2n+k+1} \end{bmatrix}, \ 2 \le k \le n-1.$$

Lemma 3.3 ([8, Proposition 2.3 (v)]). *Let A be a $k \times k$ matrix, $\mathbf{b} \in \mathbb{C}^k$ and $c \in \mathbb{C}$. Assume $A \ge 0$ and A is invertible. Then a 2×2 operator matrix*

$$\widetilde{A} := \begin{bmatrix} A & \mathbf{b} \\ \mathbf{b}^* & c \end{bmatrix} \ge 0$$

if and only if $\det \widetilde{A} \ge 0$.

We will discuss the largest possible values ϵ_k and δ_k in Steps I and II, respectively. The sharp number should be $\epsilon_k = \max\{h \in \mathbb{R}_+ : F_k(h) \ge 0\}$ and $\delta_k = \max\{h \in \mathbb{R}_+ : G_k(h) \ge 0\}$; we will prove them in the following lemma.

Lemma 3.4. *Taking positive real values $\widehat{\epsilon_k}$ and $\widehat{\delta_k}$ such that $\det F_k(\widehat{\epsilon_k}) = 0$ $(1 \le k \le n-2)$ and $\det G_k(\widehat{\delta_k}) = 0$ $(2 \le k \le n-1)$, we get*

$$\widehat{\epsilon_k} = \max\{h \in \mathbb{R}_+ : F_k(h) \ge 0\} \text{ and } \widehat{\delta_k} = \max\{h \in \mathbb{R}_+ : G_k(h) \ge 0\}.$$

Moreover, we have

$$\widehat{\epsilon_k} = \frac{3xy(n-k+3)[(k-1)!]^2[(n-k+1)!]^2}{[(n+2)!]^2}, \ 1 \le k \le n-2,$$
$$\widehat{\delta_k} = \frac{3xy(2n-k+1)[k!]^2[n!]^2[(n-k)!]^2}{[(2n+1)!]^2}, \ 2 \le k \le n-1.$$

Proof. By interchanging rows and columns (even number-times) from $F_k(h)$, we have

$$\widetilde{F_k}(h) := 3xy \left[\begin{array}{cccc|c} & & & & \frac{1}{n-k+2} \\ & & & & \frac{1}{n-k+4} \\ & M_{k,n-k,2} & & & \frac{1}{n-k+5} \\ & & & & \vdots \\ & & & & \frac{1}{n+k+1} \\ \hline \frac{1}{n-k+2} & \frac{1}{n-k+4} & \frac{1}{n-k+5} \ \cdots & \frac{1}{n+k+1} & \frac{1}{n-k+3} - \frac{h}{3xy} \end{array} \right].$$

By Lemma 3.1, all upper-left corner submatrices of $M_{k,n-k,2}$ have positive determinants, and then $M_{k,n-k,2} \ge 0$ and $M_{k,n-k,2}$ is invertible. It follows from Lemma 3.3, we have

$$F_k(h) \ge 0 \Longleftrightarrow \widetilde{F_k}(h) \ge 0 \Longleftrightarrow \det \widetilde{F_k}(h) \ge 0 \Longleftrightarrow \det F_k(h) \ge 0,$$

that is,

$$\{h \in \mathbb{R}_+ : F_k(h) \ge 0\} = \{h \in \mathbb{R}_+ : \det F_k(h) \ge 0\}.$$

Since $\det F_k(h) = (3xy)^{k+1} \det H_{k,n-k} - h \cdot (3xy)^k \det M_{k,n-k,2}$, we obtain easily

$$\widehat{\epsilon_k} = 3xy \frac{\det H_{k,n-k}}{\det M_{k,n-k,2}} = \frac{3xy(n-k+3)[(k-1)!]^2[(n-k+1)!]^2}{[(n+2)!]^2},$$

which satisfies $\widehat{\epsilon_k} = \max\{h \in \mathbb{R}_+ : F_k(h) \geq 0\}$. The case of $G_k(h)$ is similar to the above. Then we obtain

$$\widehat{\delta_k} = 3xy\frac{\det H_{n,k}}{\det M_{n,k,n-k+1}} = \frac{3xy(2n-k+1)[k!]^2[n!]^2[(n-k)!]^2}{[(2n+1)!]^2}.$$

Hence the proof is complete. □

If we apply the weight sequence $\alpha(x,y)$ to (2.7), the function $G_1(0)$ is represented by

$$G_1(0) := 3xy\begin{bmatrix} \frac{1}{3y} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+2} \end{bmatrix}.$$

To find the sufficient and necessary condition for the positivity of $G_1(0)$, using Lemma 3.3, we obtain that

$$\det\left(\frac{1}{3xy}G_1(0)\right) = \det\left[\begin{array}{c|ccc} \left(\frac{1}{3y}-\frac{1}{2}\right)+\frac{1}{2} & 0+\frac{1}{3} & \cdots & 0+\frac{1}{n+2} \\ \hline \frac{1}{3} & & & \\ \vdots & & H_{n-1,3} & \\ \frac{1}{n+2} & & & \end{array}\right]$$
$$= \left(\frac{1}{3y}-\frac{1}{2}\right)\det H_{n-1,3} + \det H_{n,1} \geq 0$$

if and only if

$$y \leq \frac{\det H_{n-1,3}}{\frac{3}{2}\det H_{n-1,3} - 3\det H_{n,1}} = \frac{2(n+2)^2(n+1)^2}{3n(n+3)(n^2+3n+4)}.$$

To discuss the main results of this section, we begin with a computational lemma.

Lemma 3.5. *Under the above notation, if y satisfies the inequality*

$$0 < y \leq s_n := \frac{2(n+2)^2(n+1)^2}{3n(n+3)(n^2+3n+4)},$$

then $G_1(0) \geq 0$.

Theorem 3.6. *Suppose $n \geq 3$. Let $\alpha(x,y)$ be given in* (3.1) *and let $W_{\alpha(x,y)}$ be the associated weighted shift. Then $\mathcal{CWH}^{\langle n \rangle}_{\alpha(x,y)}$ consists of pairs (x,y) such that*

(i) $0 < x \leq y \leq s_n$,

(ii) $\psi_n(x,y,\phi) \geq 0$ *for any* $\phi = \{\phi_i\}_{i=1}^n$ *in* $\mathbb{C}$ *with* $\phi_1 = 1$, *where*

$$\psi_n(x,y,\phi) = |\phi_n|^2\left(\det A_n(x,y) + \widehat{\epsilon}\det B_n(x,y)\right) + \widehat{\delta}\det A_{n-1}(x,y) + \widehat{\epsilon\delta}\det B_{n-1}(x,y) \quad (3.4)$$

with $\widehat{\epsilon} = \sum_{l=1}^{n-2}\widehat{\epsilon_l}\left|\phi_{n-l+1}\right|^2$ *and* $\widehat{\delta} = \sum_{l=2}^{n-1}\widehat{\delta_l}\left|\phi_{n-l}\right|^2$.

Proof. Applying the weight sequence $\alpha(x,y)$ to (2.9), we get

$$\det\Delta_n(x,y) := 3^{n+1}\det\begin{bmatrix} \frac{1}{3xy} & \frac{1}{3y} & \frac{1}{3} & \cdots & \frac{\phi_n}{n+1} \\ \frac{1}{3y} & \frac{1}{3}+\frac{\widehat{\epsilon}}{3xy} & \frac{1}{4} & \cdots & \frac{\phi_n}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{\phi_n}{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\overline{\phi_n}}{n+1} & \frac{\overline{\phi_n}}{n+2} & \frac{\overline{\phi_n}}{n+3} & \cdots & \frac{|\phi_n|^2}{2n+1}+\frac{\widehat{\delta}}{3xy} \end{bmatrix}$$
$$= 3^{n+1}\psi_n(x,y,\phi).$$

The submatrix obtained by deleting the first row and column from $\Delta_n(x,y)$ has positive determinant as below:

$$|\phi_n|^2 \det H_{n-1,2} + \frac{\widehat{\epsilon}}{3xy}|\phi_n|^2 \det H_{n-2,4} + \frac{\widehat{\delta}}{3xy} \det H_{n-2,2} + \frac{\widehat{\epsilon}}{3xy}\frac{\widehat{\delta}}{3xy} \det H_{n-3,4} > 0.$$

Similarly, its all upper-left corner submatrices have positive determinants, it follows from Lemma 3.3 that $\Delta_n(x,y) \geq 0$ if and only if $\psi_n(x,y,\phi) \geq 0$. By Lemma 2.4 and Lemma 3.5, we have

$$\mathcal{CWH}^{\langle n\rangle}_{\alpha(x,y)} = \{(x,y) : \text{conditions (i) and (ii) hold}\}.$$

Hence the proof is complete. □

To get a useful formula for a sufficient condition of the weak n-hyponormality, we apply Theorem 3.6 with

$$\widehat{\epsilon_2} = \cdots = \widehat{\epsilon}_{n-1} = \widehat{\delta_1} = \cdots = \widehat{\delta}_{n-2} = 0;$$

we can confirm that our result covers some known results by using formulas produced in this case.

Setting $t := |\phi_n|^2$, the equation $\psi_n(x,y,\phi)$ in (3.4) is represented by

$$\begin{aligned}\psi_n(x,y,\phi) &= t\cdot \det A_n(x,y) + t^2\cdot \widehat{\epsilon_1} \det B_n(x,y) + \widehat{\delta}_{n-1} \det A_{n-1}(x,y) + t\cdot \widehat{\epsilon_1}\widehat{\delta}_{n-1} \det B_{n-1}(x,y)\\ &= \frac{\Omega_n}{n+3}\left(f_n(x,y)\,t^2 + g_n(x,y)\,t + h_n(x,y)\right),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned} f_n(x,y) &= \frac{n^2}{12(n+2)(n+1)}\left(\frac{4}{(n+1)^2} + \frac{1}{3xy} - 1\right),\\ g_n(x,y) &= \left(\frac{1}{3xy} - 1\right)\frac{n^3+20n^2+21n+6}{12\,(n+1)^3\,(n+2)(2n+1)} - \left(\frac{1}{3y} - \frac{1}{2}\right)^2 \frac{n^2(n+1)}{12}\\ &\quad - \left(\frac{1}{3y} - \frac{1}{2}\right)\frac{n}{(n+2)(n+1)} + \frac{1}{3\,(2n+1)\,(n+1)^3},\\ h_n(x,y) &= \frac{1}{(2n+1)(n+2)(n+1)}\left(\frac{1}{n^2(n+1)^2} - \frac{(n-1)\left(\frac{1}{3y}-\frac{1}{2}\right)}{n+1} - \frac{n^2(n-1)^2\left(\frac{1}{3y}-\frac{1}{2}\right)^2}{12} + \frac{\frac{1}{3xy}-1}{(n+1)^2}\right).\end{aligned}$$

We now obtain a sufficient condition for the weak n-hyponormality.

Theorem 3.7. *Let $\alpha(x,y)$ be given in* (3.1) *and let $W_{\alpha(x,y)}$ be the associated weighted shift. Suppose $n \geq 3$. If the following two conditions hold;*

(i) $0 < x < y \leq s_n$,

(ii) $0 < x \leq X_n(y) := \begin{cases} \gamma_3(y), & n = 3;\\ \gamma_n(y), & 4 \leq n \leq 15 \text{ and } 0 < y \leq \breve{s}_n;\\ \widetilde{h}_n(y), & 4 \leq n \leq 15 \text{ and } \breve{s}_n < y \leq s_n;\\ \widetilde{h}_n(y), & n \geq 16,\end{cases}$

where

$$\breve{s}_n := \frac{2n^2(n+1)(n^3-14n^2-17n-6)}{3(n-1)(n+2)(n^4-14n^3-15n^2-36n-12)}$$

and

$$\gamma_n(y) = \frac{12\left(n^4 - 50n^3 - 95n^2 - 60n - 12\right)y}{-\xi_{32}(n)y^2 + \xi_{31}(n)y - \xi_{30}(n)};\ \widetilde{h}_n(y) = \frac{144n^2 y}{(n^2-1)(\eta_2(n)y^2 - \eta_1(n)y + \eta_0(n))}$$

with

$$\begin{aligned}
\xi_{32}(n) &= 9(n-1)(n+2)(2n^7 + 15n^6 + 49n^5 + 95n^4 + 65n^3 - 54n^2 + 76n + 40),\\
\xi_{31}(n) &= 12n(n-1)(2n+1)(n+2)(n+1)\left(n^4 + 6n^3 + 15n^2 + 22n + 8\right),\\
\xi_{30}(n) &= 4n^2(2n+1)(n+2)^2(n+1)^4,\\
\eta_2(n) &= 9(n-2)(n+2)\left(n^4 + 3n^2 - 12\right),\\
\eta_1(n) &= 12n^2(n-2)(n+2)\left(n^2+3\right),\\
\eta_0(n) &= 4n^4(n+1)(n-1),
\end{aligned}$$

then $W_{\alpha(x,y)}$ *is weakly n-hyponormal.*

Proof. According to the condition (i) of Theorem 3.6, we will prove this theorem under the condition $0 < y \le s_n$. To see the positivity of $\psi_n(x, y, \phi)$ in (3.5) for $n \ge 3$, we define a function $\varphi_n(x, y, t)$ by

$$\varphi_n(x,y,t) := f_n(x,y)t^2 + g_n(x,y)t + h_n(x,y),\ \ n \ge 3,\ \ t \ge 0.$$

Since $\varphi_n(x, y, t)$ is a quadratic polynomial in $t \ge 0$, the equivalent condition for $\varphi_n(x, y, t) \ge 0$ ($t \ge 0$) about x and y is one of the following two cases:
Case 1. $f_n(x, y) \ge 0, g_n(x, y) \ge 0$ and $h_n(x, y) \ge 0$;
Case 2. $f_n(x, y) \ge 0, g_n(x, y) < 0$ and $g_n(x, y)^2 - 4f_n(x, y)h_n(x, y) \le 0$.

To check Case 1, we observe that

$$\begin{aligned}
f_n(x,y) \ge 0 &\Longleftrightarrow 0 < x \le \widetilde{f_n}(y) := \frac{(n+1)^2}{3y(n-1)(n+3)},\\
g_n(x,y) \ge 0 &\Longleftrightarrow 0 < x \le \widetilde{g_n}(y) := \frac{12\left(n^3 + 20n^2 + 21n + 6\right)y}{(n+1)(\zeta_2(n)y^2 - \zeta_1(n)y + \zeta_0(n))},\\
h_n(x,y) \ge 0 &\Longleftrightarrow 0 < x \le \widetilde{h}_n(y),
\end{aligned}$$

where

$$\begin{aligned}
\zeta_2(n) &= 9(n-1)(n+2)\left(2n^5 + 9n^4 + 18n^3 + 23n^2 - 24n + 4\right),\\
\zeta_1(n) &= 12n(n-1)(2n+1)(n+3)(n+1)\left(n^2 + 2n + 4\right),\\
\zeta_0(n) &= 4n^2(n+2)(2n+1)(n+1)^3.
\end{aligned}$$

By a simple computation, we get $\widetilde{f_n}(y) \ge \widetilde{g_n}(y)$ for $0 < y \le s_n$. Given a fixed $y \in (0, s_n)$, we obtain a range of x satisfying Case 1 is $0 < x \le \min\{\widetilde{g_n}(y), \widetilde{h}_n(y)\}$.

To check Case 2, we put $D_n := g_n(x, y)^2 - 4f_n(x, y)h_n(x, y)$ for the discriminant of quadratic polynomial. Then we obtain

$$D_n = \frac{\left(x\Theta_n^{(1)}(y) + \Theta_n^{(2)}(y)\right)\left(x\Theta_n^{(3)}(y) + \Theta_n^{(4)}(y)\right)}{186624x^2y^4(n+2)^2(2n+1)^2(n+1)^6},$$

where

$$\Theta_n^{(1)}(y) = \xi_{12}(n)y^2 - \xi_{11}(n)y + \xi_{10}(n),$$
$$\Theta_n^{(2)}(y) = 12\left(n^2 - 6n - 3\right)y,$$
$$\Theta_n^{(3)}(y) = \xi_{32}(n)y^2 - \xi_{31}(n)y + \xi_{30}(n),$$
$$\Theta_n^{(4)}(y) = 12\left(n^4 - 50n^3 - 95n^2 - 60n - 12\right)y,$$

with

$$\xi_{12}(n) = 9(n-1)(n+1)\left(2n^5 + 9n^4 + 18n^3 + 23n^2 - 8n - 28\right),$$
$$\xi_{11}(n) = 12n(n-1)(2n+1)(n+1)\left(n^3 + 4n^2 + 7n + 8\right),$$
$$\xi_{10}(n) = 4n^2(2n+1)(n+1)^4.$$

If $D_n = 0$, we can obtain that $x = \delta_n(y)$ or $x = \gamma_n(y)$, where

$$\delta_n(y) = \frac{12\left(n^2 - 6n - 3\right)y}{-\xi_{12}(n)y^2 + \xi_{11}(n)y - \xi_{10}(n)}, \quad \gamma_n(y) = \frac{12\left(n^4 - 50n^3 - 95n^2 - 60n - 12\right)y}{-\xi_{32}(n)y^2 + \xi_{31}(n)y - \xi_{30}(n)}.$$

Firstly, we suppose that $\widetilde{h}_n(y) \le \widetilde{g}_n(y)$. Considering Case 2, since $h_n(x,y) < 0$ when $g_n(x,y) < 0$, we get $D_n > 0$, which is impossible. i.e., $X_n(y) = \widetilde{h}_n(y)$. Secondly, we may assume that $\widetilde{h}_n(y) \ge \widetilde{g}_n(y)$. Observe that $\Theta_n^{(3)}(y) > 0$. If $\Theta_n^{(1)}(y) \ge 0$, by some technical computations we have that $\delta_n(y) < \widetilde{g}_n(y) < \gamma_n(y) < \widetilde{h}_n(y)$, i.e., a range of x satisfying Case 2 becomes $\tilde{g}_n(y) \le x \le \gamma_n(y)$. On the other hand, if $\Theta_n^{(1)}(y) < 0$, then $\widetilde{g}_n(y) < \gamma_n(y) < \widetilde{h}_n(y) < \delta_n(y)$, and we have the same range in this case also. Therefore $X_n(y) = \gamma_n(y)$.

By direct computations, we get $\widetilde{h}_3(y) \ge \widetilde{g}_3(y)$ and $\widetilde{h}_n(y) \le \widetilde{g}_n(y)$ for $n \ge 16$, which induce $X_3(y) = \gamma_3(y)$ and $X_n(y) = \widetilde{h}_n(y)$ for $n \ge 16$. For $4 \le n \le 15$, we have the following

$$\widetilde{h}_n(y) \ge \widetilde{g}_n(y) \Longleftrightarrow 0 < y \le \check{s}_n \text{ and } \widetilde{h}_n(y) \le \widetilde{g}_n(y) \Longleftrightarrow \check{s}_n \le y \le s_n.$$

Thus $\varphi_n(x,y,t) \ge 0$, and so $\psi_n(x,y,\phi) \ge 0$ for $n \ge 3$. Hence the proof is complete. □

We now discuss distinctions for the weak n-hyponormality and the n-hyponormality of a weighted shift $W_{\alpha(x,y)}$ with the weight sequence $\alpha(x,y)$ in (3.1). Recall an equivalent condition for the n-hyponormality of the weighted shift $W_{\alpha(x,y)}$ from [13] or [15] as below.

Proposition 3.8 ([13, Theorem 3.6], [15, p.1371]). *Let $\alpha(x,y)$ be given in* (3.1) *and let $W_{\alpha(x,y)}$ be the associated weighted shift. Then $W_{\alpha(x,y)}$ is n-hyponormal if and only if it holds that*

$$0 < y \le \frac{2(n+1)^2(n+2)^2}{3n(n+3)(n^2+3n+4)}; \quad 0 < x \le \frac{144(n+1)^2 y}{n(n+2)(9\varphi_{n2}y^2 - 12\varphi_{n1}y + 4\varphi_{n0})} =: t_n,$$

where

$$\varphi_{n0} = n(n+2)(n+1)^4,$$
$$\varphi_{n1} = (n-1)(n+3)\left(n^2+2n+4\right)(n+1)^2$$
$$\varphi_{n2} = (n-1)(n+3)\left(n^4+4n^3+9n^2+10n-8\right).$$

According to Theorem 3.7 and Proposition 3.8, we may obtain the following corollary which is an improvement of [25, Theorem 4.1].

Corollary 3.9. *Let $\alpha(x,y)$ be given in* (3.1) *and let $W_{\alpha(x,y)}$ be the associated weighted shift. Then it holds that*

$$\{(x,y) : 0 < y \le s_n,\ t_n < x \le X_n(y)\} \subset \mathcal{WH}_{\alpha(x,y)}^{\langle n\rangle} \backslash \mathcal{SH}_{\alpha(x,y)}^{\langle n\rangle}, \quad n \ge 3,$$

where $X_n(y)$ is as in Theorem 3.7, s_n is as in Lemma 3.5 and t_n is as in Proposition 3.8.

4. Further examples

It follows from [11, Theorem 2.7] and [4, Theorem 2.4] that if a weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is given by

$$\alpha_n = \sqrt{\frac{an+b}{cn+d}} \ \ (n \geq 0),$$

where $a, b, c, d > 0$ with $ad - bc > 0$, then the associate weighted shift $W_\alpha \equiv S(a, b, c, d)$ is subnormal with the Berger measure

$$d\mu(t) = \left(\frac{c}{d}\right)^{b/a} \frac{\Gamma(\frac{d}{c})}{\Gamma(\frac{b}{a})\Gamma\left(\frac{d}{c} - \frac{b}{a}\right)} t^{b/a-1} \left(1 - \frac{ct}{a}\right)^{d/c-b/a-1} dt$$

with support $\left[0, \frac{a}{c}\right]$. This operator $S(a, b, c, d)$ covers several known examples, for example, if $a = b = c = 1$ and $d = 2$, the associated weighted shift $S(1, 1, 1, 2)$ is Bergman shift. In Section 3, we applied Bergman shift $S(1, 1, 1, 2)$ to Lemma 2.4 to study Problem 1.2. We may follow the same technique in Section 3 with a sequence $\alpha(x, y)$ defined by

$$\alpha(x, y): \ \sqrt{x}, \ \sqrt{y}, \ \sqrt{\frac{2a+b}{2c+d}}, \ \sqrt{\frac{3a+b}{3c+d}}, \ldots,$$

where x and y are positive real numbers, and will find a nonempty subregion of $\mathcal{WH}_{\alpha(x,y)}^{\langle n\rangle} \setminus \mathcal{SH}_{\alpha(x,y)}^{\langle n\rangle}$ for $n \geq 3$. For a simple computation, we consider a sequence $\alpha(x) := \alpha\left(x, \frac{1}{2}\right)$ defined by

$$\alpha_0 := \sqrt{x} \ \text{ and } \ \alpha_n := \sqrt{\frac{2n+1}{2n+4}}, \ \ n \in \mathbb{N}, \tag{4.1}$$

where x is a positive real number. Observe that the moment sequence $\gamma := \{\gamma_n\}_{n=0}^{\infty}$ of $\alpha(x)$ is given by

$$\gamma_n = \begin{cases} 1, & n = 0; \\ \frac{1}{4^{n-1}} C_n x, & n \geq 1, \end{cases} \tag{4.2}$$

with $C_n := \frac{1}{n+1}\binom{2n}{n}$, which is called the *Catalan number* ([2]). It follows from [28] that determinants of Hankel matrices of C_n are given by

$$\det[C_{i+j+k}]_{i,j=0}^{n} = \begin{cases} 1, & k = 0, 1; \\ \prod_{1 \leq i \leq j \leq k-1} \frac{i+j+2n+2}{i+j}, & k \geq 2. \end{cases} \tag{4.3}$$

We now find the elements of the set $\mathcal{SH}_{\alpha(x,\frac{1}{2})}^{\langle n\rangle}$ for $n \geq 2$ as the following proposition.

Proposition 4.1. *Let $\alpha(x)$ be a sequence as in* (4.1) *and let $W_{\alpha(x)}$ be the associate weighted shift. Then $W_{\alpha(x)}$ is n-hyponormal if and only if $0 < x \leq \frac{n+1}{4n}$, namely,*

$$\mathcal{SH}_{\alpha(x,\frac{1}{2})}^{\langle n\rangle} = \left(0, \frac{n+1}{4n}\right] \times \left\{\frac{1}{2}\right\}.$$

Moreover, $W_{\alpha(x)}$ is subnormal if and only if $0 < x \leq \frac{1}{4}$.

Proof. Let $\gamma = \{\gamma_i\}_{i=0}^{\infty}$ be as in (4.2). Recall that $W_{\alpha(x)}$ is n-hyponormal if and only if the Hankel matrix $H_{n,k}(\gamma)$ is positive for all $k \geq 0$ ([6, Theorem 4]). By (4.3), we first observe that

$$\begin{aligned}\det H_{n,0}(\gamma) &= \left(\frac{1}{4}\right)^{n^2-1} x^{n+1} \det \begin{bmatrix} \frac{1}{4x} & C_1 & C_2 & \cdots & C_n \\ C_1 & C_2 & C_3 & \cdots & C_{n+1} \\ C_2 & C_3 & C_4 & \cdots & C_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n+1} & C_{n+2} & \cdots & C_{2n} \end{bmatrix} \\ &= \left(\frac{1}{4}\right)^{n^2-1} x^{n+1} \left(\left(\frac{1}{4x} - C_0\right) \det[C_{i+j+2}]_{i,j=0}^{n-1} + \det[C_{i+j}]_{i,j=0}^{n}\right) \\ &= \left(\frac{1}{4}\right)^{n^2-1} x^{n+1} \left(\frac{n+1}{4x} - n\right)\end{aligned}$$

is positive if and only if $0 < x < \frac{n+1}{4n}$. By (4.3), we get $\det[C_{i+j+2}]_{i,j=0}^{n-1} = n+1 > 0$, and applying Lemma 3.3, we obtain that $H_{n,0}(\gamma) \geq 0$ if and only if $x \leq \frac{n+1}{4n}$. The "moreover" part is obvious. □

Now we use the technique in Section 2 with $\widehat{\epsilon_2} = \cdots = \widehat{\epsilon}_{n-1} = \widehat{\delta_1} = \cdots = \widehat{\delta}_{n-2} = 0$ to get a formula for the sufficient condition for the weak n-hyponormality of $W_{\alpha(x)}$. Applying $\alpha(x)$ to (2.5) and (2.7), we obtain the matrix-valued functions

$$F_1(h) = \begin{bmatrix} \frac{1}{4^{n-2}} C_{n-1} x & \frac{1}{4^{n-1}} C_n x \\ \frac{1}{4^{n-1}} C_n x & \frac{1}{4^n} C_{n+1} x - h \end{bmatrix} = \frac{x}{4^{n-2}} \begin{bmatrix} C_{n-1} & \frac{1}{4} C_n \\ \frac{1}{4} C_n & \frac{1}{16} C_{n+1} - \frac{4^{n-2}}{x} h \end{bmatrix}$$

and

$$G_{n-1}(h) = \frac{x}{4^{n-2}} \begin{bmatrix} C_{n-1} & \frac{1}{4} C_n & \frac{1}{4^2} C_{n+1} & \cdots & \frac{1}{4^n} C_{2n-1} \\ \frac{1}{4} C_n & \frac{1}{4^2} C_{n+1} - \frac{4^{n-2}}{x} h & \frac{1}{4^3} C_{n+2} & \cdots & \frac{1}{4^{n+1}} C_{2n} \\ \frac{1}{4^2} C_{n+1} & \frac{1}{4^3} C_{n+2} & \frac{1}{4^4} C_{n+3} & \cdots & \frac{1}{4^{n+2}} C_{2n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4^n} C_{2n-1} & \frac{1}{4^{n+1}} C_{2n} & \frac{1}{4^{n+2}} C_{2n+1} & \cdots & \frac{1}{4^{2n}} C_{3n-1} \end{bmatrix}.$$

Take $\widehat{\epsilon_1} > 0$ such that $\det F_{n-1}(\widehat{\epsilon_1}) = 0$, i.e.,

$$\widehat{\epsilon_1} = \frac{3 C_n x}{2^{2n-1}(n+1)(n+2)}.$$

Observe that

$$\det G_{n-1}(h) = \frac{x}{4^{n^2+2n-2}} \det \begin{bmatrix} C_{n-1} & C_n & C_{n+1} & \cdots & C_{2n-1} \\ C_n & C_{n+1} - \frac{4^n}{x} h & C_{n+2} & \cdots & C_{2n} \\ C_{n+1} & C_{n+2} & C_{n+3} & \cdots & C_{2n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{2n-1} & C_{2n} & C_{2n+1} & \cdots & C_{3n-1} \end{bmatrix}.$$

Under the authors' knowledge, it looks difficult to estimate the exact value $\widehat{\delta}_{n-1} > 0$ such that $\det G_{n-1}(\widehat{\delta}_{n-1}) = 0$ with respect to the general number $n \in \mathbb{N}$. (Nevertheless we can find the value $\widehat{\delta}_{n-1} > 0$ in some low numbers $n \in \mathbb{N}$ by using the computer software; for examples, $n = 3, 4, ..., 20$, and more, etc.) Since $\det[C_{i+j+n-1}]_{i,j=0}^{n} > 0$, there exists a unique value $\widehat{\delta}_{n-1} > 0$ such that $\det G_{n-1}(\widehat{\delta}_{n-1}) = 0$. Hence the hypothesis of Proposition 4.2 below is valid.

Proposition 4.2. *Let $\alpha(x)$ be given in* (4.2) *and let $W_{\alpha(x)}$ be the associated weighted shift. Suppose*

$$\psi_n(x,t) := \det \begin{bmatrix} \frac{1}{4x} & C_1 & C_2 & \cdots & C_n\sqrt{t} \\ C_1 & C_2+\kappa_n t & C_3 & \cdots & C_{n+1}\sqrt{t} \\ C_2 & C_3 & C_4 & \cdots & C_{n+2}\sqrt{t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n\sqrt{t} & C_{n+1}\sqrt{t} & C_{n+2}\sqrt{t} & \cdots & C_{2n}t+\nu_n \end{bmatrix} \geq 0 \text{ for all } t \geq 0,$$

with $\kappa_n := \frac{\widehat{\epsilon_1}}{4x} = \frac{3C_n}{2^{2n+1}(n+1)(n+2)}$ and $\nu_n := \frac{4^{2n-1}}{x}\widehat{\delta}_{n-1}$, where $\widehat{\delta}_{n-1}$ is some positive real number such that $\det G_{n-1}(\widehat{\delta}_{n-1}) = 0$. *Then $W_{\alpha(x)}$ is weakly n-hyponormal.*

Proof. In Lemma 2.2, if we put $t := |\phi_n|^2$ and $\phi_1 = 1$, then $\epsilon = \widehat{\epsilon_1}t$ and $\delta = \widehat{\delta}_{n-1}$. By some determinant properties, we get

$$\det \Phi_n(\epsilon,\delta) = \det \Phi_n(\widehat{\epsilon_1}t, \widehat{\delta}_{n-1}) = \frac{x^{n+1}}{4^{n^2-1}}\psi_n(x,t).$$

Similarly to the proof of Theorem 3.6, the submatrix obtained by deleting the first row and column from $\Phi_n(\epsilon,\delta)$ has positive determinant and its all upper-left corner submatrices have positive determinants. Hence, by Lemma 3.3, we have $\Phi_n(\epsilon,\delta) \geq 0$ if and only if $\det \Phi_n(\epsilon,\delta) \geq 0$ if and only if $\psi_n(x,t) \geq 0$. Applying to Lemma 2.4, we obtain this proposition. □

In Proposition 4.2, if $\widehat{\delta}_{n-1}$ vanishes, then we obtain that $\psi_n(x,t) \geq 0$ for all $t \geq 0$ if and only if $x \leq \frac{n+1}{4n}$, which is the sufficient and necessary condition for the n-hyponormality. In this situation we can not distinguish between the n-hyponormality and the weak n-hyponormality. To avoid such undesirable situation, we have to find $\widehat{\delta}_{n-1} > 0$ such that $\det G_{n-1}(\widehat{\delta}_{n-1}) = 0$.

Recall an element fact that if a Hankel matrix $\left[s_{i+j}\right]_{i,j=0}^{n}$ has rank r $(1 \leq r \leq n)$, it holds that

$$\det\left[s_{i+j}\right]_{i,j=0}^{r-1} \det\left[s_{i+j+m}\right]_{i,j=0}^{r-1} = \det\left[s_{i+j+1}\right]_{i,j=0}^{r-1} \det\left[s_{i+j+m-1}\right]_{i,j=0}^{r-1}, \tag{4.4}$$

for all $1 \leq m \leq 2n-2r+2$; see [19, p.60]. By using this recurrence formula, we see the gap between the n-hyponormality and weak n-hyponormality as following lemma.

Lemma 4.3. *Let $\alpha(x)$ be given in* (4.1) *and let $W_{\alpha(x)}$ be the associated weighted shift. Then $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,\frac{1}{2})} \backslash \mathcal{SH}^{\langle n\rangle}_{\alpha(x,\frac{1}{2})} \neq \emptyset$ for $n \geq 3$.*

Proof. Let $\gamma = \{\gamma_i\}_{i=0}^{\infty}$ be the associated moment sequence of $\alpha(x)$ as in (4.2). For our convenience, we let $M_n(x) := H_{n,0}(\gamma)$ and $M_n'(x)$ be the submatrix obtained by deleting the second row and column from $M_n(x)$. By a direct computation, we have

$$\psi_n(x,t) = \kappa_n \det M_n'(x) t^2 + \left(\det M_n(x) + \kappa_n\nu_n \det M_{n-1}'(x)\right)t + \nu_n \det M_{n-1}(x). \tag{4.5}$$

Since the range of x for the weak n-hyponormality contains the interval $\left(0, \frac{n+1}{4n}\right]$, we may assume that $\psi_n(x,t) \geq 0$ for all $t \geq 0$ and $0 < x \leq \frac{n+1}{4n}$. If $x = \frac{n+1}{4n}$ in (4.5), it follows from the proof of Proposition 4.1 that

$$\psi_n\left(\frac{n+1}{4n}, t\right) = \kappa_n \det M_n'\left(\frac{n+1}{4n}\right)t^2 + \kappa_n\nu_n \det M_{n-1}'\left(\frac{n+1}{4n}\right)t + \nu_n \det M_{n-1}\left(\frac{n+1}{4n}\right),$$

and $\det M_{n-1}\left(\frac{n+1}{4n}\right) > 0$. Since $M_{n-1}\left(\frac{n+1}{4n}\right) \geq 0$, its all submatrices have positive determinants. i.e., $\det M_{n-1}'\left(\frac{n+1}{4n}\right) > 0$. To show $\psi_n\left(\frac{n+1}{4n}, t\right) > 0$ for all $t \geq 0$, we claim that $\det M_n'\left(\frac{n+1}{4n}\right) > 0$. Let $r = \operatorname{rank} M_n\left(\frac{n+1}{4n}\right)$. Since $\det M_n\left(\frac{n+1}{4n}\right) = 0$, we have $1 \leq r \leq n$. Then

$$\det M_{r-1}\left(\frac{n+1}{4n}\right) = \left(\frac{n}{n+1} - C_0\right)\det\left[C_{i+j+2}\right]_{i,j=0}^{r-2} + \det\left[C_{i+j}\right]_{i,j=0}^{r-1} = \frac{n-r+1}{n+1},$$

$$\det H_{r-1,1}(\gamma) = \det \left[C_{i+j+1} \right]_{i,j=0}^{r-1} = 1,$$

and

$$\det H_{r-1,2}(\gamma) = \det \left[C_{i+j+2} \right]_{i,j=0}^{r-1} = r+1.$$

By (4.4) with $m = 2$, we obtain that $\frac{n-r+1}{n+1}(r+1) = 1$, i.e., $r = n$. Thus the rank of $M_n\left(\frac{n+1}{4n}\right)$ becomes n, which implies $\det M_n'\left(\frac{n+1}{4n}\right) > 0$. Since there is $\delta > 0$ such that $\det M_n'(x) > 0$ on $\left(0, \frac{n+1}{4n} + \delta\right]$, we have that for each $x \in \left(0, \frac{n+1}{4n} + \delta\right]$, $\psi_n(x,t)$ has the minimum on $[0,\infty)$. We consider a continuous function $m_n(x)$ on $\left(0, \frac{n+1}{4n} + \delta\right)$ defined by

$$m_n(x) = \min\{\psi_n(x,t) : t \geq 0\}.$$

Since $m_n(\frac{n+1}{4n})$ is strictly positive, there exists $\varepsilon \in (0,\delta)$ such that $m_n(x) \geq 0$ on $\left(0, \frac{n+1}{4n} + \varepsilon\right]$, i.e., for each $x \in \left(0, \frac{n+1}{4n} + \varepsilon\right]$, $\psi_n(x,t) \geq 0$ for all $t \geq 0$. Therefore we have this Lemma. □

We now obtain a nonempty subregion of $\mathcal{WH}_{\alpha(x,y)}^{\langle n\rangle} \setminus \mathcal{SH}_{\alpha(x,y)}^{\langle n\rangle}$ for $n \geq 3$.

Theorem 4.4. *Let $\alpha(x,y)$ be a sequence defined by*

$$\alpha_0 := \sqrt{x}, \alpha_1 = \sqrt{y}, \ \ \alpha_k := \sqrt{\frac{2k+1}{2k+4}}, \ k \geq 2,$$

and let $W_{\alpha(x,y)}$ be the associated weighted shift. Then for each $n \geq 3$, there exist $\varepsilon_n > 0$ and a continuous positive real function $\sigma_n(x)$ such that

$$\left\{(x,y) : \frac{n+1}{4n} < x < \frac{n+1}{4n} + \varepsilon_n \text{ and } \frac{1}{2} - \sigma_n(x) < y < \frac{1}{2} + \sigma_n(x)\right\} \subset \mathcal{WH}_{\alpha(x,y)}^{\langle n\rangle} \setminus \mathcal{SH}_{\alpha(x,y)}^{\langle n\rangle}.$$

Proof. The existence of $\varepsilon_n > 0$ follows from Lemma 4.3, and so it is sufficient to find $\sigma_n(x) > 0$ for $n \geq 3$. Suppose $n \geq 3$. Let $s \in \left(\frac{n+1}{4n}, \frac{n+1}{4n} + \varepsilon_n\right)$ be fixed. We apply a weight sequence $\alpha(s,y)$ to Algorithm 2.3. Firstly, we check for the positivity of $G_1(0)$ in (2.7). Set $f(y) := \det G_1(0) = \det[\gamma_{i+j+1}]_{i,j=0}^{n}$, where γ_k are moments of $\alpha(s,y)$. Then

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{4}\right)^{n(n+1)} s \det[C_{i+j+1}]_{i,j=0}^{n} = s\left(\frac{1}{4}\right)^{n(n+1)} > 0.$$

By the continuity of f, there exists $\sigma_n^{(1)}(s) > 0$ such that $f(y) > 0$ for

$$\frac{1}{2} - \sigma_n^{(1)}(s) < y < \frac{1}{2} + \sigma_n^{(1)}(s).$$

Secondly, we check for positivity of $\Delta_n(\sqrt{s}, \sqrt{y})$ in (2.9) with $\widehat{\epsilon_i} = 0 = \widehat{\delta_i}$ $(i = 2, \ldots, n-2)$. Set $g(y) := \det \Delta_n(\sqrt{s}, \sqrt{y})$. Then g is a rational function in y and it is continuous obviously. According to the proof of Lemma 4.3, we see that $g(\frac{1}{2}) > 0$. Similarly to the first case, there exists $\sigma_n^{(2)}(s) > 0$ such that $g(y) > 0$ for $\frac{1}{2} - \sigma_n^{(2)}(s) < y < \frac{1}{2} + \sigma_n^{(2)}(s)$. Taking $\sigma_n(s) := \min\{\sigma_n^{(1)}(s), \sigma_n^{(2)}(s)\}$, we can see that

$$\left\{(x,y) : \frac{n+1}{4n} < x < \frac{n+1}{4n} + \varepsilon_n, \ \frac{1}{2} - \sigma_n(x) < y < \frac{1}{2} + \sigma_n(x)\right\} \subset \mathcal{WH}_{\alpha(x,y)}^{\langle n\rangle} \setminus \mathcal{SH}_{\alpha(x,y)}^{\langle n\rangle},$$

which proves the theorem. □

Remark 4.5. As seeing in this section, we can find several weighted shifts $W_{\alpha(x,y)}$ such that $\mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)}\backslash\mathcal{SH}^{\langle n\rangle}_{\alpha(x,y)}$ has a nonempty subregion for any $n \geq 3$. We leave such an attempt to the interesting readers.

We close this paper with a concluding open problem.

Problem 4.6. Let $\alpha(x,y)$ be given in (3.1) and let $W_{\alpha(x,y)}$ be the associated weighted shift.

$$\textit{Find the full range of } \mathcal{WH}^{\langle n\rangle}_{\alpha(x,y)}\backslash\mathcal{SH}^{\langle n\rangle}_{\alpha(x,y)} \textit{ for } n \geq 2.$$

This problem is closely related to the long-standing open problems; *"find a concrete weighted shift that is polynomially hyponormal but not subnormal"* and *"whether a polynomially hyponormal weighted shift but not 2-hyponormal exists?"*.

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