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Stability results on non-instantaneous impulsive fractional integro-differential equations with multipoint boundary conditions

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Abstract. The Ulam-Hyers stability for non-instantaneous impulsive fractional integro-differential equations in a Banach space with Caputo-Katugampola fractional derivative is the main focus of this paper. The Krasnoselskii fixed point theorem and the contraction principle play a role in establishing sufficient conditions for existence and uniqueness results. An application is also shown.

1. Introduction

Fractional differential equations are suitable to model the process with hereditary property. It is used in a variety of areas, including biology, physics, economics, and control theory. We suggest the following papers [1, 6, 19, 23] and their references for more information on the theory and implementations. This theory has received a lot of attention from scientists and mathematicians because of these applications.

In dynamical structures such as pharmacotherapy, physical, social sciences, medicine, and mechanical engineering, impulsive fractional differential equations are used to make abrupt changes [2, 7, 11]. It is categorized into two kinds: one is instantaneous impulses, which are short-term perturbations with a negligible duration in comparison to the interval of the entire processes. Noninstantaneous impulses are the other form of change that occurs unexpectedly and lasts for a short period of time. In this way, the study of impulsive fractional differential equations in various aspects for several researchers [3, 4, 9, 25, 26] and references therein.

The investigation of stability is one of the tool of research. The study of this area has become one of the central themes of mathematical analysis. In [24] Yu, discussed the existence and β- Ulam-Hyers stability of fractional differential equations with involving of noninstantaneous impulses. The new class types of Ulam-Heyrs stability of fractional integral boundary conditions was studied in [27]. Selvam et.al. in [21] discussed the Ulam Hyers stability of fractional Duffing equation. In [29] Zada et.al, established the Ulam Stability on Caputo sense of multipoint boundary conditions with noninstantaneous impulsive. In [28] Zada et.al, discussed the Stability sense of fractional differential equations with noninstantaneous

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boundary conditions:

$$\begin{aligned}
&{}^{C}D^{q}y(t) = f(t, y(t)),\ t \in (t_j, s_j], q \in (0, 1],\\
&\quad y(t) = G_i(t, y(t)),\ \ t \in (s_{j-1}, t_j],\ \ i = 1, ..., n,\\
&\quad y(0) = I^{q}y(t)|_{t=0} = 0\\
&\quad y(T) = I^{q}y(t)|_{t=T}.
\end{aligned}$$

Where ${}^{C}D^{q}$ and I^{q}-is Caputo derivative and Riemann-Liouville fractional integral.

Recently, In [3] Agarwal et.al, established the Caputo fractional differential equations with non-instantaneous impulsive and boundary conditions. Non instantaneous impulses with the fractional boundary value problems was referred in [25]. In [10] Gupta et.al discussed nonlinear fractional boundary value with non-instantaneous using Caputo fractional derivative. In [17] Long et.al, discussed the new boundary value problem for non instantaneous impulses with fractional differential equations:

$$\begin{aligned}
&{}^{C}D^{p}_{0,t}w(t) = f(t, w(t)),\ t \in (s_i, t_{i+1}] \subset [0, T], p \in (0, 1),\\
&\quad w(t) = H_i(t, w(t)),\ \ t \in (t_i, s_i],\ \ i = 1, ..., m,\\
&\quad w(T) = w(0) + \chi \int_0^T w(s)ds.
\end{aligned}$$

where f, H_i- is continuous and χ-is constant.

In [22], Thaiprayoon et.al, studied the Langevin equation of Katugampola multipoint integral boundary conditions:

$$\begin{aligned}
&D^{p_1}(D^{p_2} + \omega)w(t) = f(t, w(t)),\ \ 0 < t < T,\\
&w(0) = 0, \quad w(T) = \sum_{i=1}^{\mathrm{n}} \alpha_i \frac{\rho_i^{1-q_i}}{\Gamma(q_i)} \int_0^{\epsilon_i} \frac{s^{p_i - 1}w(s)}{(t^{\rho_i} - s^{\rho_i})^{1-q_i}}ds := \sum_{i=1}^{n} \alpha_i {}^{\rho_i}I^{q_i}w(\epsilon_i),
\end{aligned}$$

where D^{p_i}-Riemann-Liouville fractional derivative, ${}^{\rho_i}I^{q_i}$ be the Katugampola fractional integral operator,and the function f is continuous.

In [18], Mahmudov et.al discussed the following Caputo sense with Katugampola integral conditions:

$$\begin{aligned}
&{}^{C}D^{\alpha_1}w(t) = f(t, w(t)),\ t \in [0, T],\ \ 2 < \alpha_1 \leq 3\\
&\quad w(T) = \vartheta {}^{\varrho}I^{q}w(\tau), 0 < \tau < T,\\
&\quad w'(T) = \chi {}^{\varrho}I^{q}w'(F), 0 < F < T,\\
&\quad w''(T) = \iota {}^{\varrho}I^{q}w''(\zeta), 0 < \zeta < T,
\end{aligned}$$

where $D^{\alpha_1}-$ Caputo fractional derivative, ${}^{\varrho}I^{q}-$ Katugampola integral and f is a continuous.

Inspired by above literature, we consider a Caputo fractional integro- differential equations with non instantaneous impulsive involving Katugampola multi-point integral boundary conditions:

$$ {}^{C}D^{p}w(t) = f(t, w(t), \Psi w(t)), t \in (s_i, t_{i+1}] \subset [0, T], 1 < p \leq 2, \tag{1}$$

$$ w(t) = H_i(t, w(t)),\ \ t \in (t_i, s_i],\ \ i = 1, ..., m, \tag{2}$$

$$ w(0) = 0, \quad w(T) = \sum_{i=1}^{n} \alpha_i \frac{\rho_i^{1-q_i}}{\Gamma(q_i)} \int_0^{\epsilon_i} \frac{s^{\rho_i - 1}w(s)}{(t^{\rho_i} - s^{\rho_i})^{1-q_i}}ds := \sum_{i=1}^{n} \alpha_i {}^{\rho_i}I^{q_i}w(\epsilon_i), \tag{3}$$

where ${}^{C}D^{p}$ is the Caputo fractional derivatives of order p, ${}^{\rho_i}I^{q_i}$- Katugampola integral of order $\rho_i > 0, q_i > 0$, and $\epsilon_i \in (0, T), \alpha_i \in \mathbb{R}$, and $0 = s_0 < t_1 \leq t_2 < ... < t_m \leq s_m \leq s_{m+1} = T$,- pre-fixed, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $H_i : [t_i, s_i] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Moreover, $\Psi w(t) = \int_0^t k(t, s)w(s)ds$ and $k \in C(D, \mathbb{R}^+)$ with domain $D = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$.

Let the space $PC([0,T],\mathbb{R}) = \{w : [0,T] \to \mathbb{R} : w \in C(t_k, t_{k+1}], \mathbb{R}\}$ be continuous and there exist $w(t_k^-)$ and $w(t_k^+)$ with $w(t_k^-) = w(t_k^+)$ with the norm $\|w\|_{PC} = \sup\{|w(t)| : 0 \le t \le T\}$. Now define

$$PC^1([0,T],\mathbb{R}) := \{w \in PC([0,T],\mathbb{R}) : w' \in PC([0,T],\mathbb{R})\}$$

with the norm $\|w\|_{PC^1} := \max\{\|w\|_{PC}, \|w'\|_{PC}\}$. Clearly, $PC^1([0,T],\mathbb{R})$ induced with the norm $\|.\|_{PC^1}$ is a Banach space.

The structure of this article is organised as follows: Section 2 is devoted to the basic definitions and lemmas which will be used in proving results. In Section 3, we establish the system's existence and uniqueness of solution (1.1)- (1.3) under suitable conditions. In Section 4, we examine at the stability of Ulam under various circumstances. Application is also presented in section 5.

2. Supporting Notes

The definitions mentioned below are from [18].

Definition 2.1. *The Riemann-Liouville fractional derivative of order $q > 0$ for a continuous function f is given by*

$$D_{0^+}^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-p-1} f(s)ds, \qquad n-1 < p < n.$$

Definition 2.2. *The Riemann-Liouville fractional integral of order $p > 0$ for a continuous function f is given by*

$$J^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s)ds.$$

where Γ is defined by $\Gamma(p) = \int_0^\infty e^{-s} s^{p-1} ds$.

Definition 2.3. *For the function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of order p is defined as*

$$^{C}D^p f(t) = \frac{1}{\Gamma(n-p)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{p+1-n}} ds = I^{n-p} f^{(n)}(t), \qquad t > 0, n-1 < p < n.$$

Definition 2.4. *Katugampola fractional integral of order $\mathfrak{p} > 0$ and $\varrho > 0$, of a given function $\mathfrak{F}$ is defined by*

$$^{\varrho}I^p f(t) = \frac{\varrho^{1-p}}{\Gamma(p)} \int_0^t \frac{s^{\varrho-1} f(s)}{(t^\varrho - s^\varrho)^{1-p}} ds.$$

Lemma 2.5. *[10] Let $p > 0$, then $^{C}D^p K(t) = 0$ has solutions $K(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{q-1} t^{q-1}$, and $I^{p\,C}D^p K(t) = K(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{q-1} t^{q-1}$, where $c_i \in \mathbb{R}, i = 0, 1, 2, ..., q-1, q = [p] + 1$.*

Lemma 2.6. *A function $w \in PC([0,T],\mathbb{R})$ is given by,*

$$w(t) = \begin{cases} H_i(s_m) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \omega(s)ds \\ \quad + \frac{1}{\Omega}\Big[\sum_{i=1}^n \alpha_i{}^{\rho_i} I^{p+q_i} \omega(\epsilon_i) d\epsilon_i - {}^{\rho} I^p \omega(s)ds\Big], \;\; t \in [0, t_1], \\ H_i(t), \qquad t \in (t_i, s_i], i = 1, 2, ..., m, \\ H_i(s_i) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \omega(s)ds \\ \quad - \frac{1}{\Gamma(p)} \int_0^{s_i} (s_i - s)^{p-1} \omega(s)ds, \;\; t \in (s_i, t_{i+1}], i = 1, 2, ..., m. \end{cases} \tag{4}$$

is a solution of following system

$$\begin{aligned} &^{C}D^p w(t) = \omega(t) \;\; t \in (s_i, t_{i+1}] \subset [0,T], 1 < p \le 2, \\ &\quad w(t) = H_i(t), \;\; t \in (t_i, s_i], \;\; i = 1, ..., m, \\ &\quad w(0) = 0, \quad w(T) = \sum_{i=1}^n \alpha_i \frac{\rho_i^{1-q_i}}{\Gamma(q_i)} \int_0^{\epsilon_i} \frac{s^{\rho_i - 1} w(s)}{(t^{\rho_i} - s^{\rho_i})^{1-q_i}} ds := \sum_{i=1}^n \alpha_i{}^{\rho_i} I^{q_i} w(\epsilon_i). \end{aligned} \tag{5}$$

Proof. :Assume that $w(t)$ is satisfies for equation (2.2). If $t \in [0, t_1]$, (2.2)-integrating of first equation, then

$$w(t) = w(T) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \omega(s) ds. \tag{6}$$

On otherhand, if $t \in (s_i, t_{i+1}], i = 1, 2, ..., m$ and again integrate of 1^{st} equation, we have

$$w(t) = w(s_i) + \frac{1}{\Gamma(p)} \int_{s_i}^t (t-s)^{p-1} \omega(s) ds. \tag{7}$$

Now, we applying impulsive condition, $w(t) = H_i(t), \ t \in (t_i, s_i]$, we get,

$$w(s_i) = H_i(s_i). \tag{8}$$

Consequently, from (2.4) and (2.5), we get

$$w(t) = H_i(s_i) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \omega(s) ds. \tag{9}$$

and

$$w(t) = H_i(s_i) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \omega(s) ds - \frac{1}{\Gamma(p)} \int_0^{s_i} (s_i - s)^{p-1} \omega(s) ds. \tag{10}$$

Now, By using the boundary conditions:

$$w(T) = \sum_{i=1}^{n} \alpha_i {}^{\rho_i}I^{q_i} w(\epsilon_i) := -\frac{1}{|\Omega|} \Big[\sum_{i=1}^{n} \alpha_i {}^{\rho_i}I^{p+q_i} \omega(\epsilon_i) d\epsilon_i -{}^{\rho} I^p \omega(s) ds \Big]. \tag{11}$$

where

$$\begin{aligned} |\Omega| &= \left[1 - \sum_{i=1}^{n} \sigma_i \frac{\epsilon^{\rho q_i}}{\rho^{q_i} \Gamma(q_i + 1)} \right], \Omega \neq 0, \\ {}^{\rho_i}I^{p+q_i} &= \left\{ \sum_{i=1}^{n} |\alpha_i| \frac{\epsilon^{\rho(p+q_i)}}{\rho^{p+q_i} \Gamma(p + q_i + 1)} \right\}, \\ {}^{\rho}I^p &= \frac{T^{\rho p}}{\rho^p \Gamma(\alpha + 1)}. \end{aligned}$$

Hence, by using the fractional derivatives, integral definitions and Lemmas. Now it's clear that (2.3),(2.7) and (2.8) ⇒ (2.1). □

3. Main Results

We list the assumptions which are required to show the major results of this paper.
(Al_1): There is a positive constant L, G, L_{h_i} such that

$$\begin{aligned} \left| f(t, w_1, \omega_1) - f(t, w_2, \omega_2) \right| &\leq L |w_1 - w_2| + G |\omega_1 - \omega_2|, \texttt{ for } t \in [0, T], \ w_1, w_2, \omega_1, \omega_2 \in \mathbb{R}, \\ |k(t, s, \vartheta) - k(t, s, \nu)| &\leq M |\vartheta - \nu|, \texttt{for } t \in [t_i, s_i] \ \vartheta, \nu \in \mathbb{R}, \\ |H_i(t, v_1) - H_i(t, v_2)| &\leq L_{h_i} |v_1 - v_2|, \texttt{for } v_1, v_2 \in \mathbb{R}. \end{aligned}$$

Theorem 3.1. *Under the assumption* (Al_1) *and if*

$$Z : \max\left\{\max_{i=1,2,\dots,m} L_{h_i} + \frac{(L+GM)}{\Gamma(p+1)}(t_{i+1}^p + s_i^p), L_{h_i} + \frac{(L+GM)}{\Gamma(p+1)} + \frac{(L+GM)}{|\Omega|}\Big[\sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{p+q_i} +^{\rho} I^p\Big]\right\} < 1,$$

then the problems (1.1) – (1.3) *has a unique solution on* $[0, T]$.

Proof. : Let us define an operator $N : PC([0,T],\mathbb{R}) \longrightarrow PC([0,T],\mathbb{R})$ by

$$(Nw)(t) = \begin{cases} H_m(s_m, w(s_m)) + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s, w(s), \Psi w(s))ds \\ +\frac{1}{\Omega}\Big[\sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{p+q_i} f(\epsilon_i, w(\epsilon_i), \Psi w(\epsilon_i))d\epsilon_i -^{\rho} I^p f(s, w(s), \Psi w(s))ds\Big], \quad t \in [0, t_1], \\ H_i(t), \quad t \in (t_i, s_i], i = 1, 2, \dots, m, \\ H_i(s_i) + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s, w(s), \Psi w(s))ds \\ -\frac{1}{\Gamma(p)}\int_0^{s_i} (s_i - s)^{p-1} f(s, w(s), \Psi w(s))ds, \quad t \in (s_i, t_{i+1}], i = 1, 2, \dots, m. \end{cases}$$

One can observe that N is well defined and $(Nw) \in PC([0,T],\mathbb{R})$. Now, we prove that N is a contraction mapping.

Case:1 For $w, \wp \in PC([0,T],\mathbb{R})$ and $t \in [0, t_1]$, we get

$$\begin{aligned} &|(Nw)(t) - (N\wp)(t)| \\ &= L_{h_i}|w(s_m) - \wp(s_m)|\,ds + \frac{(L+GM)}{\Gamma(p+1)}|w - \wp|\,ds \\ &+ \frac{1}{\Omega}\Big[\sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{p+q_i}(L+GM)|w-\wp|\,d\epsilon_i -^{\rho} I^p (L+GM)|w-\wp|\,ds\Big] \\ &\le \left[L_{h_i} + \frac{(L+GM)}{\Gamma(p+1)} + \frac{(L+GM)}{\Omega}\Big[\sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{p+q_i} -^{\rho} I^p\Big]\right]\|w-\wp\|_{PC}. \end{aligned}$$

Case:2 For $t \in (t_i, s_i]$, we obtain

$$\begin{aligned} |(Nw)(t) - (N\wp)(t)| &\le |H_i(t, w(t)) - H_i(t, \wp(t))|, \\ &\le L_{h_i}\|w - \wp\|_{PC}. \end{aligned}$$

Case:3 For $t \in (s_i, t_{i+1}]$, we get

$$\begin{aligned} &|(Nw)(t) - (N\wp)(t)| \\ &\le |H_i(s_i, w(s_i) - H_i(s_i, \wp(s_i)| + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}\Big|f(s, w(s), \Psi w(s)) - f(s, w(s), \Psi w(s))\Big|ds \\ &+ \frac{1}{\Gamma(p)}\int_0^{s_i} (s_i - s)^{p-1}\Big|f(s, w(s), \Psi w(s)) - f(s, w(s), \Psi w(s))\Big|ds, \\ &\le \left[L_{h_i} + \frac{(L+GM)}{\Gamma(p+1)}(t_{i+1}^p + s_i^p)\right]\|w - \wp\|_{PC}. \end{aligned}$$

The above equation $|(Nw)(t) - (N\wp)(t)|_{PC} \le Z\|w - \wp\|_{PC}$, where Z is less than one, therefore N is a contraction. Hence the problem stated in (1.1) – (1.3) has a unique on $w \in PC([0,T],\mathbb{R})$. □

Theorem 3.2. *Suppose that the condition* (Al_1) *is satisfied and the following assumptions hold*
(Al_2)*:There is a constant* $L_{g_i} > 0$*, such that*

$$\Big|f(t, W_1, \omega_1)\Big| \le L_{g_i}(1 + |W_1| + |\omega_1|), \quad t \in [s_i, t_{i+1}], \forall\ W_1, \omega_1 \in \mathbb{R}.$$

(Al_3):*There is a function* $\kappa_i(t), i = 1, 2, ..., m$, *such that*

$$|H_i(t, W_1, \omega_1)| \leq \kappa_i(t), \quad t \in [t_i, s_i], \forall \; W_1, \omega_1 \in \mathbb{R}.$$

Also assume that $M_i : \sup_{t\in[t_i,s_i]} \kappa_i(t) < \infty$, *and* $K := \max\{L_{h_i}\} < 1$, *for all* $i = 1, 2, .., m$. *Then the problems* $(1.1) - (1.3)$ *has at least one solution on* $[0, T]$.

Proof. Consider $B_{p,r} = \{w \in PC([0, T], \mathbb{R}) : \|w\|_{PC} \leq r\}$. Let Q and R be the two operators explained on $B_{p,r}$ by

$$Qw(t) = \begin{cases} H_m(s_m, w(s_m)), & t \in [0, t_1], \\ H_i(t, w(t)), & t \in (t_i, s_i], i = 1, 2, ..., m, \\ H_i(s_i, w(s_i)), & t \in (s_i, t_{i+1}], i = 1, 2, ..., m. \end{cases}$$

and

$$Rw(t) = \begin{cases} \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, w(s), \Psi w(s)) ds \\ +\frac{1}{\Omega}\Big[\sum_{i=1}^n \alpha_i{}^{\rho_i} I^{p+q_i} f(\epsilon_i, w(\epsilon_i), \Psi w(\epsilon_i)) d\epsilon_i -^{\rho} I^p f(s, w(s), \Psi w(t)) ds\Big], & t \in [0, t_1], \\ 0, t \in (t_i, s_i], i = 1, 2, ..., m, \\ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, w(s), \Psi w(s)) ds \\ -\frac{1}{\Gamma(p)} \int_0^{s_i} (s_i - s)^{p-1} f(s, w(s), \Psi w(s)) ds, & t \in (s_i, t_{i+1}], i = 1, 2, ..., m. \end{cases}$$

Step:1 For $w \in B_{p,r}$ then $Qw + Rw \in B_{p,r}$.
Case:1 For t $\in [0, t_1]$, we get

$$\begin{aligned} \|Qw + R\wp\| &\leq |H_m(s_m, w(s_m))| + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Big|f(s, w(s), \Psi w(s))\Big| ds \\ &+ \frac{1}{\Omega}\Big[\sum_{i=1}^n \alpha_i{}^{\rho_i} I^{p+q_i} \Big|f(\epsilon_i, w(\epsilon_i), \Psi w(\epsilon_i))\Big| d\epsilon_i -^{\rho} I^p \Big|f(s, w(s), \Psi w(t))\Big| ds\Big], \\ &\leq \left[M_m + \frac{L_{g_i}}{\Gamma(p+1)} + \frac{L_{g_i}}{\Omega}\Big[\sum_{i=1}^n \alpha_i{}^{\rho_i} I^{p+q_i} -^{\rho} I^p\Big]\right](1+r) \leq r. \end{aligned}$$

Case:2 For each $t \in (t_i, s_i]$, we have

$$\|Qw + R\wp\| \leq |H_i(t, W_1(t))| \leq M_i.$$

Case:3 For each $t \in (s_i, t_{i+1}]$, we obtain

$$\begin{aligned} \|Qw(t) + R\wp(t)\| &\leq |H_i(s_i, w(s_i))| + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Big|f(s, w(s), \Psi w(s))\Big| ds \\ &+ \frac{1}{\Gamma(p)} \int_0^{s_i} (s_i - s)^{p-1} \Big|f(s, w(s), \Psi w(s))\Big| ds, \\ &\leq M_i + \left[\frac{L_{g_i}(s_i^p + t_{i+1}^p)}{\Gamma(p+1)}\right](1+r) \leq r. \end{aligned}$$

Thus

$$Qw + Rw \in B_{p,r}.$$

Step:2 Q is contraction on $B_{p,r}$.
Case:1 $w_1, w_2 \in B_{p,r}$ and $t \in [0, t_1]$, we have

$$|Qw_1(t) - Qw_2(t)| \leq L_{g_m} |w_1(s_m) - w_2(s_m)| \leq L_{g_m} |w_1 - w_2|_{PC}.$$

Case:2. For each $t \in (t_i, s_i], i = 1, 2, ..., m$, we occur

$$|Qw_1(t) - Qw_2(t)| \leq L_{g_i} |w_1 - w_2|_{PC}.$$

Case:3 For $t \in (s_i, t_{i+1}]$,we get

$$|Qw_1(t) - Qw_2(t)| \leq L_{g_i} |w_1 - w_2|_{PC}.$$

From the above inequalities, we obtain

$$|Qw_1(t) - Qw_2(t)| \leq K |w_1 - w_2|_{PC}.$$

Hence, Q is a contraction.Now we move to the next step.
Step:3 We prove that R is continuous.
Let w_n be sequence $\ni w_n \to \wp$ in $PC([0, T], \mathbb{R})$.
Case:1 For each $t \in [0, t_1]$, we have

$$\|Qw_n(t) - Qw(t)\| \leq \left[\frac{1}{\Gamma(p+1)} + \frac{1}{\Omega}\left[\sum_{i=1}^{n} {\alpha_i}^{\rho_i} I^{p+q_i} -^{\rho} I^p\right]\right] \left\| f(., w_n(.), .,) - f(., w(.), .,)\right\|_{PC}.$$

Case:2 For each $t \in (t_i, s_i]$, we obtain

$$\|Qw_n(t) - Qw(t)\| = 0.$$

Case:3 For each $t \in (s_i, t_{i+1}], i = 1, 2, ..., m$, we get

$$\|Qw_n(t) - Qw(t)\| \leq \frac{(t_{i+1} - s_i)}{\Gamma(p+1)} \left\| f(., w_n(.), .,) - f(., w(.), .,)\right\|_{PC}.$$

Thus, we conclude that the above cases $\|Qw_n(t) - Qw(t)\|_{PC} \longrightarrow 0$ as $n \longrightarrow \infty$.
Step:4 We prove that Q is compact.
Firstly observe that Q is uniformly bounded on $B_{p,r}$. Since $\|Qw\| \leq \frac{L_{g_i}(T)}{\Gamma(1+p)} < r$.
Next, prove that Q maps bounded set into equicontinuous set of $B_{p,r}$.
Case:1 For interval $t \in [0, t_1], 0 \leq E_1 \leq E_2 \leq t_1, w \in B_r$, we obtain

$$|QE_2 - QE_1| \leq \frac{L_{g_i}(1+r)}{\Gamma(p+1)}(E_2 - E_1).$$

Case:2 For each $t \in (t_i, s_i], t_i < E_1 < E_2 \leq s_i, w \in B_{p,r}$, we obtain

$$|QE_2 - QE_1| = 0.$$

Case:3 For each $t \in (s_i, t_{i+1}], s_i < E_1 < E_2 \leq t_{i+1}, w \in B_{p,r}$, we establish

$$|QE_2 - QE_1| \leq \frac{L_{g_i}(1+r)}{\Gamma(p+1)}(E_2 - E_1).$$

From the above, we get $|QE_2 - QE_1| \longrightarrow 0$ as $E_2 \longrightarrow E_1$ and Q is equicontinuous. Thus $Q(B_{p,r})$- relatively compact, so by using Ascoli-Arzela theorem, Q is compact. Hence the considered problem (1.1) – (1.3) have at least one fixed point on $[0, T]$. $\square$

4. Hyers-Ulam stability

The definitions of generalized Ulam-Hyers stable for the problem (1.1) − (1.2) and inequalities

$$\begin{cases} \left|{}^{C}D^{p}w(t) - f(t, w(t), \Psi w(t))\right| \leq \iota_r, \\ |w(t) - H_i(t, w(t))| \leq \iota_r, \quad t \in (t_i, s_i], \quad i = 1, ..., m. \end{cases} \tag{12}$$

$$\begin{cases} \left|{}^{C}D^{p}w(t) - f(t, w(t), \Psi w(t))\right| \leq \varpi(t), \\ |w(t) - H_i(t, w(t))| \leq \nu, \quad t \in (t_i, s_i], \quad i = 1, ..., m. \end{cases} \tag{13}$$

and

$$\begin{cases} \left|{}^{C}D^{p}w(t) - f(t, w(t), \Psi w(t))\right| \leq \iota_r\varpi(t), \\ |w(t) - H_i(t, w(t))| \leq \iota_r\nu, \quad t \in (t_i, s_i], \quad i = 1, ..., m. \end{cases} \tag{14}$$

Definition 4.1 The equation (1.1 − 1.2) is Ulam-Hyers-stable if a real number $C_{f,\varphi} > 0$ exists such that for each solution $\wp \in PC$ of (4.1) there exists a mild solution $w \in PC$ of Equation (1.1 − 1.2) with

$$|w(t) - \wp(t)| \leq C_{f,\varphi}\iota_r \quad t \in J, \ \iota_r > 0.$$

Definition 4.2 The equation (1.1 − 1.2) has been generalised Ulam-Hyers-stable if $\vartheta_{f,\varphi} \in PC(\mathbb{R}_+, \mathbb{R}_+)$ with $\vartheta_{f,\varphi}(0) = 0$ exists such that for each solution $\wp \in PC$ of (4.1) there exists a mild solution $w \in PC$ of Equation (1.1 − 1.2) with

$$|w(t) - \wp(t)| \leq \vartheta_{f,\varphi}(\iota_r) \quad t \in J.$$

Remark: 4.3 If there is a function $\wp \in PC$ and a sequence $g_i, i = 1, 2, ..., k$,(which depend on $\wp$)∋, the function $\wp \in PC$ is a solution of inequality (4.1).

(*i*) $|H| \leq \iota_r$ and $|H_i| \leq \iota_r$, `for all` $i = 1, 2, ..., k$,
(*ii*) ${}^{C}D^{p}w(t) = f(t, w(t), \Psi w(t)) + g(t), t \in (s_i, t_{i+1}] \subset [0, T]$,
(*iii*) $w(t) = H_i(t, w(t)) + H_i(t), \ t \in (t_i, s_i], \ i = 1, ..., m$,

Lemma: 4.4 If $w \in PC$ is an inequality solution (4.1), then the inequalities below satisfy:

$$\begin{cases} \left|w(t) - H_i(s_m) - \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s)ds\right| \leq \frac{\iota_r}{\Gamma(p+1)}t^p, \ \ t \in [0, t_1], \\ |w(t) - H_i(t)| \leq \iota_r, \qquad t \in (t_i, s_i], i = 1, 2, ..., m, \\ \left|w(t) - H_i(s_i) + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s)ds \right. \\ \left. -\frac{1}{\Gamma(p)}\int_0^{s_i} (s_i - s)^{p-1} f(s)ds\right| \leq \iota_r + \frac{\iota_r}{\Gamma(p+1)}t^p, \ \ t \in (s_i, t_{i+1}], i = 1, 2, ..., m. \end{cases} \tag{15}$$

Proof: For any $i = 1, 2, ..., m$ and $t \in (s_i, t_{i+1}]$, applying the Remark (4.4) and (2.1), the solution is given by

$$w(t) - H_i(s_i) + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s)ds - \frac{1}{\Gamma(p)}\int_0^{s_i} (s_i - s)^{p-1} f(s)ds$$
$$= H_i + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s)ds.$$

This proves the claim of Lemma 4.5.

Theorem 4.1. *Let the assumption* (Al_1) *holds. Then the problems* (1.1) − (1.3) *is Ulam-Hyers stable.*

Proof: Let us denote by w, the unique solution of

$$\begin{aligned} &{}^{C}D^{p}w(t) = f(t, w(t), \Psi w(t)), t \in (s_i, t_{i+1}] \subset [0, T], 1 < p \le 2, \\ &\quad w(t) = H_i(t, w(t)), \;\; t \in (t_i, s_i], \;\; i = 1, ..., m, \\ &\quad w(0) = 0, \quad w(T) = \sum_{i=1}^{n} \alpha_i \frac{\rho_i^{1-q_i}}{\Gamma(q_i)} \int_0^{\epsilon_i} \frac{s^{\rho_i - 1} w(s)}{(t^{\rho_i} - s^{\rho_i})^{1-q_i}} ds := \sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{q_i} w(\epsilon_i). \end{aligned} \tag{16}$$

Then, we get

$$w(t) = \begin{cases} H_i(s_m) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \omega(s) ds \\ + \frac{1}{\Omega} \Big[\sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{p+q_i} \omega(\epsilon_i) d\epsilon_i -^{\rho} I^{p} \omega(s) ds \Big], \;\; t \in [0, t_1], \\ H_i(t), \qquad t \in (t_i, s_i], i = 1, 2, ..., m, \\ H_i(s_i) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \omega(s) ds \\ - \frac{1}{\Gamma(p)} \int_0^{s_i} (s_i - s)^{p-1} \omega(s) ds, \;\; t \in (s_i, t_{i+1}], i = 1, 2, ..., m. \end{cases}$$

Let $\wp \in PC(J, \mathbb{R})$ be a solution of (4.1). According to (4.7), for each $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned} &\Big| \wp(t) - H_i(s_i, \wp(s_i)) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, \wp(s), \Psi\wp(s)) ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^{s_i} (s_i - s)^{p-1} f(s, \wp(s), \Psi\wp(s)) ds \Big| \\ &\quad \le \iota_r + \frac{(t_{i+1} - s_i)^p}{\Gamma(p+1)} \epsilon, \end{aligned}$$

and for $(t_i, s_i], i = 1, 2, ..., m$, we obtain

$$|w(t) - H_i(t, w(t))| \le \iota_r.$$

Now for each $t \in [0, t_1]$, we have

$$\begin{aligned} &\Big| \wp(t) - H_i(s_m, \wp(s_m)) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, \wp(s), \Psi\wp(s)) ds \\ &\quad + \frac{1}{|\Omega|} \Big[\sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{p+q_i} f(\epsilon_i, \wp(\epsilon_i), \Psi\wp(\epsilon_i)) d\epsilon_i -^{\rho} I^{p} f(s, \wp(s), \Psi\wp(s)) ds \Big] \Big|, \\ &\quad \le \frac{t_1^p}{\Gamma(p+1)} \iota_r. \end{aligned}$$

Now we discuss several cases.

Case:1 For each t $\in [0, t_1]$, we get

$$\begin{aligned}
|\wp(t) - w(t)| \leq & \Big|\wp(t) - H_i(s_m, \wp(s_m)) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, \wp(s), \Psi\wp(s))ds \\
& - \frac{1}{|\Omega|}\Big[\sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{p+q_i} f(\epsilon_i, \wp(\epsilon_i), \Psi\wp(\epsilon_i))d\epsilon_i -^{\rho} I^p f(s, \wp(s), \Psi\wp(s))ds\Big]\Big|, \\
& + \Big|\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, \wp(s), \Psi\wp(s))ds \\
& - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, \wp(s), \Psi\wp(s))ds\Big|, \\
& \leq \frac{t_1^p}{\Gamma(p+1)} \iota_r + L_{h_i} + \frac{(L_{h_i} + GM)}{\Gamma(p)} \int_0^t (t-s)^{p-1} |\wp(s) - w(s)| \, ds \\
& \leq \frac{t_1^p}{\Gamma(p+1)} \iota_r + L_{h_i} + \frac{(L_{h_i} + GM)t_1^p}{\Gamma(p+1)} |\wp - w|_{PC} .
\end{aligned}$$

This implies

$$\left[1 - \frac{(L_{h_i} + GM)t_1^p}{\Gamma(p+1)}\right] |\wp - w|_{PC} \leq \frac{T^p}{\Gamma(p+1)} + L_{h_i}.$$

$$|\wp(t) - w(t)| \leq C_{f,\varphi}\iota_r, \quad t \in [0, t_1], \tag{17}$$

where

$$C_{f,\varphi} := \frac{\frac{T^p}{\Gamma(p+1)} + L_{h_i}}{1 - \frac{(L+GM)t_1^p}{\Gamma(p+1)}}.$$

Case 2: For $t \in (t_i, s_i], i = 1, 2, ..., m$, we have

$$\begin{aligned}
|\wp(t) - w(t)| \leq & |\wp(t) - H_i(t, \wp(t))| \\
& + |H_i(t, \wp(t)) - H_i(t, w(t))| , \\
& \iota_r + \frac{L_{h_i} s_i^p}{\Gamma(p+1)} |\wp - w|_{PC} ,
\end{aligned}$$

which further implies

$$\left[1 - \frac{L_{h_i} s_i^p}{\Gamma(p+1)}\right] |\wp - w|_{PC} \leq \iota_r.$$

Thus, we obtain

$$|\wp(t) - w(t)| \leq C_{f,\varphi}\iota_r, \ t \in (t_i, s_i], \ i = 1, 2, ..., m, \tag{18}$$

where

$$C_{f,\varphi} = \frac{1}{\left(1 - \frac{L_{h_i} s_i^p}{\Gamma(p+1)}\right)}.$$

Case 3: For $t \in (s_i, t_{i+1}], i = 1, 2, ..., m$, we have

$$\begin{aligned}
|\wp(t) - w(t)| \leq & \Big| H_i(s_i) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, \wp(s), \Psi\wp(s)) ds \\
& + \frac{1}{\Gamma(p)} \int_0^{s_i} (s_i - s)^{p-1} f(s, \wp(s), \Psi\wp(s)) ds \Big| \\
& + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left| f(s, \wp(s), \Psi\wp(s)) - f(s, w(s), \Psi w(s)) \right| ds \\
& + \frac{1}{\Gamma(p)} \int_0^{s_i} (s_i - s)^{p-1} \left| f(s, \wp(s), \Psi\wp(s)) - f(s, w(s), \Psi w(s)) \right| ds, \\
\leq & \left(\iota_r + \frac{(t_{i+1}^p - s_i^p \iota_r}{\Gamma p + 1} \right) + \frac{(L_{h_i} + GM) t_{i+1}^p - s_i^p}{\Gamma(p+1)} |\wp - w|_{PC}.
\end{aligned}$$

Hence, we get

$$\left[1 - \frac{(L_{h_i} + GM) t_{i+1}^p - s_i^p}{\Gamma(p+1)} \right] |\wp - w|_{PC} \leq \left(1 + \frac{t_{i+1}^p - s_i^p}{\Gamma(p+1)} \right) \iota_r |\wp - w|_{PC}.$$

Further computation shows that

$$|\wp(t) - w(t)| \leq C_{f,\varphi} \iota_r, \tag{19}$$

where,

$$C_{f,\varphi} := \frac{1 + \frac{t_{i+1}^p - s_i^p}{\Gamma p + 1}}{1 - \frac{(L_{h_i} + GM) t_{i+1}^p - s_i^p}{\Gamma(p+1)}}.$$

Summarizing (4.8), (4.9) and (4.9), we conclude that (1) is Ulam-Hyers stable with respect to ι_r.

5. Application

Let us consider the Caputo-Katugampola multipoint boundary value problem,

$$D^p w(t) = \frac{e^{-t} |w|}{12 + e^t (1 + |w|)} + \frac{1}{3} \int_0^t e^{-(s-t)} w(s) ds, \quad t \in (0, 1] \cup (2, 3], \tag{20}$$

$$w(t) = \frac{|w(t)|}{2(1 + |w(t)|)}, \ t \in (1, 2], \tag{21}$$

$$w(0) = 0 \quad w(1) = 2\ {}^{1/3} I^{2/3} w(3/5) + 1^{1/3} I^{2/7} w(2/5), \tag{22}$$

where

$$|\Omega| = \left[1 - \sum_{i=1}^n \sigma_i \frac{\epsilon_i^{\rho q_i}}{\rho^{q_i} \Gamma(q_i + 1)} \right], \Omega \neq 0,$$

and $L = G = \frac{1}{12}$, $M = \frac{1}{3}$, $p = \frac{5}{7}$ $n = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $\epsilon_1 = \frac{5}{12}$, $\epsilon_2 = \frac{6}{13}$, $q_1 = \frac{2}{3}, q_2 = \frac{2}{7}$ $\rho = \frac{1}{3}$, $L_{h_1} = \frac{1}{3}$. Using the given data, $|\Omega| = 3.81$, by using theorem (3.1), we determine that

$$L_{h_i} + \frac{(L + GM)}{\Gamma(p+1)} (t_{i+1}^p + s_i^p) \approx 0.41 < 1.$$

and

$$\left\{ L_{h_i} + \frac{(L + GM)}{\Gamma(p+1)} + \frac{(L + GM)}{|\Omega|} \Big[\sum_{i=1}^n {\alpha_i}^{\rho_i} I^{p+q_i} +^{\rho} I^p \Big] \right\} \approx 0.54 < 1.$$

Hence, all assumptions of Theorem 3.1 is satisfied, so that the problem (4.1) − (4.3) has a unique solution $[0, T]$.
Further, we take the solution w of the problem, (5.1) − (5.3) given by

$$\begin{aligned} w(t) =& \frac{|w(t)|}{2(1+|w(t)|)} + \frac{1}{\Gamma\frac{5}{7}} \int_0^t (t-s)^{\frac{-2}{7}} + \frac{e^{-t}|w|}{12+e^t(1+|w|)} \\ &+ \frac{1}{3}\int_0^t e^{-(s-t)} w(s)ds + 2^{1/3} I^{2/3} \left[\frac{e^{-t}|w|}{12+e^t(1+|w|} \right] + 1^{1/3} I^{2/7} \Big[\frac{e^{-t}|w|}{12+e^t(1+|w|)} \\ &+ \frac{1}{3}\int_0^t e^{-(s-t)} w(s)ds \Big], \quad t \in (0,1], \\ w(t) =& \frac{|w(t)|}{2(1+|w(t)|)}, \quad t \in (1,2], \end{aligned}$$

$$\begin{aligned} w(t) =& \frac{|w(t)|}{2(1+|w(t)|)} + \frac{1}{\Gamma\frac{5}{7}} \int_0^t (t-s)^{\frac{-2}{7}} + \frac{e^{-t}|w|}{12+e^t(1+|w|)} \\ &+ \frac{1}{3}\int_0^t e^{-(s-t)} w(s)ds - \frac{1}{\Gamma\frac{5}{7}} \int_0^2 (2-s)^{\frac{-2}{7}} + \frac{e^{-t}|w|}{12+e^t(1+|w|} \\ &+ \frac{1}{3}\int_0^t e^{-(s-t)} w(s)ds, \quad t \in (1,2]. \end{aligned}$$

For $t \in (0, 1]$, we obtain

$$|\wp(t) - w(t)| \le \frac{t_1^p}{\Gamma(p+1)}\epsilon + L_{h_i} + \frac{(L+GM)t_1^p}{\Gamma(p+1)} \le 1.23.$$

For $t \in (1, 2]$, we get

$$|\wp(t) - w(t)| \le \left(\iota_r + \frac{(t_{i+1}^p - s_i^p \iota_r}{\Gamma p + 1} \right) + \frac{(L_{h_i} + GM)t_{i+1}^p - s_i^p}{\Gamma(p+1)} \le 1.778,$$

which shows that (5.1) − (5.3) is Ulam-Hyers stable with respect to $\iota_r = 1$.
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