Filomat 37:19 (2023), 6289–6306
https://doi.org/10.2298/FIL2319289L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Special affine biorthogonal wavelets on $\mathbb{R}$ and logarithmic regression curves

Waseem Z. Lone[a], **Firdous A. Shah**[a]

[a]*Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India*

Abstract. In the article "Special affine multiresolution analysis and the construction of orthonormal wavelets in $L^2(\mathbb{R})$", [Appl Anal. 2022; D.O.I: 10.1080/00036811.2022.2030723], we introduced the notion of multiresolution analysis (MRA) in the realm of the special affine Fourier transform. In continuation to the study, our aim is to present the construction of special affine biorthogonal wavelets in $L^2(\mathbb{R})$. Besides, we provide a complete characterization for the biorthogonality of the translates of the scaling functions of two special affine MRA's and the associated special affine biorthogonal wavelet families. We show that the wavelets associated with the biorthogonal special affine MRA's are also biorthogonal in nature. To extend the scope of the present study, we present the biorthogonal special affine MRA and its biorthogonal properties on a logarithmic regression curve $\mathscr{C}$.

1. Introduction

The premiere development in the theory of wavelet analysis was reported in 1986, when Stéphans Mallat and Yves Meyer came up with the remarkable discovery of a new formalism, known as the multiresolution analysis, for the construction of orthogonal wavelet bases [1]. Mallat's brilliant work served as the pedestal for many subsequent developments, including the construction of orthogonal spline wavelets [2]. Using MRA, wavelet spaces are constructed by splitting the frequency domain dyadically and their bases are obtained with the help of translated and dialated form of a single function. Some of the prominent wavelets obtained via the multiresolution analysis include Shannon wavelet, Meyer wavelet, Franklin wavelet, spline wavelets, nonuniform wavelets, harmonic wavelets, and Daubechies wavelets [3, 4]. Despite the remarkable success over the past few decades, compactly supported orthonormal wavelets suffer from certain apparent limitations. For instance, they lack symmetry, that is, the processing filters are non-symmetric and do not possess a linear phase property. The lack of these properties puts a strong limitation on the construction of symmetric, orthogonal and compactly supported wavelets and also results in severe undesirable phase distortion in signal processing. To overcome such limitations, Cohen *et al.* [5] developed another elegant approach in the form of biorthogonal multiresolution analysis. Unlike the classical multiresolution analysis, the biorthogonal MRA has the prime feature of efficiently resolving linear phased finite impulse response filters adapted to the fast wavelet transform. As of now, the theory of biorthogonal wavelets has fascinated the scientific, engineering and research communities both with their versatile applicability and lucid mathematical framework [6, 7].

2020 *Mathematics Subject Classification*. Primary 42C40; Secondary 42C15,42A38.
Keywords. Special affine wavelet; Special affine MRA; Biorthogonal wavelets; Scaling function.
Received: 22 April 2022; Accepted: 13 February 2023
Communicated by Hari M. Srivastava
Email addresses: lone.waseem.393@gmail.com (Waseem Z. Lone), fashah@uok.edu.in (Firdous A. Shah)

On the other hand, the special affine Fourier transform (SAFT) is a recent addition in the context of phase-space transforms which embodies a wider class of integral transforms ranging from the classical Fourier to the much recent linear canonical transforms [8]. The SAFT is an integral transformation associated with a general inhomogeneous lossless linear mapping in phase-space that depends on six parameters independent of the phase-space coordinates. The six-parameters constitute an augmented matrix $M = (A, B, C, D : p, q) = [\Lambda \,|\, \lambda]$ consisting of a 2×2 unimodular matrix $\Lambda = (A, B, C, D)$ and a real 2×1 augmentation vector $\lambda = (p\,,\,q)$. Due to the offset by the vector λ, the transformation is also referred as the offset linear canonical transform [9–12]. As of now, the special affine Fourier transforms has received immense attention from researchers working in different branches of science and engineering, including harmonic analysis, sampling, signal and image processing, and so on. For instance, Srivastava et al. [13] investigated the convolution operations associated with the Bessel wavelet transform and obtained the bounds of the normalized Bessel wavelet transform on the generalized Sobolev space $B^{\mu}_{p,k}(I)$ via the theory of Hankel transformation. Moreover, the authors dig deep into the localization operators associated with the integral representation of locally compact groups and study their Schatten-von Neumann properties [14]. Shah et al. [15] proposed the notion of the linear canonical wavelet transform in the framework of quantum mechanics and derived the inner product relation and inversion formula for the linear canonical wavelet transform in the realm of quantum mechanics. Mishra et al. [16] presented a systematic study of various characteristics and properties of the continuous and discrete fractional Bessel wavelet transform. Srivastava et al. [17] formulated a novel and efficient collocation method based on Fibonacci wavelets for the numerical solution of the non-linear Hunter–Saxton equation, where the operational matrices of integration associated with the Fibonacci wavelets are constructed by following the strategy of Chen and Hsiao. Srivastava and Shah extended a unified treatment for the continuum and digital realm of multivariate data by establishing several sufficient conditions under which the AB-wavelet system constitutes a frame for $L^2(R^n)$ [18]. Panday et al. [19] defined a continuous wavelet transform of a Schwartz-tempered distribution $f \in S'(R^n)$ with wavelet kernel $\psi \in S(R^n)$ and then derived the corresponding wavelet inversion formula interpreting convergence in the weak topology of $S'(R^n)$.

Much recently, we introduced the notion of multiresolution analysis in the framework of special affine Fourier transform [20]. This article focuses on the introduction and study of the continuous special affine wavelet transform. It covers the orthogonality relation and inversion formula, the construction of orthonormal special affine wavelets in $L^2(\mathbb{R})$, and a fast wavelet transform associated with the special affine MRA. In particular, it allows a smoother construction of orthonormal discrete special affine wavelets in a simple and insightful way. However, much to dismay, the theoretical manifestation of the special affine MRA is still in its infancy and needs to be explored exclusively. Taking this opportunity, we are deeply motivated to initiate an exclusive study of the biorthogonal wavelets associated with the special affine MRA. To facilitate the narrative, we shall briefly recall the prerequisite and then introduce the notion of the biorthogonal wavelets associated with the special affine MRA in $L^2(\mathbb{R})$. We show that, if the translates of the scaling functions of the special affine multiresolution analyses are biorthogonal, then the associated special affine wavelet families are also biorthogonal. Moreover, we extend the scope of biorthogonal special affine MRA on the logarithmic regression curves. The special affine biorthogonal wavelets are expected to provide better compression and representation capabilities as they can represent a wider range of functions. Additionally, special affine biorthogonal wavelets also offer more freedom in design and construction, which can lead to special affine wavelets better suited for specific applications.

The highlights of the article are given below:

- To introduce the special affine biorthogonal multiresolution analysis in $L^2(\mathbb{R})$.
- To investigate the characterization for the biorthogonality of the translates of the scaling functions of a pair of special affine MRA's.
- To study the biorthogonal properties of a dual special affine wavelets.
- To study the biorthogonal properties of a dual special affine wavelets on a logarithmic curve $\mathscr{C}$.

The layout of the article is as follows: Section 2 is completely devoted for the exposition of the preliminaries including the notion of multiresolution analysis associated with the SAFT. Section 3 is exclusively concerned for the establishment of necessary and sufficient conditions for the translates of a function to form a Riesz basis for its closed linear span. In Section 4, we show that the wavelets associated with the biorthogonal special affine MRA's are also biorthogonal in nature. Section 5 is solely concerned with the notion of biorthogonal special affine MRA on a logarithmic regression curve. Finally, a conclusion is extracted in Section 6.

2. Special Affine Fourier Transform and the Associated MRA

In this section, we shall formally recall the fundamentals of special affine Fourier transform and the associated multiresolution analysis, which serves as a corner stone for the development of the subsequent sections.

2.1. Special Affine Fourier Transform

Here, we shall briefly present the notion of the special affine Fourier transform and some of its fundamental properties.

Definition 2.1. *For any $f \in L^2(\mathbb{R})$, the special affine Fourier transform with respect to a real, augmented matrix $M = (A, B, C, D : p, q)$ is defined by*

$$\mathscr{L}_M\Big[f\Big](\omega) = \int_{\mathbb{R}} f(t)\, \mathcal{K}_M(t, \omega)\, \mathrm{dt}, \tag{1}$$

where $\mathcal{K}_M(t, \omega)$ denotes the kernel of the SAFT given by

$$\mathcal{K}_M(t,\omega) = \begin{cases} \dfrac{1}{\sqrt{2\pi i B}} \exp\left\{ \dfrac{i\Big(At^2 + 2t(p-\omega) - 2\omega(Dp - Bq) + D\big(\omega^2 + p^2\big)\Big)}{2B} \right\}, & B \neq 0 \\[2ex] \sqrt{D} \exp\left\{ \dfrac{iCD(\omega - p)^2}{2} + iq\omega \right\} f\Big(D(\omega - p)\Big), & B = 0. \end{cases}$$

Throughout the article, we shall only consider the case $B \neq 0$, since for the case $B = 0$, the SAFT (1) is just a chirp multiplication operation. We also note that phase-space transform defined in (1) is lossless if and only if the matrix $M = (A, B, C, D)$ is unimodular, that is, $AD - BC = 1$.

From (1), we observe that the special affine Fourier transform $\mathscr{L}_M\big[f\big](\omega)$ of any function $f \in L^2(\mathbb{R})$ can be recast as:

$$\begin{aligned} \mathscr{L}_M\Big[f\Big](\omega) &= \frac{1}{\sqrt{2\pi i B}} \exp\left\{ \frac{i\Big(D\big(\omega^2 + p^2\big) - 2\omega(Dp - Bq)\Big)}{2B} \right\} \int_{\mathbb{R}} e^{iAt^2/2B} f(t)\, e^{-i(\omega - p)t/B}\, \mathrm{dt} \\ &= \frac{1}{\sqrt{iB}} \exp\left\{ \frac{i\Big(D\big(\omega^2 + p^2\big) - 2\omega(Dp - Bq)\Big)}{2B} \right\} \mathscr{F}\Big[F\Big]\left(\frac{\omega - p}{B}\right), \end{aligned} \tag{2}$$

where $\mathscr{F}[F]$ represents the Fourier transform of F and $F(t) = e^{iAt^2/2B} f(t)$.

The Plancheral and inversion formulae corresponding to (1) are given by

$$\Big\langle f(t), g(t) \Big\rangle_2 = \Big\langle \mathscr{L}_M\Big[f\Big](\omega), \mathscr{L}_M\Big[g\Big](\omega) \Big\rangle_2, \quad \forall f, g \in L^2(\mathbb{R}) \quad \text{and} \tag{3}$$

$$f(t) = \mathrm{I_M} \int_{\mathbb{R}} O_M\Big[f\Big](\omega)\, \mathcal{K}_{M^{-1}}(t,\omega)\, \mathrm{d}\omega, \tag{4}$$

where

$$\mathrm{I_M} = \exp\left\{ \frac{i\Big(CDp^2 + ABq^2 - 2ADPq\Big)}{2} \right\}, \tag{5}$$

$M^{-1} = \Big[\Lambda^{-1}|\lambda^{-1}\Big]$ with $\Lambda^{-1} = (D, -B, -C, A)$ and $\lambda^{-1} = (Dp - Bq; Aq - Cp)^T$. Besides, the kernel $\mathcal{K}_M(t,\omega)$ satisfies the following properties:

(i). $\mathcal{K}_{M^{-1}}(t,\omega) = \bar{I}_M\, \overline{\mathcal{K}_M(t,\omega)}$,

(ii). $\displaystyle\int_{\mathbb{R}} \mathcal{K}_M(t,\omega)\, \mathcal{K}_{M^{-1}}(t,\mu)\, \mathrm{dt} = \bar{I}_M\, \delta(\omega - \mu)$,

(iii). $\displaystyle\int_{\mathbb{R}} \mathcal{K}_M(t,\omega)\, \mathcal{K}_{M^{-1}}(z,\omega)\, \mathrm{d}\omega = \bar{I}_M\, \delta(t - z)$.

2.2. *Special Affine Multiresolution Analysis in $L^2(\mathbb{R})$*

Much recently, we introduced the notion of multiresolution analysis in the framework of special affine Fourier transform and then studied the construction of more flexible orthogonal wavelets coined as special affine wavelets [20]. Here, our main aim is to recall the definition of multiresolution analysis associated with the SAFT and some of its results, which serve as a building block for the construction of biorthogonal special affine wavelets in $L^2(\mathbb{R})$.

Definition 2.2. *Given a real parametric matrix $M = (A, B, C, D : p, q)$, $B \neq 0$, an associated special affine multiresolution is a collection $\{V_j^M : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties:*

(i). *$V_j^M \subset V_{j+1}^M$, for all $j \in \mathbb{Z}$;*

(ii). *$\bigcup_{j\in\mathbb{Z}} V_j^M$ is dense in $L^2(\mathbb{R})$;*

(iii). *$\bigcap_{j\in\mathbb{Z}} V_j^M = \Big\{0\Big\}$;*

(iv). *$f(t) \in V_j^M$ if and only if $e^{i3At^2/2B} f(2t) \in V_{j+1}^M$, for all $j \in \mathbb{Z}$;*

(v). *There exists a function $\phi(t) \in L^2(\mathbb{R})$ in V_0^M such that*

$$\phi_{0,k}^M(t) = \phi(t-k)\, \exp\left\{ \frac{-i\Big(At^2 + Dp^2 - Ak^2\Big)}{2B} \right\}, \quad k \in \mathbb{Z} \tag{6}$$

is an orthonormal basis of subspace V_0^M.

The function ϕ whose existence is guaranteed in (v) is called a scaling function corresponding to the given special affine MRA. Now, if we assume that set of functions $\Big\{\phi_{0,k}^M(t) : k \in \mathbb{Z}\Big\}$, then

$$\phi_{j,k}^M(t) = 2^{j/2}\phi\Big(2^j t - k\Big) \exp\left\{ \frac{-i\Big(At^2 + Dp^2 - Ak^2\Big)}{2B} \right\} \tag{7}$$

forms a complete orthonormal basis for V_j^M, $j \in \mathbb{Z}$. Given a special affine multiresolution analysis $\Big\{V_j^M : j \in \mathbb{Z}\Big\}$, we define another sequence $\Big\{W_j^M : j \in \mathbb{Z}\Big\}$ of closed subspaces of $L^2(\mathbb{R})$ by $V_{j+1}^M = V_j^M \oplus W_j^M$, $j \in \mathbb{Z}$. Followed by Definition (2.2), these subspaces inherit the scaling property of $\Big\{V_j^M : j \in \mathbb{Z}\Big\}$, namely

$$f(t) \in V_j^M \quad \text{if and only if} \quad e^{3iAt^2/2B} f(2t) \in W_{j+1}^M,\ j \in \mathbb{Z}. \tag{8}$$

Moreover the subspaces W_j^M are mutually orthogonal with the following decomposition formula:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j^M. \tag{9}$$

Note that condition (9) means that any orthonormal basis for $L^2(\mathbb{R})$ can be constructed by finding out an orthonormal basis for the subspace W_j^M.

Undoubtedly, orthogonal wavelets enjoy many desirous properties, including compact support, good frequency localization, and vanishing moments. However, they suffer from certain apparent limitations due to the lack of continuous symmetry. For example, in medical imaging, noise reduction, image compression and signal processing. The biorthogonal special affine wavelets achieve symmetry where the orthogonality is replaced by the biorthogonality. As such, we define the biorthogonal special affine scaling functions and the associated biorthogonal wavelets as follows:

Definition 2.3. *A pair of special affine MRA's* $\left\{V_j^M : j \in \mathbb{Z}\right\}$ *and* $\{\widetilde{V}_j^M : j \in \mathbb{Z}\}$ *with scaling functions* ϕ *and* $\widetilde{\phi}$*, respectively are said to be biorthogonal to each other if* $\left\{\phi_{0,k}^M(t) = \phi(t-k)\, e^{-i\left(At^2+Dp^2-Ak^2\right)/2B} : k \in \mathbb{Z}\right\}$ *and* $\left\{\widetilde{\phi}_{0,k}^M(t) = \widetilde{\phi}(t-k)\, e^{-i\left(At^2+Dp^2-Ak^2\right)/2B} : k \in \mathbb{Z}\right\}$ *are biorthogonal. The functions* ϕ *and* $\widetilde{\phi}$ *are called a pair of biorthogonal special affine scaling functions.*

Assume that $\{V_j^M : j \in \mathbb{Z}\}$ and $\{\widetilde{V}_j^M : j \in \mathbb{Z}\}$ are the biorthogonal special affine MRA's in $L^2(\mathbb{R})$. Then, for any fixed matrix $M = (A, B, C, D : p, q)$, $B \neq 0$, the associated scaling functions satisfy the following pair of equations:

$$\phi_{0,0}^M(t) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k\, \phi(2t-k)\, e^{-i\left(At^2+Dp^2-Ak^2\right)/2B} \tag{10}$$

$$\widetilde{\phi}_{0,0}^M(t) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k\, \widetilde{\phi}(2t-k)\, e^{-i\left(At^2+Dp^2-Ak^2\right)/2B} \tag{11}$$

On taking the special affine Fourier transform on both sides of (10), we obtain

$$\widehat{\phi}\left(\frac{\omega-p}{B}\right) = \Lambda_0\left(\frac{\omega-p}{2B}\right)\widehat{\phi}\left(\frac{\omega-p}{2B}\right) \tag{12}$$

$$\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}\right) = \widetilde{\Lambda}_0\left(\frac{\omega-p}{2B}\right)\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}\right), \tag{13}$$

where

$$\Lambda_0\left(\frac{\omega-p}{B}\right) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h_k^M\, e^{-ik(\omega-p)/B}, \quad h_k^M = h_k\, e^{iAk^2/B} \tag{14}$$

$$\widetilde{\Lambda}_0\left(\frac{\omega-p}{B}\right) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \widetilde{h}_k^M\, e^{-ik(\omega-p)/B}, \quad \widetilde{h}_k^M = h_k\, e^{iAk^2/B}. \tag{15}$$

Equation (14) is a $2\pi B-$periodic function and is called the biorthogonal special affine low-pass filter.

Similar to the special affine refinement equation (10), we have the biorthogonal special affine wavelet equations of the form

$$\psi_{0,0}^M(t) = \sqrt{2} \sum_{k\in\mathbb{Z}} d_k\, \psi(2t-k)\, e^{-i\left(At^2+Dp^2-Ak^2\right)/2B} \tag{16}$$

$$\widetilde{\psi}_{0,0}^M(t) = \sqrt{2} \sum_{k\in\mathbb{Z}} d_k\, \widetilde{\psi}(2t-k)\, e^{-i\left(At^2+Dp^2-Ak^2\right)/2B}. \tag{17}$$

Implementing SAFT on both sides of (16), we obtain

$$\widehat{\psi}\left(\frac{\omega-p}{B}\right)=\Lambda_1\left(\frac{\omega-p}{2B}\right)\widehat{\psi}\left(\frac{\omega-p}{2B}\right) \tag{18}$$

$$\widehat{\widetilde{\psi}}\left(\frac{\omega-p}{B}\right)=\widetilde{\Lambda}_1\left(\frac{\omega-p}{2B}\right)\widehat{\widetilde{\psi}}\left(\frac{\omega-p}{2B}\right), \tag{19}$$

where

$$\Lambda_1\left(\frac{\omega-p}{B}\right)=\frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} d_k^M\, e^{-ik(\omega-p)/B},\quad d_k^M=d_k\, e^{iAk^2/B} \tag{20}$$

$$\widetilde{\Lambda}_1\left(\frac{\omega-p}{B}\right)=\frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} \widetilde{d}_k^M\, e^{-ik(\omega-p)/B},\quad \widetilde{d}_k^M=d_k\, e^{iAk^2/B}. \tag{21}$$

Equation (20) is a $2\pi B$−periodic function and is called the biorthogonal special affine high-pass filter.

3. Riesz Basis of Translates

The orthogonality property puts a strong limitation on the construction of wavelets. For instance, orthogonal MRA cannot produce symmetric scaling filter coefficients. The generalization to biorthogonal structure has been considered to gain more flexibility. Here, a biorthogonal scaling function $\widetilde{\phi}_{0,0}^M$ and a biorthogonal wavelet $\widetilde{\psi}_{0,0}^M$ exist that generate a biorthogonal special affine MRA with subspaces $\widetilde{V}_j^M$ and $\widetilde{W}_j^M$ such that $\widetilde{V}_j^M \perp W_j^M$ and $V_j^M \perp \widetilde{W}_j^M$.

Theorem 3.1. *Assume that ϕ and $\widetilde{\phi}$ are square integrable functions. Then, the collections $\left\{\phi_{0,k}^M(t)=\phi(t-k)\, e^{-i(At^2+Dp^2-Ak^2)/2B} : k\in\mathbb{Z}\right\}$ and $\left\{\widetilde{\phi}_{0,k}^M(t)=\widetilde{\phi}(t-k)\, e^{-i(At^2+Dp^2-Ak^2)/2B} : k\in\mathbb{Z}\right\}$ are biorthogonal to each other with respect to a parametric matrix $M=(A,B,C,D:p,q)$, $B\neq 0$ if and only if*

$$\sum_{n\in\mathbb{Z}}\widehat{\phi}\left(\frac{\omega-p}{B}+2n\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}+2n\pi\right)}=1,\qquad a.e. \tag{22}$$

Proof. We have

$$\begin{aligned}
\mathscr{L}_M\left[\phi_{0,k}^M\right](\omega) &= \frac{1}{\sqrt{2\pi iB}}\int_{\mathbb{R}}\phi_{0,k}^M(t)\, e^{i\left(At^2-2t(\omega-p)-2\omega(Dp-Bq)+D\left(\omega^2+p^2\right)\right)/2B}\,\mathrm{dt}\\
&= \frac{1}{\sqrt{2\pi iB}}\, e^{i\left(Ak^2-2\omega(Dp-Bq)+D\omega^2\right)/2B}\int_{\mathbb{R}}\phi(t-k)\, e^{-2it(\omega-p)/2B}\,\mathrm{dt}\\
&= \frac{1}{\sqrt{2\pi iB}}\, e^{i\left(Ak^2-2k(\omega-p)-2\omega(Dp-Bq)+D\omega^2\right)/2B}\int_{\mathbb{R}}\phi(z)\, e^{-iz(\omega-p)/B}dz\\
&= \frac{1}{\sqrt{iB}}\, e^{i\left(Ak^2-2k(\omega-p)-2\omega(Dp-Bq)+D\omega^2\right)/2B}\,\widehat{\phi}\left(\frac{\omega-p}{B}\right).
\end{aligned}$$

By invoking the orthonormality of ϕ, we have

$$\left\langle\phi_{0,k}^M(t),\widetilde{\phi}_{0,\ell}^M(t)\right\rangle=\delta_{k,\ell}. \tag{23}$$

Therefore, by virtue of (23) and the Parsevals formula (3), we obtain

$$\delta_{k,\ell}=\left\langle\phi_{0,k}^M(t),\widetilde{\phi}_{0,\ell}^M(t)\right\rangle$$

$$
\begin{aligned}
&= \left\langle \mathscr{L}_M\left[\phi^M_{0,k}\right](\omega), \mathscr{L}_M\left[\widetilde{\phi}^M_{0,\ell}\right](\omega)\right\rangle \\
&= \frac{1}{|B|}\, e^{i\left(A\left(k^2-\ell^2\right)+2(k-\ell)p\right)/2B} \int_{\mathbb{R}} e^{-i(k-\ell)\omega/B}\, \widehat{\phi}\left(\frac{\omega-p}{B}\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}\right)}\mathrm{d}\omega \\
&= \frac{1}{|B|}\, e^{i\left(A\left(k^2-\ell^2\right)+2(k-\ell)p\right)/2B} \int_0^{2\pi B} e^{-i(k-\ell)\omega/B} \sum_{n\in\mathbb{Z}} \widehat{\phi}\left(\frac{\omega-p}{B}+2n\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}+2n\pi\right)}\mathrm{d}\omega.
\end{aligned}
$$

Hence, we deduce that

$$
\sum_{n\in\mathbb{Z}} \widehat{\phi}\left(\frac{\omega-p}{B}+2n\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}+2n\pi\right)} = 1, \qquad \text{a.e.}
$$

This completes the proof of Theorem 3.1. □

Remark 3.2. *For the suitable choices of the parametric matrix $M = (A, B, C, D : p, q), B \neq 0$, Theorem 3.1 boils down to their counterparts for the respective integral transforms.*

Lemma 3.3. *Assume that for any square integrable function ϕ, there exist positive constants η_1 and η_2 such that*

$$
\eta_1 \leq \sum_{n\in\mathbb{Z}} \left|\widehat{\phi}\left(\frac{\omega-p}{B}+2n\pi\right)\right|^2 \leq \eta_2, \quad \forall\, \omega \in \mathbb{R}. \tag{24}
$$

Then, the collection $\left\{\phi^M_{0,k}(t) = \phi(t-k)\, e^{-i\left(At^2+Dp^2-Ak^2\right)} : k \in \mathbb{Z}\right\}$ is linearly independent.

Proof. In order to prove the result, it is sufficient to find another function $\widetilde{\phi}$ whose translates are biorthogonal to the translates of ϕ. We define $\widetilde{\phi}$ by

$$
\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}\right) = \frac{\widehat{\phi}\left(\frac{\omega-p}{B}\right)}{\sum_{m\in\mathbb{Z}}\left|\widehat{\phi}\left(\frac{\omega-p}{B}+2m\pi\right)\right|^2}.
$$

Then, relation (24) implies that $\widetilde{\phi}$ is well defined and

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} \widehat{\phi}\left(\frac{\omega-p}{B}+2n\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}\right)} &= \sum_{n\in\mathbb{Z}} \widehat{\phi}\left(\frac{\omega-p}{B}+2n\pi\right) \frac{\overline{\widehat{\phi}\left(\frac{\omega-p}{B}+2n\pi\right)}}{\sum_{m\in\mathbb{Z}}\left|\widehat{\phi}\left(\frac{\omega-p}{B}+2(m+n)\pi\right)\right|^2} \\
&= \frac{\sum_{n\in\mathbb{Z}}\left|\widehat{\phi}\left(\frac{\omega-p}{B}+2n\pi\right)\right|^2}{\sum_{\ell\in\mathbb{Z}}\left|\widehat{\phi}\left(\frac{\omega-p}{B}+2\ell\pi\right)\right|^2} \\
&= 1.
\end{aligned}
$$

Therefore, it follows from Lemma 3.1 that the collection $\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$ is linearly independent.

This completes the proof of the Lemma 3.3. □

Proposition 3.4. *Assume that the scaling function ϕ satisfies the relation* (24). *Further, assume that $f = \sum_{k\in\mathbb{Z}} h_k\,\phi^M_{0,k}(t)$, where $f \in span\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$ and $h_k \in \ell^2(\mathbb{Z})$ is a finite sequence. Then, we have*

$$\eta_1 \int_0^{2\pi B} \left|\widehat{h}_M(\omega)\right|^2 \mathrm{d}\omega \le \left\|f\right\|_2^2 \le \eta_2 \int_0^{2\pi B} \left|\widehat{h}_M(\omega)\right|^2 \mathrm{d}\omega, \tag{25}$$

where $\widehat{h}_M(\omega) = \sum_{k\in\mathbb{Z}} h_k\, e^{i\left(At^2 - 2t(\omega-p) - 2\omega(Dp-Bq) + D\left(\omega^2+p^2\right)\right)/2B}$ is the discrete-time SAFT of the sequence h_k.

Proof. Invoking the Plancherel theorem for SAFT, we have

$$\begin{aligned}
\int_{\mathbb{R}} \left|f(t)\right|^2 dt &= \int_{\mathbb{R}} \left|\mathscr{L}_M\left[f\right](\omega)\right|^2 \mathrm{d}\omega \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} h_k\,\phi^M_{0,k}(t)\,\mathcal{K}_M(t,\omega)\,\mathrm{dt}\right|^2 \mathrm{d}\omega \\
&= \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi i B}} \sum_{k\in\mathbb{Z}} h_k\, e^{i\left(Ak^2 - 2\omega(Dp-Bq) + D\omega^2\right)/2B} \int_{\mathbb{R}} \phi(t-k)\, e^{-it(\omega-p)/B}\,\mathrm{dt}\right|^2 \mathrm{d}\omega \\
&= \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi i B}} \sum_{k\in\mathbb{Z}} h_k\, e^{i\left(Ak^2 - 2k(\omega-p) - 2\omega(Dp-Bq) + D\omega^2\right)/2B}\, \widehat{\phi}\left(\frac{\omega-p}{B}\right)\right|^2 \mathrm{d}\omega \\
&= \int_{\mathbb{R}} \left| \sum_{k\in\mathbb{Z}} \widehat{h}_M(\omega)\,\widehat{\phi}\left(\frac{\omega-p}{B}\right)\right|^2 \mathrm{d}\omega \\
&= \int_{\mathbb{R}} \left|\widehat{h}_M(\omega)\right|^2 \left|\widehat{\phi}\left(\frac{\omega-p}{B}\right)\right|^2 \mathrm{d}\omega \\
&= \int_0^{2\pi B} \left|\widehat{h}_M(\omega)\right|^2 \sum_{n\in\mathbb{Z}} \left|\widehat{\phi}\left(\frac{\omega-p}{B} + 2n\pi\right)\right|^2 \mathrm{d}\omega.
\end{aligned} \tag{26}$$

By implementing Lemma 3.3 in (26), the desired result follows. □

Theorem 3.5. *Let $\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$ be a Reisz basis for its closed linear span. Assume that there exists a function $\widetilde{\phi}$ such that $\left\{\widetilde{\phi}^M_{0,k}(t) : k \in \mathbb{Z}\right\}$ is biorthogonal to $\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$. Then for every $M = (A, B, C, D : p, q)$, $B \neq 0$ and $f \in span\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$, we have*

$$f(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \widetilde{\phi}^M_{0,k}\right\rangle \phi^M_{0,k} \tag{27}$$

and there exist positive constants η_1 and η_2 such that for every $f \in span\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$, we have

$$\eta_1 \left\|f\right\|_2^2 \le \sum_{n=1}^{\infty} \left|\left\langle f, \widetilde{\phi}^M_{0,k}\right\rangle\right|^2 \le \eta_2 \left\|f\right\|_2^2. \tag{28}$$

Proof. Since $\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$, forms a Riesz basis for its closed linear span, then there exist positive constants η_1 and η_2 such that (24) holds true. We first prove the results for $f \in \mathrm{span}\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$ and then generalize the established results to $\mathrm{span}\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$. Suppose that $f \in \mathrm{span}\left\{\phi^M_{0,k}(t) : k \in \mathbb{Z}\right\}$, then there exists a finite sequence h_k such that

$$f(t) = \sum_{k\in\mathbb{Z}} h_k\,\phi^M_{0,k}(t).$$

Invoking the condition of biorthogonality, we have

$$\left\langle f, \widetilde{\phi}_{0,k}^{M} \right\rangle = \left\langle \sum_{k\in\mathbb{Z}} h_k\, \phi_{0,k}^{M}, \widetilde{\phi}_{0,k}^{M} \right\rangle = \sum_{k\in\mathbb{Z}} h_k \left\langle \phi_{0,k}^{M}, \widetilde{\phi}_{0,k}^{M} \right\rangle = h_k,$$

which evidently proves (27).

Since equation (24) holds. Therefore by virtue of Proposition 3.4, we have

$$\frac{1}{\eta_2} \left\| f \right\|_2^2 \le \int_0^{2\pi B} \left| \widehat{h}_M(\omega) \right|^2 \mathrm{d}\omega \le \frac{1}{\eta_1} \left\| f \right\|_2^2.$$

By using Plancherel formula of the SAFT and the fact that $h_k = \left\langle f(t), \widetilde{\phi}_{0,k}^{M}(t) \right\rangle$, we have

$$\int_0^{2\pi B} \left| \widehat{h}_M(\omega) \right|^2 d\omega = \sum_{k\in\mathbb{Z}} \left| h_k \right|^2 = \sum_{k\in\mathbb{Z}} \left| \left\langle f, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2,$$

which establishes (28).

Finally, we proceed to generalize the results to $\operatorname{span}\left\{ \phi_{0,k}^{M}(t) : k \in \mathbb{Z} \right\}$. For $f \in \operatorname{span}\left\{ \widetilde{\phi}_{0,k}^{M}(t) : k \in \mathbb{Z} \right\}$, there exists a sequence $f_m \in \ell^2(\mathbb{Z})$ in $\operatorname{span}\left\{ \widetilde{\phi}_{0,k}^{M}(t) : k \in \mathbb{Z} \right\}$ such that $\lim_{m\to\infty} f_m = f$. Thus for each $k \in \mathbb{Z}$, we have

$$\left\langle f_m, \widetilde{\phi}_{0,k}^{M} \right\rangle \to \left\langle f, \widetilde{\phi}_{0,k}^{M} \right\rangle \quad \text{as } m \to \infty.$$

Hence, the result holds for each f_m. Thus, we have

$$\begin{aligned} \sum_{k=-N}^{N} \left| \left\langle f, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2 &= \sum_{k=-N}^{N} \lim_{m\to\infty} \left| \left\langle f_m, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2 \\ &= \lim_{m\to\infty} \sum_{k=-N}^{N} \left| \left\langle f_m, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2 \\ &\le \eta_2 \lim_{m\to\infty} \left\| f_m \right\|_2^2 \\ &= \eta_2 \left\| f \right\|_2^2. \end{aligned} \tag{29}$$

Letting $N \to \infty$ in (29), we obtain

$$\sum_{k\in\mathbb{Z}} \left| \left\langle f, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2 \le \eta_2 \left\| f \right\|_2^2.$$

Thus, the upper bound appearing in (28) holds. Moreover, by the Cauchy Schwarz inequality for sequences, we have

$$\left\{ \sum_{k\in\mathbb{Z}} \left| \left\langle f_m, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2 \right\}^{1/2} \le \left\{ \sum_{k\in\mathbb{Z}} \left| \left\langle f_m - f, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2 \right\}^{1/2} + \left\{ \sum_{k\in\mathbb{Z}} \left| \left\langle f, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2 \right\}^{1/2}.$$

Since the upper bound appearing in (28) holds for each $f_m - f$ and the lower bound holds for each f_m, we have

$$\eta_1^{1/2} \left\| f_m \right\|_2 \le \eta_2^{1/2} \left\| f_m - f \right\|_2 + \left\{ \sum_{k\in\mathbb{Z}} \left| \left\langle f, \widetilde{\phi}_{0,k}^{M} \right\rangle \right|^2 \right\}^{1/2}. \tag{30}$$

Therefore, from (30) we conclude that

$$A\left\|f\right\|_2^2 \leq \sum_{k\in\mathbb{Z}} \left|\left\langle f, \widetilde{\phi}_{0,k}^M\right\rangle\right|^2,$$

which establishes (28) completely. On the similar lines, we can prove (27) for $\text{span}\left\{\phi_{0,k}^M(t) : k \in \mathbb{Z}\right\}$.

This completes the proof of Theorem 3.5. □

4. Biorthogonal Properties of Special Affine Wavelets

Biorthogonal properties performs a key role in virtually all standard approaches when analyzing or synthesizing higher-level signal proceedings. In this section, we shall investigate biorthogonal properties of the special affine wavelets.

Let ϕ and $\widetilde{\phi}$ be scaling functions associated with the biorthogonal special affine MRA's $\left\{V_j^M : j \in \mathbb{Z}\right\}$ and $\left\{\widetilde{V}_j^M : j \in \mathbb{Z}\right\}$, respectively. For each $j \in \mathbb{Z}$, we define a pair of operators $\left\{\mathcal{P}_j^M, \widetilde{\mathcal{P}}_j^M\right\}$ and $\left\{\mathcal{Q}_j^M, \widetilde{\mathcal{Q}}_j^M\right\}$ on $L^2(\mathbb{R})$ by

$$\mathcal{P}_j^M f = \sum_{k\in\mathbb{Z}} \left\langle f, \widetilde{\phi}_{j,k}^M\right\rangle \phi_{j,k}^M \tag{31}$$

$$\widetilde{\mathcal{P}}_j^M f = \sum_{k\in\mathbb{Z}} \left\langle f, \phi_{j,k}^M\right\rangle \widetilde{\phi}_{j,k}^M \tag{32}$$

and

$$\mathcal{Q}_j^M f = \sum_{k\in\mathbb{Z}} \left\langle f, \widetilde{\psi}_{j,k}^M\right\rangle \psi_{j,k}^M \tag{33}$$

$$\widetilde{\mathcal{Q}}_j^M f = \sum_{k\in\mathbb{Z}} \left\langle f, \psi_{j,k}^M\right\rangle \widetilde{\psi}_{j,k}^M, \tag{34}$$

respectively. It is easy to verify that both these operators are uniformly bounded on $L^2(\mathbb{R})$ and both the series are convergent in $L^2(\mathbb{R})$.

Remark 4.1. *The operators $\mathcal{P}_j^M$ and $\widetilde{\mathcal{P}}_j^M$ satisfy the following properties:*

(i) $\mathcal{P}_j^M f = f$ *if and only if* $f \in V_j^M$ *and* $\widetilde{\mathcal{P}}_j^M = f$ *if only if* $f \in \widetilde{V}_j^M$;

(ii) $\lim_{j\to\infty} \left\|\mathcal{P}_j^M f - f\right\|_2 = 0$ *and* $\lim_{j\to-\infty} \left\|\mathcal{P}_j^M f\right\|_2 = 0$ *for every* $f \in L^2(\mathbb{R})$.

Theorem 4.2. *Let ϕ and $\widetilde{\phi}$ be the scaling functions associated with the biorthogonal special affine MRA's $\left\{V_j^M : j \in \mathbb{Z}\right\}$ and $\left\{\widetilde{V}_j^M : j \in \mathbb{Z}\right\}$, respectively. If ψ and $\widetilde{\psi}$ are the associated special affine wavelets satisfying the matrix condition*

$$\mathbb{M}(\omega)\,\overline{\widetilde{\mathbb{M}}}(\omega) = \mathbb{I}_{2\times 2}, \tag{35}$$

where

$$\mathbb{M}(\omega) = \begin{pmatrix} \Lambda_0\left(\dfrac{\omega - p}{2B}\right) & \Lambda_0\left(\dfrac{\omega - p}{2B} + \pi\right) \\ \Lambda_1\left(\dfrac{\omega - p}{2B}\right) & \Lambda_1\left(\dfrac{\omega - p}{2B} + \pi\right) \end{pmatrix}.$$

Then, we have

(i) $\left\{\widetilde{\psi}^M_{0,k} : k \in \mathbb{Z}\right\}$ *is biorthogonal to* $\left\{\psi^M_{0,\ell} : \ell \in \mathbb{Z}\right\}$*;*

(ii) $\left\langle \psi^M_{0,k}, \widetilde{\phi}^M_{0,\ell} \right\rangle = \left\langle \widetilde{\psi}^M_{0,k}, \phi^M_{0,\ell} \right\rangle$, $\forall\, k, \ell \in \mathbb{Z}$.

Proof. We have,

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} \widehat{\psi}\left(\frac{\omega-p}{B}+2n\pi\right)\overline{\widehat{\widetilde{\psi}}\left(\frac{\omega-p}{B}+2n\pi\right)}\\
&=\sum_{n\in\mathbb{Z}}\left[\Lambda_1\left(\frac{\omega-p}{2B}+n\pi\right)\overline{\widetilde{\Lambda}_1\left(\frac{\omega-p}{2B}+n\pi\right)}\widehat{\phi}\left(\frac{\omega-p}{2B}+n\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+n\pi\right)}\right]\\
&=\Lambda_1\left(\frac{\omega-p}{2B}\right)\overline{\widetilde{\Lambda}_1\left(\frac{\omega-p}{2B}\right)}\left[\sum_{k\in\mathbb{Z}}\widehat{\phi}\left(\frac{\omega-p}{2B}+2k\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+2k\pi\right)}\right]\\
&+\Lambda_1\left(\frac{\omega-p}{2B}+\pi\right)\overline{\widetilde{\Lambda}_1\left(\frac{\omega-p}{2B}+\pi\right)}\left[\sum_{k\in\mathbb{Z}}\widehat{\phi}\left(\frac{\omega-p}{2B}+(2k+1)\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+(2k+1)\pi\right)}\right]\\
&=\left[\Lambda_1\left(\frac{\omega-p}{2B}\right)\overline{\widetilde{\Lambda}_1\left(\frac{\omega-p}{2B}\right)}+\Lambda_1\left(\frac{\omega-p}{2B}+\pi\right)\overline{\widetilde{\Lambda}_1\left(\frac{\omega-p}{2B}+\pi\right)}\right]\\
&=1.
\end{aligned}
$$

Hence by virtue of Theorem 3.1, $\left\{\psi^M_{0,k} : k \in \mathbb{Z}\right\}$ is biorthogonal to $\left\{\widetilde{\psi}^M_{0,k} : k \in \mathbb{Z}\right\}$.

We shall now proceed prove (ii). For any fixed constants $k, \ell \in \mathbb{Z}$, an application of Plancherel formula for SAFT yields

$$
\begin{aligned}
\left\langle \psi^M_{0,k}, \widetilde{\phi}^M_{0,\ell}\right\rangle &= \left\langle \mathscr{L}_M\left[\psi^M_{0,k}\right](\omega), \mathscr{L}_M\left[\widetilde{\phi}^M_{0,\ell}\right](\omega)\right\rangle\\
&=\int_{\mathbb{R}}\widehat{\psi}\left(\frac{\omega-p}{B}\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}\right)}\mathcal{K}(k,\omega)\,\overline{\mathcal{K}(\ell,\omega)}\,\mathrm{d}\omega\\
&=\frac{1}{2\pi B}\,e^{i\left(A\left(k^2-\ell^2\right)+2p(k-\ell)\right)/2B}\int_{\mathbb{R}}\widehat{\psi}\left(\frac{\omega-p}{B}\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{B}\right)}e^{-i(k-\ell)\omega/B}\,\mathrm{d}\omega\\
&=\frac{1}{2\pi B}\,e^{i\left(A\left(k^2-\ell^2\right)+2p(k-\ell)\right)/2B}\int_{\mathbb{R}}\Lambda_1\left(\frac{\omega-p}{2B}\right)\overline{\widetilde{\Lambda}_0\left(\frac{\omega-p}{2B}\right)}\widehat{\phi}\left(\frac{\omega-p}{2B}\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}\right)}e^{-i(k-\ell)\omega/B}\,\mathrm{d}\omega\\
&=\frac{1}{2\pi B}\,e^{i\left(A\left(k^2-\ell^2\right)+2p(k-\ell)\right)/2B}\int_0^{2\pi B}\left[\sum_{n\in\mathbb{Z}}\Lambda_1\left(\frac{\omega-p}{2B}+n\pi\right)\overline{\widetilde{\Lambda}_0\left(\frac{\omega-p}{2B}+n\pi\right)}\right.\\
&\qquad\left.\times\,\widehat{\phi}\left(\frac{\omega-p}{2B}+n\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+n\pi\right)}\right]e^{-i(k-\ell)\omega/B}\,\mathrm{d}\omega\\
&=\frac{1}{2\pi B}\,e^{i\left(A\left(k^2-\ell^2\right)+2p(k-\ell)\right)/2B}\int_0^{2\pi B}\left[\Lambda_1\left(\frac{\omega-p}{2B}\right)\overline{\widetilde{\Lambda}_0\left(\frac{\omega-p}{2B}\right)}\right.\\
&\qquad\times\left[\sum_{k\in\mathbb{Z}}\widehat{\phi}\left(\frac{\omega-p}{2B}+2k\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+2k\pi\right)}\right]\\
&\qquad\left.+\;\Lambda_1\left(\frac{\omega-p}{2B}+\pi\right)\overline{\widetilde{\Lambda}_0\left(\frac{\omega-p}{2B}+\pi\right)}\left[\sum_{k\in\mathbb{Z}}\widehat{\phi}\left(\frac{\omega-p}{2B}+\pi\right)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+\pi\right)}\right]\right]e^{-i(k-\ell)\omega/B}\,\mathrm{d}\omega
\end{aligned}
$$

$$= \frac{1}{2\pi B}\, e^{i\left(A\left(k^2-\ell^2\right)+2p(k-\ell)\right)/2B} \int_0^{2\pi B} \left[\Lambda_1\left(\frac{\omega-p}{2B}\right)\overline{\widetilde{\Lambda}_0\left(\frac{\omega-p}{2B}\right)} + \Lambda_1\left(\frac{\omega-p}{2B}+\pi\right)\overline{\widetilde{\Lambda}_0\left(\frac{\omega-p}{2B}+\pi\right)}\right] e^{-i(k-\ell)\omega/B}\, d\omega$$

$$= 0.$$

On the similar lines, we can show that

$$\left\langle \widetilde{\psi}^M_{0,k}, \phi^M_{0,\ell}\right\rangle = 0, \quad \forall\, k,\ell \in \mathbb{Z}.$$

This completes the proof of Theorem 4.2. □

Theorem 4.3. *Let $\phi, \widetilde{\phi}, \psi$ and $\widetilde{\psi}$ be as in Theorem 4.2 such that $\psi_0 = \phi$ and $\widetilde{\psi}_0 = \widetilde{\phi}$. Then for every $M = (A, B, C, D : p, q)$, $B \neq 0$ and $f \in L^2(\mathbb{R})$, we have*

(i).

$$\mathcal{Q}^M_1 f = \mathcal{P}^M_0 f + \sum_{k\in\mathbb{Z}} \left\langle f, \widetilde{\psi}^M_{0,k}\right\rangle \psi^M_{0,k} \tag{36}$$

$$\widetilde{\mathcal{Q}}^M_1 f = \widetilde{\mathcal{P}}^M_0 f + \sum_{k\in\mathbb{Z}} \left\langle f, \psi^M_{0,k}\right\rangle \widetilde{\psi}^M_{0,k}, \tag{37}$$

where the series (36) *and* (37) *converges in $L^2(\mathbb{R})$.*

(ii). *The collection $\left\{\psi^M_{j,k} : j,k \in \mathbb{Z}\right\}$ is biorthogonal to $\left\{\widetilde{\psi}^M_{j,k} : j,k \in \mathbb{Z}\right\}$.*

Proof. We shall only prove (36) as the proof of (37) follows in the similar manner. Moreover, It is sufficient to prove (36) in the weak sense, that is, for all $f, g \in L^2(\mathbb{R})$

$$\left\langle \mathcal{Q}^M_1 f, g\right\rangle = \left\langle \mathcal{P}^M_0 f, g\right\rangle + \sum_{k\in\mathbb{Z}} \left\langle f, \widetilde{\psi}^M_{0,k}\right\rangle \overline{\left\langle g, \psi^M_{0,k}\right\rangle} = \sum_{k\in\mathbb{Z}} \left\langle f, \widetilde{\psi}^M_{0,k}\right\rangle \overline{\left\langle g, \psi^M_{0,k}\right\rangle}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left\langle f, \widetilde{\psi}^M_{0,k}\right\rangle \overline{\left\langle g, \psi^M_{0,k}\right\rangle} \\
&= \frac{1}{|B|} \sum_{k\in\mathbb{Z}} \left\{ \int_{\mathbb{R}} \mathscr{L}_M\big[f\big](\omega)\, e^{-i\left(Ak^2-2k(\omega-p)-2\omega(Dp-Bq)+D\omega^2\right)/2B}\, \overline{\widehat{\widetilde{\psi}}\left(\frac{\omega-p}{B}\right)} d\omega \right\} \\
&\quad \times \left\{ \int_{\mathbb{R}} \overline{\mathscr{L}_M\big[g\big]}(\omega)\, e^{i\left(Ak^2-2k(\omega-p)-2\omega(Dp-Bq)+D\omega^2\right)/2B}\, \widehat{\psi}\left(\frac{\omega-p}{B}\right) d\omega \right\} \\
&= \frac{1}{|B|} \sum_{k\in\mathbb{Z}} \left\{ \int_0^{2\pi B} \sum_{m\in\mathbb{Z}} \mathscr{L}_M\big[f\big](\omega+2m\pi B)\, \overline{\widehat{\widetilde{\psi}}\left(\frac{\omega-p}{B}+2m\pi\right)} d\omega \right\} \\
&\quad \times \left\{ \int_0^{2\pi B} \sum_{n\in\mathbb{Z}} \overline{\mathscr{L}_M\big[g\big]}(\omega+2n\pi B)\, \widehat{\psi}\left(\frac{\omega-p}{B}+2n\pi\right) d\omega \right\} \\
&= \frac{1}{|B|} \int_0^{2\pi B} \sum_{m\in\mathbb{Z}} \mathscr{L}_M\big[f\big](\omega+2m\pi B)\, \overline{\widetilde{\Lambda}_1\left(\frac{\omega-p}{2B}+m\pi\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+m\pi\right)} \\
&\quad \times \sum_{n\in\mathbb{Z}} \overline{\mathscr{L}_M\big[f\big]}(\omega+2m\pi B)\, \Lambda_1\left(\frac{\omega-p}{2B}+n\pi\right) \widehat{\phi}\left(\frac{\omega-p}{2B}+n\pi\right) d\omega
\end{aligned}$$

$$= \frac{1}{|B|}\int_0^{2\pi B}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathscr{L}_M\big[f\big](\omega+2m\pi B)\,\overline{\mathscr{L}_M\big[f\big]}(\omega+2m\pi B)\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+m\pi\right)}\widehat{\phi}\left(\frac{\omega-p}{2B}+n\pi\right)\mathrm{d}\omega. \quad (38)$$

Similarly, we can show that

$$\sum_{k\in\mathbb{Z}}\left\langle f,\widetilde{\phi}^M_{1,k}\right\rangle\overline{\left\langle g,\phi^M_{1,k}\right\rangle} = \frac{1}{|B|}\int_0^{2\pi B}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathscr{L}_M\big[f\big](\omega+2m\pi B)\,\overline{\mathscr{L}_M\big[f\big]}(\omega+2m\pi B) \times\overline{\widehat{\widetilde{\phi}}\left(\frac{\omega-p}{2B}+m\pi\right)}\widehat{\phi}\left(\frac{\omega-p}{2B}+n\pi\right)\mathrm{d}\omega. \quad (39)$$

From equations (38) and (39), the desired result follows.

In order to prove that the collections $\left\{\psi^M_{j,k}: j,k\in\mathbb{Z}\right\}$ and $\left\{\widetilde{\psi}^M_{j,k}: j,k\in\mathbb{Z}\right\}$ are biorthogonal to each other, we shall show that for each $j\in\mathbb{Z}$

$$\left\langle\psi^M_{j,k},\widetilde{\psi}^M_{j,k'}\right\rangle = \delta_{k,k'}. \quad (40)$$

The case $j=0$ is ascertained by Theorem 4.2. Further for $j\neq 0$, we have

$$\left\langle\psi^M_{j,k},\widetilde{\psi}^M_{j,k'}\right\rangle = \left\langle\delta_{-j}\,\psi^M_{0,k},\,\delta_{-j}\,\widetilde{\psi}^M_{0,k'}\right\rangle = \left\langle\psi^M_{0,k},\widetilde{\psi}^M_{0,k'}\right\rangle = \delta_{k,k'}.$$

Let $k,k'\in\mathbb{Z}$ be fixed and $j,j'\in\mathbb{Z}$ with $j<j'$. We show that

$$\left\langle\psi^M_{j,k},\widetilde{\psi}^M_{j',k'}\right\rangle = 0.$$

Since $\psi^M_{0,k}\in V^M_1$, hence $\psi^M_{j,k}=\delta_{-j}\,\psi^M_{0,k}\in V^M_{j+1}\subseteq V^M_{j'}$. Therefore, it is sufficient to show that $\widetilde{\psi}^M_{j',k'}$ is orthogonal to every element of $V^M_{j'}$. Let $f\in V^M_{j'}$. Since $\left\{\phi^M_{j',k'}: k\in\mathbb{Z}\right\}$ is a Riesz basis for $V^M_{j'}$. Hence, there exists a sequence $d_k\in\ell^2(\mathbb{Z})$ such that $f=\sum_{k\in\mathbb{Z}} d_k\,\phi^M_{j',k'}$ in $L^2(\mathbb{R})$. By virtue of Theorem 4.2 (ii), we have

$$\left\langle\widetilde{\psi}^M_{j',k'},\phi^M_{j',k}\right\rangle = \left\langle\delta_{-j'}\,\widetilde{\psi}^M_{0,k'},\,\delta_{-j'}\,\phi^M_{0,k}\right\rangle = \left\langle\widetilde{\psi}^M_{0,k'},\phi^M_{0,k}\right\rangle = 0. \quad (41)$$

Hence,

$$\left\langle\widetilde{\psi}^M_{j',k'},f\right\rangle = \left\langle\widetilde{\psi}^M_{j',k'},\sum_{k\in\mathbb{Z}} d_k\,\phi^M_{j',k'}\right\rangle = \sum_{k\in\mathbb{Z}}\bar{d}_k\left\langle\widetilde{\psi}^M_{j',k'},\phi^M_{j',k}\right\rangle = 0, \quad (42)$$

which evidently proves that the collections $\left\{\psi^M_{j,k}: j,k\in\mathbb{Z}\right\}$ and $\left\{\widetilde{\psi}^M_{j,k}: j,k\in\mathbb{Z}\right\}$ are biorthogonal to each other. This completes the proof of Theorem 4.4. □

Theorem 4.4. *Let $\phi,\widetilde{\phi},\psi$ and $\widetilde{\psi}$ be defined as in Theorem 4.5. Then, for every $f\in L^2(\mathbb{R})$, we have*

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left\langle f,\widetilde{\psi}^M_{j,k}\right\rangle\psi^M_{j,k} = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left\langle f,\psi^M_{j,k}\right\rangle\widetilde{\psi}^M_{j,k}, \quad (4.8)$$

where the series converges in $L^2(\mathbb{R})$.

Proof. The result asserted by Theorem 3.1 follows immediately by using Remark 4.1 and Theorem 4.3. □

5. Special Affine Biorthogonal Wavelets on a Logarithmic Regression Curve

In this section, we first recall the fundamentals of logarithmic regression curve $\mathscr{C}$ and then introduce the notion of special affine biorthogonal wavelets on a logarithmic regression curve. Moreover, we provide a complete characterization of the biorthogonal functions corresponding to two special affine MRA's.

5.1. Logarithmic Regression Curve and the Associated Special Affine MRA

Undoubtedly logarithmic regression trend curve has witnessed a great deal of development in the economic and financial models, such as Cobb-Douglas production function given by $y = AK^{\alpha}x^{\beta}e^{\mu}$ [21]. On taking log on both sides of this function yields

$$\ln y = \ln A + \alpha \ln K + \beta \ln x + \mu, \tag{43}$$

where μ is the white noise and x is a trend variable. For $A = 1, K = e$, relation (43) reduces to

$$\ln y = \alpha + \beta \ln x + \mu. \tag{44}$$

The formula given by (44) is called a one-dimensional logarithmic model. Moreover, if the dependent variable y and the trend variable x satisfy the relation

$$y = \alpha + \beta \ln x + \mu, \tag{45}$$

then the equation (45) is known as one-dimensional semi-logarithmic regression model and $\widetilde{y} = \widetilde{\alpha} + \widetilde{\beta} \ln x$ is known as logarithmic regression curve, where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the estimators of α and β, respectively.

Assume that a logarithmic regression trend curve

$$\mathscr{C} : \begin{cases} x = x \\ y = a \ln x \end{cases}, \quad a \in \mathbb{R},\, x \in \mathbb{R}^{+} \tag{46}$$

satisfies the parametric equation

$$\xi = \xi(x, y) = \big(x(t), y(t)\big), \quad t \in \mathbb{R} \tag{47}$$

where x and y are the functions of parameter t. Moreover, we consider the length preserving projection $\mathscr{P} : (x, y) \to (\ell, 0) = (L(x), 0) = (X, 0)$, so that the length element $\mathrm{dL}(\xi)$ of $\mathscr{C}$ is equal to the length element dX of $\mathbb{R}$. For all arbitrary functions $\widetilde{f}, \widetilde{g} \in L^2(\mathscr{C})$, we have [22]

$$\left\langle \widetilde{f}, \widetilde{g} \right\rangle_{L^2(\mathscr{C})} = \left\langle \widetilde{f} \circ \mathscr{P}^{-1}, \widetilde{g} \circ \mathscr{P}^{-1} \right\rangle_2 \quad \text{and} \quad \left\langle f, g \right\rangle_2 = \left\langle f \circ \mathscr{P}, g \circ \mathscr{P} \right\rangle_{L^2(\mathscr{C})}. \tag{48}$$

The Fourier transform on the space $L^2(\mathscr{C})$ is given by [23]:

$$\widehat{\widetilde{f}}(\widetilde{\omega}) = \frac{1}{\sqrt{2\pi}} \int_{\mathscr{C}} \widetilde{f}(\eta)\, e^{-i\mathscr{P}(\widetilde{\omega})\cdot\mathscr{P}(\eta)}\, \mathrm{dL}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widetilde{f}\left(\mathscr{P}^{-1}(\mathrm{X})\right) e^{i\omega \mathrm{X}}\, \mathrm{dX}. \tag{49}$$

By virtue of the relations (2) and (49), we have the following definition of SAFT on the smooth curve $\mathscr{C}$.

Definition 5.1. *The special affine Fourier transform of any function $\widetilde{f} \in L^2(\mathscr{C})$ is defined by*

$$\begin{aligned} \mathscr{L}_M\left[\widetilde{f}\right](\widetilde{\omega}) &= \frac{1}{\sqrt{2\pi i B}}\, e^{i\left(D\left(\mathscr{P}(\widetilde{\omega})^2 + p^2\right) - 2\mathscr{P}(\widetilde{\omega})(Dp - Bq)\right)/2B} \int_{\mathscr{C}} e^{iA\mathscr{P}(\eta)^2}\, \widetilde{f}(\eta)\, e^{-i\left(\mathscr{P}(\widetilde{\omega}) - p\right)\mathscr{P}(\eta)}\, \mathrm{d}\eta \\ &= \frac{1}{\sqrt{2\pi i B}}\, e^{i\left(D\left(\omega^2 + p^2\right) - 2\omega(Dp - Bq)\right)/2B} \int_{\mathbb{R}} e^{iA\mathrm{X}^2/2B}\, \widetilde{f}\left(\mathscr{P}^{-1}(\mathrm{X})\right) e^{-i\omega \mathrm{X}}\, \mathrm{dX}. \end{aligned} \tag{50}$$

For every $f \in L^2(\mathbb{R})$, the induced function $f^{\mathscr{C}} \in L^2(\mathscr{C})$ can be defined as

$$f^{\mathscr{C}} = f \circ \mathscr{P}. \tag{51}$$

Therefore, by invoking (48), we conclude that if the collection $\left\{ f^M_{j,k} : j,k \in \mathbb{Z} \right\}$ are orthogonal, so are

$$f^{M,\mathscr{C}}_{j,k} = f^M_{j,k} \circ \mathscr{P}, \quad \forall j,k \in \mathbb{Z}. \tag{52}$$

For the cases $f^M_{j,k} = \phi^M_{j,k}$ and $f^M_{j,k} = \psi^M_{j,k}$, we have the following pair of functions on the smooth curve $\mathscr{C}$.

$$\phi^{M,\mathscr{C}}_{j,k} = \phi^M_{j,k} \circ \mathscr{P} \tag{53}$$

$$\psi^{M,\mathscr{C}}_{j,k} = \psi^M_{j,k} \circ \mathscr{P}. \tag{54}$$

Therefore, the special affine MRA of $L^2(\mathscr{C})$ can be defined by virtue of the special affine MRA of $L^2(\mathbb{R})$ and the induced function as defined by (51). For every $j \in \mathbb{Z}$, we define the space v^M_j as

$$v^M_j = \left\{ f^{M,\mathscr{C}}_j = f^M_j \circ \mathscr{P} : f_j \in V^M_j \right\}. \tag{55}$$

It is immediate that the sequence $\left\{ v^M_j \right\}$ is a closed subspace of $L^2(\mathscr{C})$. Thus, we have the following definition of MRA of $L^2(\mathscr{C})$ associated with the SAFT. Prior to that, we define a translation operator $\mathcal{T}_k$ and a dilation operator $\mathcal{D}_a$ by $\mathcal{T}_k \widetilde{f}(\eta) = \left(\widetilde{f} \circ \mathscr{P}^{-1} \right) (\mathscr{P}(\eta) - k)$, $k \in \mathbb{R}$ and $\mathcal{D}_a \widetilde{f}(\eta) = \left(\widetilde{f} \circ \mathscr{P}^{-1} \right) (a\mathscr{P}(\eta))$, $a \in \mathbb{R}^+$, respectively.

Definition 5.2. *Given a real parametric matrix $M = (A, B, C, D : p, q)$, $B \neq 0$, an associated special affine multiresolution is a collection $\{v^M_j\}$ of closed subspaces of $L^2(\mathscr{C})$ satisfying the following properties:*

(i). $v^M_j \subset v^M_{j+1}$, *for all* $j \in \mathbb{Z}$;

(ii). $\bigcup_{j\in\mathbb{Z}} v^M_j$ *is dense in* $L^2(\mathscr{C})$;

(iii). $\bigcap_{j\in\mathbb{Z}} v^M_j = \left\{ 0 \right\}$;

(iv). $f^{\mathscr{C}} \in v^M_j$ *if and only if* $e^{i3A\mathscr{P}(\eta)^2/2B} \mathcal{D}_2 f^{\mathscr{C}} \in v^M_{j+1}$, *for all* $j \in \mathbb{Z}$;

(v). *There exists a function* $\phi^{\mathscr{C}} \in L^2(\mathscr{C})$ *in* v^M_0 *such that* $\left\{ e^{-i\left(A\mathscr{P}(\eta)^2 + Dp^2 - Ak^2\right)/2B} \mathcal{T}_k \phi^{\mathscr{C}} : k \in \mathbb{Z} \right\}$ *is a Riesz basis of subspace* v^M_0.

For every $j \in \mathbb{Z}$, we define another sequence w^M_j of closed subspaces of $L^2(\mathscr{C})$ by $v^M_{j+1} = v^M_j \oplus w^M_j$. Moreover, for every $j \in \mathbb{Z}$, w^M_j are mutually orthogonal and $\left\{ \psi^{M,\mathscr{C}}_{j,k} : j,k \in \mathbb{Z} \right\}$ forms an orthogonal basis of $\overline{\oplus_{j\in\mathbb{Z}} w^M_j} = L^2(\mathscr{C})$. Further, we can define $\psi^{M,\mathscr{C}}_{j,k}$ as $\psi^{M,\mathscr{C}}_{j,k} = \psi^M_{j,k} \circ \mathscr{P}$, then $\phi^{\mathscr{C}}$ is called the scaling function on the logarithmic regression trend curve $\mathscr{C}$, and $\psi^{\mathscr{C}}$ is called the corresponding wavelets on the logarithmic regression trend curve $\mathscr{C}$.

By invoking the length preserving projection operator $\mathscr{P}$, translation operator $\mathcal{T}_k$ and the dilation operator $\mathcal{D}_a$, the scaling function $\phi^{\mathscr{C}}$ in $L^2(\mathscr{C})$ can be written as

$$\phi^{M,\mathscr{C}}_{0,0}(\eta) = \sqrt{2} \sum_{k\in\mathbb{Z}} h^{\mathscr{C}}_k \left(\phi^{\mathscr{C}} \circ \mathscr{P}^{-1} \right) (2\mathscr{P}(\eta) - k)\, e^{-i\left(A\mathscr{P}(\eta)^2 + Dp^2 - Ak^2\right)/2B}, \tag{56}$$

where $h^{\mathscr{C}}_k$ is called the special affine refinement equation in $L^2(\mathscr{C})$ and it is quite easy to prove that $h^{\mathscr{C}}_k = h_k$. Implementing SAFT as given by (50) on both sides of (56), we have

$$\widehat{\phi^{\mathscr{C}}} \left(\frac{\widetilde{\omega} - p}{B} \right) = \Lambda^{\mathscr{C}}_0 \left(\frac{\widetilde{\omega} - p}{2B} \right) \widehat{\phi^{\mathscr{C}} \circ \mathscr{P}^{-1}} \left(\frac{\widetilde{\omega} - p}{2B} \right), \tag{57}$$

where $\mathscr{P}(\widetilde{\omega}) = \omega$, $\widetilde{\omega} \in \mathscr{C}$ and $\Lambda_0^{\mathscr{C}}\left(\dfrac{\widetilde{\omega}-p}{2B}\right) = \dfrac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_k^{M,\mathscr{C}}\, e^{-i\left(\mathscr{P}(\widetilde{\omega})-p\right)k/2B}$, $h_k^{M,\mathscr{C}} = h_k^{\mathscr{C}}\, e^{iAk^2/2B}$ is called the special affine low-pass filter in $L^2(\mathscr{C})$. By invoking (57) continuously, we can obtain

$$\widehat{\phi^{\mathscr{C}}}\left(\frac{\widetilde{\omega}-p}{B}\right) = \prod_{j=1}^{\infty} \Lambda_0^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2^j B}\right) \widehat{\phi^{\mathscr{C}} \circ \mathscr{P}^{-1}}\left(\frac{\widetilde{\omega}-p}{2^j B}\right), \tag{58}$$

It is immediate that we should must be able to let $n \to \infty$.

$$\widehat{\phi^{\mathscr{C}}}\left(\frac{\widetilde{\omega}-p}{B}\right) = \prod_{j=1}^{\infty} \Lambda_0^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2^j B}\right) \widehat{\phi^{\mathscr{C}} \circ \mathscr{P}^{-1}}(0),$$

since $2^{-j} \to 0$ as $j \to \infty$. Moreover, it follows that a non-trivial solution must satisfy $\widehat{\phi^{\mathscr{C}} \circ \mathscr{P}^{-1}}(0) \neq 0$. Assume that $\widehat{\phi^{\mathscr{C}} \circ \mathscr{P}^{-1}}(0) = 1$, then (58) becomes

$$\widehat{\phi^{\mathscr{C}}}\left(\frac{\widetilde{\omega}-p}{B}\right) = \prod_{j=1}^{\infty} \Lambda_0^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2^j B}\right). \tag{59}$$

Since $\widehat{\phi^{\mathscr{C}} \circ \mathscr{P}^{-1}}(0) = 1$, it follows immediately from (20) that $\Lambda_0^{\mathscr{C}}(0) = 1$, which is essential for convergence of the infinite product $\prod_{j=1}^{\infty} \Lambda_0^{\mathscr{C}}\left(\dfrac{\widetilde{\omega}-p}{2^j B}\right)$.

5.2. Special Affine Biorthogonal Wavelets on a Logarithmic Regression Trend Curve

Let $\{v_j^M : j \in \mathbb{Z}\}$ and $\{\widetilde{v}_j^M : j \in \mathbb{Z}\}$ be biorthogonal special affine MRA of $L^2(\mathscr{C})$ with scaling functions $\phi^{\mathscr{C}}$ and $\widetilde{\phi}^{\mathscr{C}}$, respectively. Then, the scaling functions $\phi^{\mathscr{C}}$ and $\widetilde{\phi}^{\mathscr{C}}$ satisfy the following pair of equations:

$$\widehat{\phi^{\mathscr{C}}}\left(\frac{\widetilde{\omega}-p}{B}\right) = \Lambda_0^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2B}\right) \widehat{\phi^{\mathscr{C}} \circ \mathscr{P}^{-1}}\left(\frac{\widetilde{\omega}-p}{2B}\right) \tag{60}$$

$$\widehat{\phi^{\mathscr{C}}}\left(\frac{\widetilde{\omega}-p}{B}\right) = \Lambda_0^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2B}\right) \widehat{\phi^{\mathscr{C}} \circ \mathscr{P}^{-1}}\left(\frac{\widetilde{\omega}-p}{2B}\right), \tag{61}$$

where

$$\Lambda_0^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2B}\right) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_k^{M,\mathscr{C}}\, e^{-i\left(\mathscr{P}(\widetilde{\omega})-p\right)k/2B},\ h_k^{M,\mathscr{C}} = h_k^{\mathscr{C}}\, e^{iAk^2/2B}$$

$$\widetilde{\Lambda}_0^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2B}\right) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} \widetilde{h}_k^{M,\mathscr{C}}\, e^{-i\left(\mathscr{P}(\widetilde{\omega})-p\right)k/2B},\ \widetilde{h}_k^{M,\mathscr{C}} = \widetilde{h}_k^{\mathscr{C}}\, e^{iAk^2/2B}.$$

Assume that there exists a pair of two-scale functions $\left\{\Lambda_0^{M,\mathscr{C}}, \widetilde{\Lambda}_0^{M,\mathscr{C}}\right\}$ and $\left\{\Lambda_1^{M,\mathscr{C}}, \widetilde{\Lambda}_1^{M,\mathscr{C}}\right\}$ such that

$$\mathbb{M}^{\mathscr{C}}(\tilde{\omega})\,\overline{\widetilde{\mathbb{M}}}^{\mathscr{C}}(\tilde{\omega}) = \mathbb{I}_{2\times 2}, \tag{62}$$

where

$$\mathbb{M}(\tilde{\omega}) = \begin{pmatrix} \Lambda_0^{\mathscr{C}}\left(\dfrac{\mathscr{P}(\widetilde{\omega})-p}{2B}\right) & \Lambda_0^{\mathscr{C}}\left(\dfrac{\mathscr{P}(\widetilde{\omega})-p}{2B}+\pi\right) \\ \Lambda_1^{\mathscr{C}}\left(\dfrac{\mathscr{P}(\widetilde{\omega})-p}{2B}\right) & \Lambda_1^{\mathscr{C}}\left(\dfrac{\mathscr{P}(\widetilde{\omega})-p}{2B}+\pi\right) \end{pmatrix}.$$

Moreover, we define the associated special affine biorthgonal wavelets as $\psi^{\mathscr{C}}$ and $\widetilde{\psi}^{\mathscr{C}}$ by

$$\widehat{\psi^{\mathscr{C}}}\left(\frac{\widetilde{\omega}-p}{B}\right)=\Lambda_1^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2B}\right)\widehat{\phi^{\mathscr{C}}\circ\mathscr{P}^{-1}}\left(\frac{\widetilde{\omega}-p}{2B}\right) \tag{63}$$

$$\widehat{\widetilde{\psi}^{\mathscr{C}}}\left(\frac{\widetilde{\omega}-p}{B}\right)=\Lambda_1^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2B}\right)\widehat{\phi^{\mathscr{C}}\circ\mathscr{P}^{-1}}\left(\frac{\widetilde{\omega}-p}{2B}\right), \tag{64}$$

where

$$\Lambda_1^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2B}\right)=\frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}g_k^{M,\mathscr{C}}\,e^{-i\left(\mathscr{P}(\widetilde{\omega})-p\right)k/2B},\ g_k^{M,\mathscr{C}}=g_k^{\mathscr{C}}\,e^{iAk^2/2B}$$

$$\widetilde{\Lambda}_1^{\mathscr{C}}\left(\frac{\widetilde{\omega}-p}{2B}\right)=\frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}\widetilde{g}_k^{M,\mathscr{C}}\,e^{-i\left(\mathscr{P}(\widetilde{\omega})-p\right)k/2B},\ \widetilde{g}_k^{M,\mathscr{C}}=\widetilde{g}_k^{\mathscr{C}}\,e^{iAk^2/2B}.$$

Definition 5.3. *A pair of special affine MRA's $\{v_j^M : j\in\mathbb{Z}\}$ and $\{\widetilde{v}_j^M : j\in\mathbb{Z}\}$ with scaling functions $\phi^{\mathscr{C}}$ and $\widetilde{\phi}^{\mathscr{C}}$, respectively, are said to be biorthogonal to each other with respect to an augmented matrix $M=(A,B,C,D:p,q)$, $B\neq 0$ if*

$$\left\{\phi_{0,k}^{M,\mathscr{C}}(\eta)=\left(\phi^{\mathscr{C}}\circ\mathscr{P}^{-1}\right)\left(\mathscr{P}(\eta)-k\right)e^{-i\left(A\mathscr{P}(\eta)^2+Dp^2-Ak^2\right)/2B}:k\in\mathbb{Z}\right\}$$

and

$$\left\{\widetilde{\phi}_{0,k}^{M,\mathscr{C}}(\eta)=\left(\widetilde{\phi}^{\mathscr{C}}\circ\mathscr{P}^{-1}\right)\left(\mathscr{P}(\eta)-k\right)e^{-i\left(A\mathscr{P}(\eta)^2+Dp^2-Ak^2\right)/2B}:k\in\mathbb{Z}\right\}$$

are biorthogonal.

Theorem 5.4. *Let $\phi^{\mathscr{C}}$ and $\widetilde{\phi}^{\mathscr{C}}$ be a pair of biorthogonal scaling functions associated with the special MRA's $\{v_j^M : j\in\mathbb{Z}\}$ and $\{\widetilde{v}_j^M : j\in\mathbb{Z}\}$, respectively. If $\psi^{\mathscr{C}}$ and $\widetilde{\psi}^{\mathscr{C}}$ are the associated special affine wavelets satisfying* (62). *Then, we have*

(i) $\left\{\widetilde{\psi}_{0,k}^{M,\mathscr{C}}:k\in\mathbb{Z}\right\}$ *is biorthogonal to* $\left\{\psi_{0,\ell}^{M,\mathscr{C}}:\ell\in\mathbb{Z}\right\}$;

(ii) $\left\langle\psi_{0,k}^{M,\mathscr{C}},\widetilde{\phi}_{0,\ell}^{M,\mathscr{C}}\right\rangle=\left\langle\widetilde{\psi}_{0,k}^{M,\mathscr{C}},\phi_{0,\ell}^{M,\mathscr{C}}\right\rangle,\ \forall\,k,\ell\in\mathbb{Z}.$

Proof. By virtue of induced function (51) and Theorem 4.2, the result follows. □

6. Conclusion

In the present study, we have accomplished two objectives. Firstly, we introduced the notion of special affine biorthogonal MRA and then studied the biorthogonal properties of the associated special affine wavelets in $L^2(\mathbb{R})$. Secondly, we accomplished the concept of special affine biorthogonal wavelets on a logarithmic regression curve $\mathscr{C}$ by formulating the notion of special affine Fourier transform in $L^2(\mathscr{C})$.

Conflict of interests:

The authors declare that they have no conflict of interest.

References

[1] L. Debnath, F. A. Shah, *Lecture Notes on Wavelet Transforms*, Birkhäuser, Boston, 2017.

[2] S. G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$*, Trans. Amer. Math. Soc. **315** (1989), 69–87.

[3] F. A. Shah, W. Z. Lone, H. Mejjaoli, *Nonuniform multiresolution analysis associated with linear canonical transform*, J. Pseudo-Differ. Oper. Appl. **12** (2021), 1–35.

[4] H. M. Srivastava, F. A. Shah, W. Z. Lone, *Fractional nonuniform multiresolution analysis in $L^2(\mathbb{R})$*, Math. Methods Appl. Sci. **44** (2021), 9351–9372.

[5] A. Cohen, I. Daubechies, J. C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Commun. Pure Appl. Math. **45** (1992), 485–560.

[6] C. K. Chui, J. Z. Wang, *On compactly supported spline wavelets and a duality principle*, Trans. Amer. Math. Soc. **330** (1992), 903–915.

[7] M. Bownik, G. Garrigos, *Biorthogonal wavelets, MRA's and shift-invariant spaces*, Studia. Math. **160** (2004), 231–248.

[8] S. Abe, J. T. Sheridan, *Generalization of the fractional Fourier transformation to an arbitrary linear lossless transformation: an operator approach*, J. Phys. **27** (1994), 4179–4187 .

[9] J. J. Healy, M. A. Kutay, Ozaktas, J. T. Sheridan, *Linear Canonical Transforms: Theory and Applications*, Springer, New York, 2016.

[10] A. Bhandari, A. I. Zayed, *Shift-invariant and sampling spaces associated with the special affine Fourier transform*, Appl. Comput. Harmon. Anal. **47** (2019), 30–52.

[11] F. A. Shah, A. A. Teali, A. Y. Tantary, *Special affine wavelet transform and the corresponding Poisson summation formula*, Int. J. Wavelets Multiresolution Inf. Process. **19** (2021), `https://doi.org/10.1142/S0219691320500861`.

[12] F. A. Shah, A. Y. Tantary, A. I. Zayed, *A convolution-based special affine wavelet transform, Integral Transforms*, Spec. Funct. **32** (2021), 780–800.

[13] H. M. Srivastava, R. Singh, S. K. Upadhyay, *The Bessel wavelet convolution involving the Hankel transformation*, J. Nonlinear Convex Anal. **23** (2022), 2649–2661.

[14] H. M. Srivastava, P. Shukla, S. K. Upadhyay, *The localization operator and wavelet multipliers involving the Watson transform*, J. Pseudo-Differ. Oper. Appl. **13** (2022), `https://doi.org/10.1007/s11868-022-00477-4`.

[15] H. M. Srivastava, F. A. Shah, A. A. Teali, *On quantum representation of the linear canonical wavelet transform*, Universe **8** (2022), `https://doi.org/10.3390/universe8090477`.

[16] H. M. Srivastava, K. K. Mishra, S. K. Upadhyay, *Characterizations of continuous fractional Bessel wavelet transforms*, Mathematics **10** (2022), `https://doi.org/10.3390/math10173084`.

[17] H. M. Srivastava, F. A. Shah, N. A. Nayied, *Fibonacci wavelet method for the solution of the non-Linear Hunter–Saxton equation*, Appl. Sci. **12** (2022), `https://doi.org/10.3390/app12157738`.

[18] H. M. Srivastava, F. A. Shah, *AB− wavelet frames in $L^2(\mathbb{R}^n)$*, Filomat 33 (2019), 3587–3597.

[19] J. N. Pandey, J. S. Maurya, S. K. Upadhyay, H. M. Srivastava, *Continuous wavelet transform of Schwartz tempered distributions in $S'(\mathbb{R}^n)$*, Symmetry **11** (2019), `https://doi.org/10.3390/sym11020235`.

[20] F. A. Shah, W. Z. Lone, *Special affine multiresolution analysis and the construction of orthonormal wavelets in $L^2(\mathbb{R})$*, Appl. Anal. (2022), `https://doi.org/10.1080/00036811.2022.2030723`.

[21] X. Zhou, G. Wang, *Biorthogonal wavelet on a logarithm curve* $\mathbb{C}$, J. Math. **2021** (2021), `https://doi.org/10.1155/2021/7985069`.

[22] D. Rosca, *Wavelet analysis on some surfaces of revolution via area preserving projection*, Appl. Comput. Harmon. Anal. 30 (2011), 262–272.

[23] Z. Zhou, *Wavelet transform on regression trend curve and its application in financial data*, Int. J. Wavelets Multiresolution Inf. Process. **18** (2020), `https://doi.org/10.1142/S021969132050040X`.