Filomat 37:19 (2023), 6319–6334
https://doi.org/10.2298/FIL2319319C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sharp inequalities related to the Adamović-Mitrinović, Cusa, Wilker and Huygens results

Chao-Ping Chen[a], Branko Malešević[b]

[a]*Henan Polytechnic University, School of Mathematics and Informatics, China*
[b]*University of Belgrade, School of Electrical Engineering, Serbia*

Abstract. In this paper, we establish sharp inequalities for trigonometric functions. For example, we consider the Wilker inequality and prove that for $0 < x < \pi/2$ and $n \geq 1$,

$$2 + \left(\sum_{j=2}^{n-1} d_{j+1}x^{2j} + \delta_n x^{2n}\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\sum_{j=3}^{n-1} d_{j+1}x^{2j} + D_n x^{2n}\right)x^3 \tan x$$

with the best possible constants

$$\delta_n = d_n \text{ and } D_n = \frac{2\pi^6 - 168\pi^4 + 15120}{945\pi^4}\left(\frac{2}{\pi}\right)^{2n} - \sum_{j=2}^{n-1} d_{j+1}\left(\frac{2}{\pi}\right)^{2n-2j},$$

where $d_k = 2^{2k+2}\Big((4k+6)\,|B_{2k+2}| + (-1)^{k+1}\Big)/(2k+3)!$ and B_k are the Bernoulli numbers ($k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$). This improves and generalizes the results given by Mortici, Nenezić and Malešević.

1. Introduction

It is known in the literature that

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} \tag{1}$$

for $0 < |x| < \pi/2$. The left-hand side inequality was obtained by Adamović and Mitrinović (see [22, p. 238]), while the right-hand side inequality was first mentioned by the German philosopher and theologian Nicolaus de Cusa (1401-1464), by a geometrical method. Huygens [14] gave a rigorous proof of the right-hand side inequality, and then used it to estimate the number π. The right-hand side inequality is now known as Cusa's inequality (see [23, 32, 37, 54]). Further interesting historical facts about the right-hand side inequality can be found in [37].

2020 *Mathematics Subject Classification*. Primary 42A10; Secondary 26D05, 26D15
Keywords. Inequalities; Double-sided Taylor's approximations; Bernoulli numbers
Received: 07 May 2022; Accepted: 13 February 2023
Communicated by Hari M. Srivastava
The first author was supported by Key Science Research Project in Universities of Henan (20B110007). The second author was financially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract number: 451-03-47/2023-01/200103.
Email addresses: chenchaoping@sohu.com (Chao-Ping Chen), malesevic@etf.rs (Branko Malešević)

The inequalities (1) have attracted much interest of many mathematicians and have motivated a large number of research papers; see, for example, [5–7, 12, 15, 23, 28, 29, 32, 33, 41, 48–51, 54] and the references cited therein.

By using inequalities involving Schwab-Borchardt mean, Neuman [29] presented the following inequality chain:

$$(\cos x)^{1/3} < \left(\cos x \frac{\sin x}{x}\right)^{1/4} < \left(\frac{\sin x}{\operatorname{arctanh}(\sin x)}\right)^{1/2} < \left(\frac{\cos x + (\sin x)/x}{2}\right)^{1/2} < \\ < \left(\frac{1+2\cos x}{3}\right)^{1/2} < \left(\frac{1+\cos x}{2}\right)^{2/3} < \frac{\sin x}{x}, \qquad 0 < x < \frac{\pi}{2}, \tag{2}$$

which improves the first inequality in (1). Yang [49] proved that for $0 < x < \pi/2$,

$$\frac{\sin x}{x} < \left(\frac{2}{3}\cos\frac{x}{2} + \frac{1}{3}\right)^2 < \cos^3\frac{x}{3} < \frac{2+\cos x}{3}, \tag{3}$$

which improves the second inequality in (1).

Motivated by (1), in Section 3 we establish sharp inequalities for trigonometric functions. By using the obtained results, we present inequality chain and improve the double inequality (1).

Wilker [39] proposed the following two open problems:

(a) Prove that if $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{4}$$

(b) Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{5}$$

for $0 < x < \pi/2$. In [38], the inequality (4) was proved, and the following inequality

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^3 \tan x, \qquad 0 < x < \frac{\pi}{2} \tag{6}$$

was also established, where the constants $(2/\pi)^4$ and $8/45$ are the best possible.

The Wilker-type inequalities (4) and (6) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs, various generalizations and improvements (cf. [4, 8, 9, 12, 13, 23–25, 27, 30–32, 34, 40, 41, 44, 45, 52–56] and the references cited therein).

A related inequality that is of interest to us is Huygens' inequality [14], which asserts that

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3, \qquad 0 < |x| < \frac{\pi}{2}. \tag{7}$$

Remark 1.1. *The first inequality in* (1) *can be re-written as*

$$\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x} > 1 \quad \left(\text{or} \quad \sqrt[3]{\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x}} > 1\right) \quad \text{for all} \quad 0 < |x| < \frac{\pi}{2}. \tag{8}$$

Baricz *and* Sándor [4] *have pointed out that inequality* (8) *implies* (4) *and* (7)*, by using the arithmetic-geometric mean inequality.*

Wu and Srivastava [44, Lemma 3] established Wilker-type inequality as follows:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \qquad 0 < |x| < \frac{\pi}{2}. \tag{9}$$

Neuman and Sándor [32, Theorem 2.3] proved that for $0 < |x| < \pi/2$,

$$\frac{\sin x}{x} < \frac{2+\cos x}{3} < \frac{1}{2}\left(\frac{x}{\sin x} + \cos x\right). \tag{10}$$

By multiplying both sides of inequality (10) by $x/\sin x$, we obtain that for $0 < |x| < \pi/2$,

$$\frac{1}{2}\left(\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right) > \frac{2(x/\sin x) + x/\tan x}{3} > 1. \tag{11}$$

Chen and Sándor [12] proved the following inequality chain:

$$\begin{aligned}&\frac{(\sin x/x)^2 + \tan x/x}{2} > \left(\tfrac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right) > \frac{2\,(\sin x/x) + \tan x/x}{3} > \\ &> \left(\frac{\sin x}{x}\right)^{2/3}\left(\frac{\tan x}{x}\right)^{1/3} > \frac{1}{2}\left(\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right) > \frac{2(x/\sin x) + x/\tan x}{3} > 1\end{aligned} \tag{12}$$

for $0 < |x| < \pi/2$.

In analogy with (6), Chen and Cheung [9] established sharp Wilker and Huygens-type inequalities. For example, these authors proved that for $0 < x < \pi/2$,

$$2 + \frac{8}{45}x^4 + \frac{16}{315}x^5\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^4 + \left(\frac{2}{\pi}\right)^6 x^5 \tan x, \tag{13}$$

where the constants $\frac{16}{315}$ and $(2/\pi)^6$ are best possible,

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^3\tan x, \tag{14}$$

where the constant $\frac{2}{45}$ is best possible, and

$$3 + \frac{3}{20}x^3\tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi}\right)^4 x^3\tan x, \tag{15}$$

where the constants $3/20$ and $(2/\pi)^4$ are best possible.

In view of (13), (14) and (15), Chen and Cheung [9] posed three conjectures. These conjectures have been proved by Chen and Paris [10, 11].

Mortici [24, Theorem 1] presented in 2014 the following double inequality:

$$\begin{aligned}&2 + \left(\frac{8}{45} - \frac{8}{945}x^2\right)x^3\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < \\ &< 2 + \left(\frac{8}{45} - \frac{8}{945}x^2 + \frac{16}{14175}x^4\right)x^3\tan x, \qquad 0 < x < 1.\end{aligned} \tag{16}$$

Nenezić et al. [25, Theorem 2.1] proved in 2016 that for $0 < x < \pi/2$,

$$\begin{aligned}&2 + \left(\frac{8}{45} - \frac{8}{945}x^2\right)x^3\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < \\ &< 2 + \left(\frac{8}{45} - \frac{8}{945}x^2 + \frac{241920 - 2688\pi^4 + 32\pi^6}{945\pi^8}x^4\right)x^3\tan x.\end{aligned} \tag{17}$$

By using power series expansions for $\sin x$ and $\cot x$, we find that

$$\begin{aligned}\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2}{x^3\tan x} &= \frac{\sin 2x}{2x^5}+\frac{1}{x^4}-\frac{2}{x^3}\cot x\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}}{(2n+1)!}x^{2n-4}+\frac{1}{x^4}-\frac{2}{x^3}\left(\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}\right)\\ &=\sum_{n=2}^{\infty}\frac{4^n\Big((-1)^n+2(2n+1)|B_{2n}|\Big)}{(2n+1)!}x^{2n-4}\\ &=\frac{8}{45}-\frac{8}{945}x^2+\frac{16}{14175}x^4+\frac{8}{467775}x^6+\frac{3184}{638512875}x^8\\ &\quad+\frac{272}{638512875}x^{10}+\frac{7264}{162820783125}x^{12}+\cdots, \end{aligned} \tag{18}$$

where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}, \qquad |z|<2\pi.$$

The formula (18) led us to claim that the upper bound in (16) should be the lower bound. Chen and Paris [11] proved that for $0 < x < \pi/2$,

$$\begin{aligned}2+\left(\frac{8}{45}-\frac{8}{945}x^2+\frac{16}{14175}x^4\right)x^3\tan x &< \left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<\\ <2+\left(\frac{8}{45}-\frac{8}{945}x^2+\frac{241920-2688\pi^4+32\pi^6}{945\pi^8}x^4\right)x^3\tan x,&\end{aligned} \tag{19}$$

where the constants $\frac{16}{14175}$ and $\frac{241920-2688\pi^4+32\pi^6}{945\pi^8}$ are the best possible.

In Section 4, we improve and generalize the double inequalities (19) and (15).

2. Taylor's approximations

Let us consider a real function $f:(a,b)\longrightarrow\mathbb{R}$ in case when exist finite limits

$$f^{(k)}(a+)=\lim_{x\to a+}f^{(k)}(x) \ \ (\text{for } k=0,1,\dots,n) \quad \text{and} \quad f(b-)=\lim_{x\to b-}f(x). \tag{20}$$

Then we consider first Taylor's polynomial

$$T_n^{f,a+}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a+)}{k!}(x-a)^k,\ n\in\mathbb{N}_0, \tag{21}$$

and the remainder

$$R_n^{f,a+}(x)=f(x)-T_{n-1}^{f,a+}(x). \tag{22}$$

Also, we consider the second Taylor's polynomial

$$\mathbb{T}_n^{f;a+,b-}(x)=\begin{cases} T_{n-1}^{f,a+}(x)+\dfrac{1}{(b-a)^n}R_n^{f,a+}(b-)(x-a)^n &, \quad n\geq 1\\ f(b-) &, \quad n=0.\end{cases} \tag{23}$$

The first TAYLOR's polynomial and the second TAYLOR's polynomial we are called the *first* TAYLOR*'s approximation for the function f in the right neighborhood of a*, and the *second* TAYLOR*'s approximation for the function f in the right neighborhood of a*, respectively.

The next Theorem on double-sided TAYLOR's approximations from [43] is applied in the papers [42], [45], [46], [47] and considered in the papers [16], [18], [19], [20], [21], [26], [35] and [36].

Theorem 2.1. ([43], Theorem 2) *Suppose that $f(x)$ is a real function on (a, b), and that n is a positive integer such that $f^{(k)}(a+)$, for $k \in \{0, 1, 2, \ldots, n\}$, exist.*

Supposing that $f^{(n)}(x)$ is increasing on (a, b), then for all $x \in (a, b)$ the following inequality also holds:

$$T_n^{f,a+}(x) < f(x) < \mathbb{T}_n^{f;a+,b-}(x). \tag{24}$$

Furthermore, if $f^{(n)}(x)$ is decreasing on (a, b), then the reversed inequality of (24) *holds.*

The condition for the application of this theorem refers to the n-th derivative of the function and it is also close to the recent papers which refer to the n-th derivative [57], [58], [59] and [60].

Remark 2.2. *In the previous inequality*

$$T_{n-1}^{f,a+}(x) + \frac{f^{(n)}(a+)}{n!}(x-a)^n < f(x) < T_{n-1}^{f,a+}(x) + \frac{1}{(b-a)^n}\left(f(b-) - T_{n-1}^{f,a+}(b-)\right)(x-a)^n, \tag{25}$$

the coefficients

$$\frac{f^{(n)}(a+)}{n!} \quad and \quad \frac{1}{(b-a)^n}\left(f(b-) - T_{n-1}^{f,a+}(b-)\right) \tag{26}$$

are the best possible constants.

In this paper we use

Theorem 2.3. ([20], Theorem 4) *Consider the real analytic functions $f : (a, b) \longrightarrow \mathbb{R}$:*

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k, \tag{27}$$

where $c_k \in \mathbb{R}$ and $c_k \geq 0$ for all $k \in \mathbb{N}_0$. Then,

$$T_0^{f,a+}(x) \leq \ldots \leq T_n^{f,a+}(x) \leq T_{n+1}^{f,a+}(x) \leq \ldots \leq f(x) \leq \ldots \leq \mathbb{T}_{m+1}^{f;a+,b-}(x) \leq \mathbb{T}_m^{f;a+,b-}(x) \leq \ldots \leq \mathbb{T}_0^{f;a+,b-}(x), \tag{28}$$

for all $x \in (a, b)$.

Elementary power series expansions. The following elementary power series expansions are useful in our investigation.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty, \tag{29}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad |x| < \infty, \tag{30}$$

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1}, \quad |x| < \frac{\pi}{2}, \tag{31}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-1}, \quad 0 < |x| < \pi, \tag{32}$$

$$\csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)|B_{2n}|}{(2n)!} x^{2n-1}, \quad |x| < \pi, \tag{33}$$

where B_n $(n = 0, 1, 2, \ldots)$ are Bernoulli numbers.

3. Sharp inequalities inspired by (1)

The first inequality in (1) is equivalent to

$$\frac{x}{\tan x} < \left(\frac{\sin x}{x}\right)^2, \qquad 0 < x < \frac{\pi}{2}. \tag{34}$$

Let us consider the following function with power series

$$\begin{aligned} f_1(x) = \left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x} &= \frac{1-\cos 2x}{2x^2} - x \cot x \\ &= \sum_{n=2}^{\infty} \left(\frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!} \right) x^{2n} \\ &= \frac{1}{15}x^4 - \frac{1}{945}x^6 + \frac{1}{2835}x^8 + \frac{8}{467775}x^{10} + \cdots \end{aligned} \tag{35}$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$a_n = \frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!}, \qquad n = 2, 3, 4, \ldots. \tag{36}$$

We use the next auxiliary statement.

Lemma 3.1. *The following are true*:

$$a_2 = \frac{1}{15} > 0, \; a_3 = -\frac{1}{945} < 0 \tag{37}$$

and

$$a_n = \frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!} > 0, \tag{38}$$

for integers $n \geq 4$.

Proof. By direct computation we obtained:

$$\begin{aligned} a_2 &= \frac{1}{15} = \lim_{x\to 0} \frac{f_1(x)}{x^4} > 0, \\ a_3 &= -\frac{1}{945} = \lim_{x\to 0} \frac{f_1(x) - \frac{1}{15}x^4}{x^6} < 0. \end{aligned} \tag{39}$$

Next, we consider the following inequalities [1, p. 805]

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \left(\frac{1}{1-2^{1-2n}} \right), \qquad n \geq 1. \tag{40}$$

Using the first inequality in (40), we obtain that for $n \geq 4$,

$$\frac{2^{2n}|B_{2n}|}{(2n)!} - \frac{2^{2n+1}}{(2n+2)!} > \frac{2^{2n}}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}} - \frac{2^{2n+1}}{(2n+2)!} = \frac{2^{2n+1}\Big((2n+2)! - (2\pi)^{2n}\Big)}{(2\pi)^{2n}\cdot(2n+2)!}.$$

By induction on n, it is easy to see that

$$(2n+2)! > (2\pi)^{2n}, \qquad n \geq 4.$$

Hence, we have

$$a_n = \frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!} > 0, \qquad n \geq 4. \qquad \square$$

Let's specify a list of Taylor's approximations

k	$T_k^{f_1,0+}(x)$	$\mathbb{T}_k^{f_1;0+,\pi/2-}(x)$
0	0	$\frac{4}{\pi^2}$
1	0	$\frac{8}{\pi^3}x$
2	0	$\frac{16}{\pi^4}x^2$
3	0	$\frac{32}{\pi^5}x^3$
4	$\frac{1}{15}x^4$	$\frac{64}{\pi^6}x^4$
5	$\frac{1}{15}x^4$	$T_4^{f_1,0+}(x)+\frac{-2\pi^6+1920}{15\pi^7}x^5$
6	$\frac{1}{15}x^4-\frac{1}{945}x^6$	$T_5^{f_1,0+}(x)+\frac{-4\pi^6+384}{15\pi^8}x^6$
7	$\frac{1}{15}x^4-\frac{1}{945}x^6$	$T_6^{f_1,0+}(x)+\frac{2\pi^8-504\pi^6+483840}{945\pi^9}x^7$
8	$\frac{1}{15}x^4-\frac{1}{945}x^6+\frac{1}{2835}x^8$	$T_7^{f_1,0+}(x)+\frac{4\pi^8-1008\pi^6+967680}{945\pi^{10}}x^8$
9	$\frac{1}{15}x^4-\frac{1}{945}x^6+\frac{1}{2835}x^8$	$T_8^{f_1,0+}(x)+\frac{-2\pi^{10}+24\pi^8-6048\pi^6+5806080}{945\pi^{11}}x^9$
10	$\frac{1}{15}x^4-\frac{1}{945}x^6+\frac{1}{2835}x^8+\frac{8}{467775}x^{10}$	$T_9^{f_1,0+}(x)+\frac{-4\pi^{10}+48\pi^8-12096\pi^6+11612160}{2835\pi^{12}}x^{10}$

Based on a method from [3] and [17] we have

Theorem 3.2. *For the function*

$$f_1(x) = \left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x} = \sum_{n=2}^{\infty}\left(\frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!}\right)x^{2n} : \left(0, \frac{\pi}{2}\right) \longrightarrow R$$

we have

$$\begin{aligned} &T_0^{f_1,0+}(x) = T_1^{f_1,0+}(x) = T_2^{f_1,0+}(x) = T_3^{f_1,0+}(x) = 0 < \\ &\quad < T_6^{f_1,0+}(x) = T_7^{f_1,0+}(x) < T_8^{f_1,0+}(x) = T_9^{f_1,0+}(x) < \\ &\qquad < T_{10}^{f_1,0+}(x) < f_1(x) < T_4^{f_1,0+}(x) = T_5^{f_1,0+}(x) \end{aligned}$$

and

$$\begin{aligned} &f_1(x) < \mathbb{T}_{10}^{f_1;0+,\pi/2-}(x) < \mathbb{T}_9^{f_1;0+,\pi/2-}(x) < \mathbb{T}_8^{f_1;0+,\pi/2-}(x) < \mathbb{T}_7^{f_1;0+,\pi/2-}(x) < \\ &\qquad < \mathbb{T}_4^{f_1;0+,\pi/2-}(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x) < \mathbb{T}_6^{f_1;0+,\pi/2-}(x) < \\ &\quad < \mathbb{T}_3^{f_1;0+,\pi/2-}(x) < \mathbb{T}_2^{f_1;0+,\pi/2-}(x) < \mathbb{T}_1^{f_1;0+,\pi/2-}(x) < \mathbb{T}_0^{f_1;0+,\pi/2-}(x), \end{aligned}$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Let us emphasize that some TAYLOR's approximatins $T_i^{f_1,0+}(x)$ and $\mathbb{T}_j^{f_1;0+,\pi/2-}(x)$ have intersections over interval $(0,\pi/2)=(0,1.570796326...)$ in exactly one point $c_{i,j}\in(0,\pi/2)$ for $i,j\in\{0,1,\ldots,10\}$. All that cases are given by the following two tables:

i,j	$f_i(x)<T_i^{f_1,0+}(x)<\mathbb{T}_j^{f_1;0+,\pi/2-}(x),\ x\in\left(0,c_{i,j}\right)$	$f_i(x)<\mathbb{T}_j^{f_1;0+,\pi/2-}(x)<T_i^{f_1,0+}(x), x\in\left(c_{i,j},\frac{\pi}{2}\right)$	$c_{i,j}$
$0,4$	$f_1(x)<T_0^{f_1,0+}(x)<\mathbb{T}_4^{f_1;0+,\pi/2-}(x),\ x\in(0,c_{0,4})$	$f_1(x)<\mathbb{T}_4^{f_1;0+,\pi/2-}(x)<T_0^{f_1,0+}(x), x\in\left(c_{0,4},\frac{\pi}{2}\right)$	1.570228574...
$0,5$	$f_1(x)<T_0^{f_1,0+}(x)<\mathbb{T}_5^{f_1;0+,\pi/2-}(x),\ x\in(0,c_{0,5})$	$f_1(x)<\mathbb{T}_5^{f_1;0+,\pi/2-}(x)<T_0^{f_1,0+}(x), x\in\left(c_{0,5},\frac{\pi}{2}\right)$	1.570228574...
$1,4$	$f_1(x)<T_1^{f_1,0+}(x)<\mathbb{T}_4^{f_1;0+,\pi/2-}(x),\ x\in(0,c_{1,4})$	$f_1(x)<\mathbb{T}_4^{f_1;0+,\pi/2-}(x)<T_1^{f_1,0+}(x), x\in\left(c_{1,4},\frac{\pi}{2}\right)$	1.570039369...
$1,5$	$f_1(x)<T_1^{f_1,0+}(x)<\mathbb{T}_5^{f_1;0+,\pi/2-}(x),\ x\in(0,c_{1,5})$	$f_1(x)<\mathbb{T}_5^{f_1;0+,\pi/2-}(x)<T_1^{f_1,0+}(x), x\in\left(c_{1,5},\frac{\pi}{2}\right)$	1.570039369...
$2,4$	$f_1(x)<T_2^{f_1,0+}(x)<\mathbb{T}_4^{f_1;0+,\pi/2-}(x),\ x\in(0,c_{2,4})$	$f_1(x)<\mathbb{T}_4^{f_1;0+,\pi/2-}(x)<T_2^{f_1,0+}(x), x\in\left(c_{2,4},\frac{\pi}{2}\right)$	1.569661027...
$2,5$	$f_1(x)<T_2^{f_1,0+}(x)<\mathbb{T}_5^{f_1;0+,\pi/2-}(x),\ x\in(0,c_{2,5})$	$f_1(x)<\mathbb{T}_5^{f_1;0+,\pi/2-}(x)<T_2^{f_1,0+}(x), x\in\left(c_{2,5},\frac{\pi}{2}\right)$	1.569661027...
$3,4$	$f_1(x)<T_3^{f_1,0+}(x)<\mathbb{T}_4^{f_1;0+,\pi/2-}(x),\ x\in(0,c_{3,4})$	$f_1(x)<\mathbb{T}_4^{f_1;0+,\pi/2-}(x)<T_3^{f_1,0+}(x), x\in\left(c_{3,4},\frac{\pi}{2}\right)$	1.568526547...
$3,5$	$f_1(x)<T_3^{f_1,0+}(x)<\mathbb{T}_5^{f_1;0+,\pi/2-}(x),\ x\in(0,c_{3,5})$	$f_1(x)<\mathbb{T}_5^{f_1;0+,\pi/2-}(x)<T_3^{f_1,0+}(x), x\in\left(c_{3,5},\frac{\pi}{2}\right)$	1.568526547...

and

i,j	$\mathbb{T}_j^{f_1;0+,\pi/2-}(x)<T_i^{f_1,0+}(x)<f_1(x),\ x\in\left(0,c_{i,j}\right)$	$T_i^{f_1,0+}(x)<\mathbb{T}_j^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{i,j},\frac{\pi}{2}\right)$	$c_{i,j}$
$4,6$	$\mathbb{T}_6^{f_1;0+,\pi/2-}(x)<T_4^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{4,6})$	$T_4^{f_1,0+}(x)<\mathbb{T}_6^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{4,6},\frac{\pi}{2}\right)$	0.3017187013...
$4,7$	$\mathbb{T}_7^{f_1;0+,\pi/2-}(x)<T_4^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{4,7})$	$T_4^{f_1,0+}(x)<\mathbb{T}_7^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{4,7},\frac{\pi}{2}\right)$	0.3017187013...
$4,8$	$\mathbb{T}_8^{f_1;0+,\pi/2-}(x)<T_4^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{4,8})$	$T_4^{f_1,0+}(x)<\mathbb{T}_8^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{4,8},\frac{\pi}{2}\right)$	0.3065585396...
$4,9$	$\mathbb{T}_9^{f_1;0+,\pi/2-}(x)<T_4^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{4,9})$	$T_4^{f_1,0+}(x)<\mathbb{T}_9^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{4,9},\frac{\pi}{2}\right)$	0.3065585396...
$4,10$	$\mathbb{T}_{10}^{f_1;0+,\pi/2-}(x)<T_4^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{4,10})$	$T_4^{f_1,0+}(x)<\mathbb{T}_{10}^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{4,10},\frac{\pi}{2}\right)$	0.3065818906...
$5,6$	$\mathbb{T}_6^{f_1;0+,\pi/2-}(x)<T_5^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{5,6})$	$T_5^{f_1,0+}(x)<\mathbb{T}_6^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{5,6},\frac{\pi}{2}\right)$	0.05795414341...
$5,7$	$\mathbb{T}_7^{f_1;0+,\pi/2-}(x)<T_5^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{5,7})$	$T_5^{f_1,0+}(x)<\mathbb{T}_7^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{5,7},\frac{\pi}{2}\right)$	0.05795414341...
$5,8$	$\mathbb{T}_8^{f_1;0+,\pi/2-}(x)<T_5^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{5,8})$	$T_5^{f_1,0+}(x)<\mathbb{T}_8^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{5,8},\frac{\pi}{2}\right)$	0.05801924550...
$5,9$	$\mathbb{T}_9^{f_1;0+,\pi/2-}(x)<T_5^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{5,9})$	$T_5^{f_1,0+}(x)<\mathbb{T}_9^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{5,9},\frac{\pi}{2}\right)$	0.05801924550...
$5,10$	$\mathbb{T}_{10}^{f_1;0+,\pi/2-}(x)<T_5^{f_1,0+}(x)<f_1(x),\ x\in(0,c_{5,10})$	$T_5^{f_1,0+}(x)<\mathbb{T}_{10}^{f_1;0+,\pi/2-}(x)<f_1(x),\ x\in\left(c_{5,10},\frac{\pi}{2}\right)$	0.05801925617...

All other TAYLOR's approximatins have no intersections.

Based on Theorem 2.3 we have

Theorem 3.3. *For the function*

$$f_1(x)=\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}=\sum_{n=2}^{\infty}\left(\frac{2^{2n}|B_{2n}|}{(2n)!}+\frac{(-1)^n 2^{2n+1}}{(2n+2)!}\right)x^{2n}:\left(0,\frac{\pi}{2}\right)\longrightarrow R$$

we have

$$T_6^{f_1,0+}(x)\le\ldots\le T_n^{f_1,0+}(x)\le T_{n+1}^{f_1,0+}(x)\le\ldots\le f_1(x)\le\ldots\le\mathbb{T}_{m+1}^{f_1;0+,\pi/2-}(x)\le\mathbb{T}_m^{f_1;0+,\pi/2-}(x)\le\ldots\le\mathbb{T}_7^{f_1;0+,\pi/2-}(x),$$

for all $x\in\left(0,\dfrac{\pi}{2}\right)$ *and* $n\ge6,\ m\ge7$.

Let us consider an empty sum as zero (elsewhere throughout this paper).

We propose the following conjecture.

Conjecture 3.4. *For $0 < x < \pi/2$ and $n \geq 2$, we have*

$$\sum_{j=2}^{n-1}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^{j}2^{2j+1}}{(2j+2)!}\right)x^{2j}+a_n x^{2n-1}\sin x<\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}<\sum_{j=2}^{n-1}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^{j}2^{2j+1}}{(2j+2)!}\right)x^{2j}+\Theta_n x^{2n-1}\sin x, \quad (41)$$

with the best possible constants

$$a_n = \frac{2^{2n}|B_{2n}|}{(2n)!}+\frac{(-1)^{n}2^{2n+1}}{(2n+2)!} \quad (42)$$

and

$$\Theta_n = \left(\frac{2}{\pi}\right)^{2n+1}-\sum_{j=2}^{n-1}a_j\left(\frac{2}{\pi}\right)^{2n-2j-1}. \quad (43)$$

Remark 3.5. *In fact, we can prove the first inequality in* (41)*. We then obtain for $0 < x < \pi/2$ and $n \geq 2$,*

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x} &> \sum_{j=2}^{n}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^{j}2^{2j+1}}{(2j+2)!}\right)x^{2j}\\ &= \sum_{j=2}^{n-1}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^{j}2^{2j+1}}{(2j+2)!}\right)x^{2j}+a_n x^{2n} \\ &> \sum_{j=2}^{n-1}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^{j}2^{2j+1}}{(2j+2)!}\right)x^{2j}+a_n x^{2n-1}\sin x.\end{aligned} \quad (44)$$

Hence, the first inequality in (41) *holds for all $n \geq 2$.*

Let us remark that the function $g_1(x) = f_1(x) - \frac{1}{15}x^4 + \frac{1}{945}x^6$ has power series with positive coefficients. Then, based on the previous Theorem we have:

Statement 3.6. *For $0 < x < \pi/2$ and $n \geq 2$,*

$$\sum_{j=2}^{n-1}a_j x^{2j}+\alpha_n x^{2n}<\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}<\sum_{j=2}^{n-1}a_j x^{2j}+A_n x^{2n} \quad (45)$$

with the best possible constants

$$\alpha_n = a_n \quad \text{and} \quad A_n = \left(\frac{2}{\pi}\right)^{2n+2}-\sum_{j=2}^{n-1}a_j\left(\frac{2}{\pi}\right)^{2n-2j}. \quad (46)$$

Next, we consider the following function

$$\begin{aligned} f_2(x) &= \frac{\left(\dfrac{\sin x}{x}\right)^2 - \dfrac{x}{\tan x}}{x^3 \sin x} \\ &= \frac{1}{x^5}\sin x + \frac{1}{x^2}\left(-\frac{\cos x}{\sin^2 x}\right) \\ &= \frac{1}{x^5}\sin x + \frac{1}{x^2}(\csc x)' \\ &= \sum_{n=2}^{\infty} \frac{2(2n-1)(2n+1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!} x^{2n-4} \qquad \text{(see (29), (33))} \\ &= \frac{1}{15} + \frac{19}{1890}x^2 + \frac{167}{113400}x^4 + \frac{479}{2494800}x^6 + \ldots \end{aligned} \tag{47}$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$b_n = \frac{2(4n^2-1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!}, \qquad n = 2, 3, 4, \ldots. \tag{48}$$

We use the next auxiliary statement.

Lemma 3.7. *The following are true*:

$$b_n = \frac{2(2n-1)(2n+1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!} > 0, \tag{49}$$

for integers $n \geq 2$.

Proof. Using the first inequality in (40), we obtain that for $n \geq 2$,

$$2(2n-1)(2n+1)(2^{2n-1}-1)|B_{2n}| > \frac{4(2n-1)(2^{2n-1}-1)\cdot(2n+1)!}{(2\pi)^{2n}} > 1 \tag{50}$$

(The second inequality in (50) can be shown by induction on n, we omit it), which implies

$$b_n > 0, \qquad n \geq 2.$$

□

Let's specify a list of Taylor's approximations for the function $f_2(x)$ over interval $(0, \pi/2)$:

k	$T_k^{f_2,0+}(x)$	$\mathbb{T}_k^{f_2;0+,\pi/2-}(x)$
0	$\frac{1}{15}$	$\frac{32}{\pi^5}$
1	$\frac{1}{15}$	$\frac{1}{15} + \frac{-2\pi^5+960}{15\pi^6}x$
2	$\frac{1}{15} + \frac{19}{1890}x^2$	$\frac{1}{15} + \frac{-4\pi^5+1920}{15\pi^7}x^2$
3	$\frac{1}{15} + \frac{19}{1890}x^2$	$\frac{1}{15} + \frac{19}{1890}x^2 + \frac{-19\pi^7-504\pi^5+241920}{945\pi^8}x^3$
4	$\frac{1}{15} + \frac{19}{1890}x^2 + \frac{167}{113400}x^4$	$\frac{1}{15} + \frac{19}{1890}x^2 + \frac{-38\pi^7-1008\pi^5+483840}{945\pi^9}x^4$

Based on Theorem 2.3 we have

Theorem 3.8. *For the function*

$$f_2(x) = \frac{\left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x}}{x^3 \sin x}$$
$$= \sum_{n=2}^{\infty} \left(\frac{2(2n-1)(2n+1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!} \right) x^{2n-4} : \left(0, \frac{\pi}{2}\right) \longrightarrow R$$

we have

$$T_0^{f_2,0+}(x) \leq \ldots \leq T_k^{f_2,0+}(x) \leq T_{k+1}^{f_2,0+}(x) \leq \ldots \leq f_2(x) \leq \ldots \leq \mathbb{T}_{k+1}^{f_2;0+,\pi/2-}(x) \leq \mathbb{T}_k^{f_2;0+,\pi/2-}(x) \leq \ldots \leq \mathbb{T}_0^{f_2;0+,\pi/2-}(x),$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Then, based on the previous Theorem we have

Statement 3.9. *For* $0 < x < \pi/2$ *and* $n \geq 0$,

$$\left(\sum_{j=0}^{n-1} b_{j+2} x^{2j} + \beta_n x^{2n} \right) x^3 \sin x < \left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x} < \left(\sum_{j=0}^{n-1} b_{j+2} x^{2j} + B_n x^{2n} \right) x^3 \sin x \tag{51}$$

with the best possible constants

$$\beta_n = b_{n+2} \quad and \quad B_n = \left(\frac{2}{\pi}\right)^{2n+5} - \sum_{j=2}^{n-1} b_{j+2} \left(\frac{2}{\pi}\right)^{2n-2j}. \tag{52}$$

Finaly, we consider the following function

$$f_3(x) = \frac{\frac{2+\cos x}{3} - \frac{\sin x}{x}}{x^3 \sin x}$$
$$= \frac{2}{3x^3} \csc x + \frac{1}{3x^3} \cot x \frac{1}{x^4}$$
$$= \sum_{n=2}^{\infty} \left(\frac{2^{2n}-4}{3\cdot(2n)!} |B_{2n}| \right) x^{2n-4} \qquad \text{(see (30), (33))}$$
$$= \frac{1}{180} + \frac{1}{1512} x^2 + \frac{1}{14400} x^4 + \frac{17}{2395008} x^6 + \ldots \tag{53}$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$c_n = \frac{2^{2n}-4}{3\cdot(2n)!} |B_{2n}|, \qquad n = 0, 1, 2, \ldots. \tag{54}$$

The next auxiliary statement is obvious.

Lemma 3.10. *The following are true*:

$$c_n = \frac{2^{2n}-4}{3\cdot(2n)!} |B_{2n}| > 0, \tag{55}$$

for integers $n \geq 0$.

Let's specify a list of TAYLOR's approximations for the function $f_3(x)$ over interval $(0, \pi/2)$:

k	$T_k^{f_3,0+}(x)$	$\mathbb{T}_k^{f_3;0+,\pi/2-}(x)$
0	$\frac{1}{180}$	$\frac{-48+16\pi}{3\pi^4}$
1	$\frac{1}{180}$	$\frac{1}{180}+\frac{-\pi^4+960\pi-2880}{90\pi^5}x$
2	$\frac{1}{180}+\frac{1}{1512}x^2$	$\frac{1}{180}+\frac{-\pi^4+960\pi-2880}{45\pi^6}x^2$
3	$\frac{1}{180}+\frac{1}{1512}x^2$	$\frac{1}{180}+\frac{1}{1512}x^2+\frac{-5\pi^6-168\pi^4+161280\pi-483840}{3870\pi^7}x^3$
4	$\frac{1}{180}+\frac{1}{1512}x^2+\frac{1}{14400}x^4$	$\frac{1}{180}+\frac{1}{1512}x^2+\frac{-5\pi^6-168\pi^4+161280\pi-483840}{1890\pi^8}x^4$

Based on Theorem 2.3 we have

Theorem 3.11. *For the function*

$$f_3(x) = \frac{\frac{2+\cos x}{3} - \frac{\sin x}{x}}{x^3 \sin x} = \sum_{n=2}^{\infty} \left(\frac{2^{2n}-4}{3\cdot(2n)!} |B_{2n}| \right) x^{2n-4} : \left(0, \frac{\pi}{2}\right) \longrightarrow R$$

we have

$$T_0^{f_3,0+}(x) \le \ldots \le T_k^{f_3,0+}(x) \le T_{k+1}^{f_3,0+}(x) \le \ldots \le f_3(x) \le \ldots \le \mathbb{T}_{k+1}^{f_3;0+,\pi/2-}(x) \le \mathbb{T}_k^{f_3;0+,\pi/2-}(x) \le \ldots \le \mathbb{T}_0^{f_3;0+,\pi/2-}(x),$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Then, based on the previous Theorem we have

Statement 3.12. *For* $0 < x < \pi/2$ *and* $n \ge 0$,

$$\left(\sum_{j=0}^{n-1} c_{j+2} x^{2j} + \gamma_n x^{2n} \right) x^3 \sin x < \frac{2+\cos x}{3} - \frac{\sin x}{x} < \left(\sum_{j=0}^{n-1} c_{j+2} x^{2j} + C_n x^{2n} \right) x^3 \sin x \tag{56}$$

with the best possible constants

$$\gamma_n = c_n \quad \text{and} \quad C_n = \frac{\pi-3}{3} \left(\frac{2}{\pi}\right)^{2n+4} - \sum_{j=2}^{n-1} c_{j+2} \left(\frac{2}{\pi}\right)^{2n-2j}. \tag{57}$$

4. Sharp Wilker and Huygens inequalities

In purpose to generalize of the double inequality (19) we consider the following function

$$\begin{aligned} f_4(x) &= \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x} - \frac{8}{45} + \frac{8}{945}x^2 \\ &= \frac{1}{x^4} + \frac{\sin 2x}{2x^5} - \frac{2\cot x}{x^3} - \frac{8}{45} + \frac{8}{945}x^2 \\ &= \frac{f(x)}{x^3} : \left(0, \frac{\pi}{2}\right) \longrightarrow R, \end{aligned} \tag{58}$$

where the function

$$f(x) = \frac{1}{x} + \frac{\sin 2x}{2x^2} - 2\cot x - \frac{8}{45}x^3 + \frac{8}{945}x^5 : \left(0, \frac{\pi}{2}\right) \longrightarrow R \tag{59}$$

is considered in the paper [35]. Therefore

$$\begin{aligned} f_4(x) &= \sum_{n=3}^{\infty} \frac{2^{2n+2}\left((4n+6)|B_{2n+2}| + (-1)^{n+1}\right)}{(2n+3)!} x^{2n-2} \\ &= \frac{16}{14175}x^4 + \frac{8}{467775}x^6 + \frac{3184}{638512875}x^8 + \frac{272}{638512875}x^{10} + \dots \end{aligned} \tag{60}$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$d_n = \frac{2^{2n+2}\left((4n+6)|B_{2n+2}| + (-1)^{n+1}\right)}{(2n+3)!}, \qquad n = 3, 4, 5, \dots. \tag{61}$$

The next auxiliary statement is obvious.

Lemma 4.1. *The following are true*:

$$d_n = \frac{2^{2n+2}\left((4n+6)|B_{2n+2}| + (-1)^{n+1}\right)}{(2n+3)!} > 0, \tag{62}$$

for integers $n \geq 3$.

Let's specify a list of Taylor's approximations for the function $f_4(x)$ over interval $(0, \pi/2)$:

k	$T_k^{f_4, 0+}(x)$	$\mathbb{T}_k^{f_4; 0+, \pi/2-}(x)$
0	0	$\frac{2\pi^6 - 168\pi^4 + 15120}{945\pi^4}$
1	0	$\frac{4\pi^6 - 336\pi^4 + 30240}{945\pi^5}x$
2	0	$\frac{8\pi^6 - 672\pi^4 + 60480}{945\pi^6}x^2$
3	0	$\frac{16\pi^6 - 1344\pi^4 + 120960}{945\pi^7}x^3$
4	$\frac{16}{14175}x^4$	$\frac{32\pi^6 - 2688\pi^4 + 241920}{945\pi^8}x^4$

Based on Theorem 2.3 we have

Theorem 4.2. *For the function*

$$\begin{aligned} f_4(x) &= \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x} - \frac{8}{45} + \frac{8}{945}x^2 \\ &= \sum_{n=3}^{\infty} \left(\frac{2^{2n+2}\left((4n+6)|B_{2n+2}| + (-1)^{n+1}\right)}{(2n+3)!} \right) x^{2n-2} : \left(0, \frac{\pi}{2}\right) \longrightarrow R \end{aligned}$$

we have

$$T_0^{f_4,0+}(x) \leq \dots \leq T_k^{f_4,0+}(x) \leq T_{k+1}^{f_4,0+}(x) \leq \dots \leq f_4(x) \leq \dots \leq \mathbb{T}_{k+1}^{f_4;0+,\pi/2-}(x) \leq \mathbb{T}_k^{f_4;0+,\pi/2-}(x) \leq \dots \leq \mathbb{T}_0^{f_4;0+,\pi/2-}(x),$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Let us remark that the function $g_4(x) = f_4(x) - \dfrac{8}{45} + \dfrac{8}{945}x^2$ has power series with positive coefficients. Then, based on the previous Theorem we have:

Statement 4.3. *For* $0 < x < \pi/2$ *and* $n \geq 4$,

$$2 + \left(\sum_{j=2}^{n-1} d_{j+1}x^{2j} + \delta_n x^{2n}\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\sum_{j=2}^{n-1} d_{j+1}x^{2j} + D_n x^{2n}\right)x^3 \tan x \tag{63}$$

with the best possible constants

$$\delta_n = d_n \quad and \quad D_n = \frac{2\pi^6 - 168\pi^4 + 15120}{945\pi^4}\left(\frac{2}{\pi}\right)^{2n} - \sum_{j=2}^{n-1} d_{j+1}\left(\frac{2}{\pi}\right)^{2n-2j}. \tag{64}$$

Finaly, we consider the following function

$$\begin{aligned} f_5(x) &= \frac{2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} - 3}{x^3 \tan x} \\ &= \frac{2}{x^4}\cos x + \frac{1}{x^4} - \frac{3}{x^3}\cot x \\ &= \sum_{n=2}^{\infty} 2\frac{(-1)^n + 3 \cdot 2^{2n+3}|B_{2n+4}|}{(2n+4)!}x^{2n} \qquad \text{(see (30), (32))} \\ &= \frac{3}{20} + \frac{1}{280}x^2 + \frac{23}{33600}x^4 + \frac{47}{739200}x^6 + \ldots \end{aligned} \tag{65}$$

over interval $\left(0, \dfrac{\pi}{2}\right)$. Let us denote

$$e_n = 2\frac{3 \cdot 2^{2n+3}|B_{2n+4}| + (-1)^n}{(2n+4)!} \qquad n = 0, 1, 2, \ldots. \tag{66}$$

The next auxiliary statement is obvious.

Lemma 4.4. *The following are true*:

$$e_n = 2\frac{3 \cdot 2^{2n+3}|B_{2n+4}| + (-1)^n}{(2n+4)!} > 0, \qquad n = 0, 1, 2, \ldots, \tag{67}$$

for integers $n \geq 0$.

Let's specify a list of Taylor's approximations for the function $f_3(x)$ over interval $(0, \pi/2)$:

k	$T_k^{f_5, 0+}(x)$	$\mathbb{T}_k^{f_5; 0+, \pi/2-}(x)$
0	$\frac{3}{20}$	$\frac{16}{\pi^4}$
1	$\frac{3}{20}$	$\frac{3}{20} + \frac{-3\pi^4 + 320}{10\pi^5}x$
2	$\frac{3}{20} + \frac{1}{280}x^2$	$\frac{3}{20} + \frac{-3\pi^4 + 320}{5\pi^6}x^2$
3	$\frac{3}{20} + \frac{1}{280}x^2$	$\frac{3}{20} + \frac{1}{280}x^2 + \frac{-\pi^6 - 168\pi^4 + 17920}{140\pi^7}x^3$
4	$\frac{3}{20} + \frac{1}{280}x^2 + \frac{23}{33600}x^4$	$\frac{3}{20} + \frac{1}{280}x^2 + \frac{-\pi^6 - 168\pi^4 + 17920}{70\pi^8}x^4$

Based on Theorem 2.3 we have

Theorem 4.5. *For the function*

$$f_5(x) = \frac{2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} - 3}{x^3 \tan x} = \sum_{n=2}^{\infty} 2\frac{(-1)^n + 3\cdot 2^{2n+3}|B_{2n+4}|}{(2n+4)!}\, x^{2n} : \left(0, \frac{\pi}{2}\right) \longrightarrow R$$

we have

$$T_0^{f_5,0+}(x) \le \ldots \le T_k^{f_5,0+}(x) \le T_{k+1}^{f_5,0+}(x) \le \ldots \le f_5(x) \le \ldots \le \mathbb{T}_{k+1}^{f_5;0+,\pi/2-}(x) \le \mathbb{T}_k^{f_5;0+,\pi/2-}(x) \le \ldots \le \mathbb{T}_0^{f_5;0+,\pi/2-}(x),$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Then, based on the previous Theorem we have

Statement 4.6. *For* $0 < x < \pi/2$ *and* $n \ge 0$,

$$3 + \left(\sum_{j=2}^{n-1} e_j x^{2j} + \eta_n x^{2n}\right) x^3 \tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\sum_{j=2}^{n-1} e_j x^{2j} + E_n x^{2n}\right) x^3 \tan x \tag{68}$$

with the best possible constants

$$\eta_n = e_n \quad \text{and} \quad E_n = \left(\frac{2}{\pi}\right)^{2n+4} - \sum_{j=2}^{n-1} e_j \left(\frac{2}{\pi}\right)^{2n-2j}. \tag{69}$$

References

[1] M. Abramowitz, I.A. Stegun (eds): *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series 55, Ninth printing, National Bureau of Standards; Washington D.C. 1972.

[2] H. Alzer: *Sharp bounds for the Bernoulli numbers*, Arch. Math. (Basel) 2000, 74:3, 207–211.

[3] B.D. Banjac: *System for automatic proving of some classes of analytic inequalities*, Doctoral dissertation (in Serbian), School of Electrical Engineering, Belgrade, May 2019. Available on: `http://nardus.mpn.gov.rs`

[4] A. Baricz, J. Sándor: *Extensions of generalized Wilker inequality to Bessel functions*, J. Math. Inequal. 2008, 2, 397–406.

[5] G. Bercu: *Padé approximant related to remarkable inequalities involving trigonometric functions*, J. Inequal. Appl. 2016, 99.

[6] B.A. Bhayo, J. Sándor: *On Jordan's, Redheffer's and Wilker's inequality*, Math. Inequal. Appl. 2016, 19:3, 823–839.

[7] C.-P. Chen, W.-S. Cheung: *Sharp Cusa and Becker-Stark inequalities*, J. Inequal. Appl. 2011, 136.

[8] C.-P. Chen, W.-S. Cheung: *Inequalities and solution to Oppenheim's problem*, Integral Transforms Spec. Funct. 2012, 23:5, 325–336.

[9] C.-P. Chen, W.-S. Cheung: *Sharpness of Wilker and Huygens type inequalities*, J. Inequal. Appl. 2012, 72.

[10] C.-P. Chen, R.B. Paris: *Series representations of the remainders in the expansions for certain trigonometric and hyperbolic functions with applications*, Math. Ineq. App. 2017, 20:4, 1003–1016.

[11] C.-P, Chen, R.B. Paris: *On the Wilker and Huygens-type inequalities*, J. Math. Inequal. 2020, 14:3, 685–705.

[12] C.-P. Chen, J. Sándor: *Inequality chains for Wilker, Huygens and Lazarević type inequalities*, J. Math. Inequal. 2014, 8:1, 55–67.

[13] B.-N. Guo, B.-M. Qiao, F. Qi, W. Li: *On new proofs of Wilker inequalities involving trigonometric functions*, Math. Inequal. Appl. 2003, 6, 19–22.

[14] C. Huygens : *Oeuvres Completes 1888-1940*, Société Hollondaise des Science, Haga.

[15] R. Klén, M. Visuri, M. Vuorinen: *On Jordan type inequalities for hyperbolic functions*, J. Inequal. Appl. 2010 Article Id. 362548.

[16] T. Lutovac, B. Malešević, M. Rašajski. *A new method for proving some inequalities related to several special functions*, Results Math. 2018, 73:100, 1–15.

[17] B. Malešević, M. Makragić: *A Method for Proving Some Inequalities on Mixed Trigonometric Polynomial Functions*, J. Math. Inequal. 10:3 (2016), 849–876.

[18] B. Malešević, T. Lutovac, M. Rašajski, C. Mortici: *Extensions of the natural approach to refinements and generalizations of some trigonometric inequalities*, Adv. Difference Equ. 2018, 90, 1–15.

[19] B. Malešević, M. Nenezić, L. Zhu, B. Banjac, M. Petrović: *Some new estimates of precision of Cusa-Huygens and huygens approximations*, Appl. Anal. Discrete Math. 2021, 15:1, 243–259.

[20] B. Malešević, M. Rašajski and T. Lutovac: *Double-sided Taylor's approximations and their applications in Theory of analytic inequalities*, Chapter in Andrica D, Rassias T (eds): *Differential and Integral Inequalities*, Springer Optimization and Its Applications, Vol. 151, pp. 569–582; Springer 2019.

[21] B. Malešević, T. Lutovac, M. Rašajski, B. Banjac: *Double-sided Taylor's approximations and their applications in Theory of analytic inequalities*, Chapter in Rassias M.T, Raigorodskii A (eds): *Trigonometric Sums and their Applications*, pp. 159–167; Springer 2020.
[22] D.S Mitrinović: *Analytic inequalities*; Springer-Verlag, Berlin, 1970.
[23] C. Mortici: *The natural approach of Wilker-Cusa-Huygens inequalities*, Math. Inequal. Appl. 2011, 14, 535–541.
[24] C. Mortici: *A subtly analysis of Wilker inequality*, Appl. Math. Comput. 2014, 231, 516–520.
[25] M. Nenezić, B. Malešević, C. Mortici: *New approximations of some expressions involving trigonometric functions*, Appl. Math. Comput. 283 (2016), 299–315.
[26] M. Nenezić, L. Zhu: *Some improvements of Jordan-Steckin and Becker-Stark inequalities*, Appl. Anal. Discrete Math. 201), 12:1, 244–256.
[27] E. Neuman: *One-and two-sided inequalities for Jacobian elliptic functions and related results*, Integral Transforms Spec. Funct. 2010, 21:6, 399–407.
[28] E. Neuman: *Inequalities for the Schwab-Borchardt mean and their applications*, J. Math. Inequal. 2011, 5:4, 601–609.
[29] E. Neuman: *Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions*, Adv. Inequal. Appl. 2012, 1:1, 1–11.
[30] E. Neuman: *On Wilker and Huygens type inequalities*, Math. Inequal. Appl. 2012, 15:2, 271–279.
[31] E. Neuman: *Wilker and Huygens-type inequalities for the generalized trigonometric and for the generalized hyperbolic functions*, Appl. Math. Comput. 2014, 230, 211–217.
[32] E. Neuman, J. Sándor: *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities*, Math. Inequal. Appl. 2010, 13:4, 715–723.
[33] Y. Nishizawa: *Sharp exponential approximate inequalities for trigonometric functions*, Results. Math. 2016, 71:3-4, 609–621.
[34] I. Pinelis: *L'Hospital rules of monotonicity and Wilker-Anglesio inequality*, Amer. Math. Monthly 2004, 111, 905–909.
[35] M. Rašajski, T. Lutovac, B. Malešević: *Sharpening and generalizations of Shafer-Fink and Wilker type inequalities: a new approach*, J. Nonlinear Sci. Appl. 2018, 11:7, 885–893.
[36] M. Rašajski, T. Lutovac, B. Malešević: *About some exponential inequalities related to the sinc function*, J. Inequal. Appl. 2018, 150, 1–10.
[37] J. Sándor and M. Bencze, *On Huygens' trigonometric inequality*, RGMIA Res. Rep. Collection 2005, 8:3, Article Id. 14.
[38] J.S. Sumner, A.A. Jagers, M. Vowe, J. Anglesio: *Inequalities involving trigonometric functions*, Amer. Math. Monthly 1991, 98, 264–267.
[39] J.B. Wilker: *Problem E 3306*, Amer. Math. Monthly 1989, 96, 55.
[40] S.-H. Wu: *On extension and refinement of Wilker's inequality*, Rocky Mountain J. Math. 2009, 39, 683–687.
[41] S.-H. Wu, A. Báricz: *Generalizations of Mitrinović, Adamović and Lazarević inequalities and their applications*, Publ. Math. Debrecen 2009, 75:(3-4), 447–458.
[42] S.-H. Wu, L. Debnath: *Jordan-type inequalities for differentiable functions and their applications*, Appl. Math. Lett. 2008, 21:8, 803–809.
[43] S.-H. Wu, L. Debnath: *A generalization of L'Hospital-type rules for monotonicity and its application*, Appl. Math. Lett. 2009, 22:2, 284–290.
[44] S.-H. Wu, H.M. Srivastava: *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transforms and Spec. Funct. 2007:8, 18, 529–535.
[45] S.-H. Wu, H.M. Srivastava: *A further refinement of Wilker's inequality*, Integral Transforms Spec. Funct. 2008, 19:(9-10), 757–765.
[46] S.-H. Wu, H.M. Srivastva: *A further refinement of a Jordan type inequality and its applications*, Appl. Math. Comput. 2008, 197, 914–923.
[47] S.-H. Wu, H.M. Srivastava, L. Debnath: *Some refined families of Jordan-type inequalities and their applications*, Integral Transforms Spec. Funct. 2008, 19:3, 183–193.
[48] Z.-H. Yang: *Renements of a two-sided inequality for trigonometric functions*, J. Math. Inequal. 2013, 7:4, 601–615.
[49] Z.-H. Yang: *Three families of two-parameter means constructed by trigonometric functions*, J. Inequal. Appl. 2013, Article Id. 541.
[50] Z.-H. Yang: *New sharp Jordan type inequalities and their applications*, Gulf J. Math. 2014, 2:1, 1–10.
[51] Z.-H. Yang, Y.-M. Chu, X.-H. Zhang: *Sharp Cusa type inequalities with two parameters and their applications*, Appl. Math. Comput. 2015, 268, 1177–1198.
[52] L. Zhang, L. Zhu: *A new elementary proof of Wilker's inequalities*, Math. Inequal. Appl. 2008, 11, 149–151.
[53] L. Zhu: *A new simple proof of Wilker's inequality*, Math. Inequal. Appl. 2005, 8), 749–750.
[54] L. Zhu: *A source of inequalities for circular functions*, Comput. Math. Appl. 2009, 58:10, 1998–2004.
[55] L. Zhu: *Some new Wilker-type inequalities for circular and hyperbolic functions*, Abstr. Appl. Anal. 2009, Article Id. 485842.
[56] L. Zhu: *Inequalities for Hyperbolic functions and their Applications*, J. Inequal. Appl. 2010, Article Id. 130821.
[57] C.-P. Chen, H.M. Srivastava: *Some inequalities and monotonicity properties associated with the gamma and psi functions and the Barnes G-function*, Integral Transforms Spec. Funct. 2011, 22:1, 1–15.
[58] F. Qi, P. Cerone, S.S. Dragomir, H.M. Srivastava: *Alternative proofs for monotonic and logarithmically convex properties of one-parameter mean values* Appl. Math. Comput. 2009, 208, 129-133.
[59] G. Chen, J. Liang, H.M. Srivastava, C. Lv: *Local Fractional Integral Hölder-Type Inequalities and Some Related Results*, Fractal Fract. 2022, 6:4, Art. no. 195, 1–16.
[60] Y. Bagul, C. Chesneau, M. Kostic, T. Lutovac, B. Malesevic, M. Rasajski, *Convexity and Double-sided Taylor's approximations*, Hacet. J. Math. Stat. (accepted: 28.09.2022), 1–12, `https://doi.org/10.15672/hujms.1096357`