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# Stabilities of multiplicative inverse quadratic functional equations arising from Pythagorean means

**B.V. Senthil Kumar[a], Hemen Dutta[b], G. Shanmugam[a], N. Balamurugan[a]**

[a]*Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai - 602 105, Tamil Nadu, India*
[b]*Department of Mathematics, Gauhati University, Guwahati - 781 014, Assam, India*

**Abstract.** In this article, we propose advanced multiplicative inverse quadratic functional equations involving arguments in rational form. The motivation to introduce these functional equations is due to inverse square law arising in gravity, electricity and radiation. We obtain their solutions and prove their stabilities in the restricted domain and non-Archimedean fields. Moreover, we provide a suitable counter-example for the failure of stability result when critical case arises. We elucidate these equations with Pythagorean means.

## 1. Introduction

The group theory has vast applications in various domains of computer science and electronics. Thanks to the query laid down in [27] concerning the stability of mathematical equations associated with group theory. A foremost answer is presented in [15] to the query raised in [27] in the framework of Banach spaces. The stability result provided in [15] is called as Ulam-Hyers stability and the method introduced to obtain this result is termed as direct method. Further, this stability problem is solved by many mathematicians and to extend the results by considering different upper bounds [3, 14, 21, 22]. Moreover, the stability results of various equations in different modern normed spaces are available in [2, 5, 10, 11, 16]. The Hyers-Ulam stability problem is first solved on restricted domains in [9].

The results on the approximation operators concerning a family of discrete probability distributions are presented in [1]. The stabilities of functional-integral equations in the setting of Banach algebra, the stability of functional-integral equations via fixed point approach and the stabilities of Jensen's functional equation in multi-normed spaces are proved in ([18, 20, 26]).

Recently, the stability problems for various rational functional equations and their applications are discussed in [4, 7, 8, 12, 23–25].

In this paper, we introduce the following multiplicative inverse quadratic functional equation

$$h\left(\frac{pq}{p+q}\right) = h(p) + h(q) + 2\sqrt{h(p)h(q)}, \tag{1}$$

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*Email addresses:* bvskumarmaths@gmail.com (B.V. Senthil Kumar), hemen_dutta08@rediffmail.com (Hemen Dutta), gsm.maths@gmail.com (G. Shanmugam), knbmurugan@gmail.com (N. Balamurugan)

multiplicative inverse quadratic adjoint functional equation

$$h\left(\frac{\mathfrak{p}\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}}\right)+h\left(\frac{2\mathfrak{p}\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}}\right)=\frac{5}{4}[h(\mathfrak{p})+h(\mathfrak{q})+2\sqrt{h(\mathfrak{p})h(\mathfrak{q})}] \tag{2}$$

and multiplicative inverse quadratic difference functional equation

$$h\left(\frac{\mathfrak{p}\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}}\right)-h\left(\frac{2\mathfrak{p}\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}}\right)=\frac{3}{4}[h(\mathfrak{p})+h(\mathfrak{q})+2\sqrt{h(\mathfrak{p})h(\mathfrak{q})}]. \tag{3}$$

We prove that equations (1), (2) and (3) are equivalent. We determine the stabilities of equations (1), (2) and (3) in the restricted domain and non-Archimedean fields. An appropriate counter-example is provided to show that the stability result fails for a singular case. We provide the relationship of these equations with Pythagorean means.

## 2. Motivation to introduce equations (1), (2) and (3)

In this section, we provide the motivation to focus on the equations (1), (2) and (3) in this study. The solutions of equations (1), (2) and (3) imply that these equations can be associated with various inverse square laws involving gravity (Newton's law), electricity (Coulomb's law) and radiation.

According to Newton's law, the force of attraction between two bodies is a function of multiplicative inverse square of the distance between them. As per Coulomb's law, the force of attraction or repulsion is a function of multiplicative inverse square of the distance between them. In accordance with the inverse square law in radiation, the light intensity is also a multiplicative inverse function.

These hypotheses motivated us to introduce and study functional equations with reciprocal inverse quadratic mapping as solution.

## 3. Mapping satisfying equation (1)

In this section, we find the mapping satisfying equation (1). Let $\mathbb{R}^{\star}$ denote the set of non-zero real numbers. Let us assume that $\mathcal{A}=\{\mathfrak{p},\mathfrak{q}\in\mathbb{R}^{\star}:\mathfrak{p}+\mathfrak{q}\neq 0\}$ for all $\mathfrak{p},\mathfrak{q}\in\mathbb{R}^{\star}$.

**Theorem 3.1.** *If $h:\mathcal{A}\longrightarrow\mathbb{R}$ is a mapping satisfying equation (1), then $h$ is a multiplicative inverse quadratic mapping of the form $h(\mathfrak{p})=\frac{1}{\mathfrak{p}^2}$ for all $\mathfrak{p}\in\mathcal{A}$.*

*Proof.* Let $h$ be a mapping satisfying (1). Letting $\mathfrak{q}=\mathfrak{p}$ in equation (1), we find that

$$h\left(\frac{\mathfrak{p}}{2}\right)=2^2h(\mathfrak{p}) \tag{4}$$

for all $\mathfrak{p}\in\mathcal{A}$. Now, replacing $\mathfrak{p}$ by $2\mathfrak{p}$ in (4), one obtains that

$$h(2\mathfrak{p})=\frac{1}{2^2}h(\mathfrak{p}) \tag{5}$$

for all $\mathfrak{p}\in\mathcal{A}$. Again, reinstating $\mathfrak{p}$ by $2\mathfrak{p}$ in (5), we get $h(2^2\mathfrak{p})=\frac{1}{2^4}h(\mathfrak{p})$ for all $\mathfrak{p}\in\mathcal{A}$. Continuing in the similar manner and using induction method for any positive integer $n$, we arrive at $h(2^n\mathfrak{p})=\frac{1}{2^{2n}}h(\mathfrak{p})$ for all $\mathfrak{p}\in\mathcal{A}$. Hence, we conclude that $h(\mathfrak{p})=\frac{1}{\mathfrak{p}^2}$ is a solution of (1). □

**Definition 3.2.** *A mapping $h:\mathcal{A}\longrightarrow\mathbb{R}$ is called a multiplicative inverse quadratic mapping if it satisfies equation (1).*

**Remark 3.3.** *One can easily obtain that equations (1), (2) and (3) are equivalent to each other by simple algebraic computations. Hence, the solution of equations (1), (2) and (3) is a reciprocal quadratic mapping.*

## 4. Stabilities of equations (1), (2) and (3) in the restricted domain

In this section, we first prove the stability of equation (1) involving a general control function $\zeta(\mathfrak{p},\mathfrak{q})$ as an upper bound and then we investigate various other stabilities involving different upper bounds. In order to obtain the required result in a simple manner, let us define the difference operator $\mathcal{D}_1\mathsf{h} : \mathcal{A}\times\mathcal{A} \longrightarrow \mathbb{R}$ as

$$\mathcal{D}_1\mathsf{h}(\mathfrak{p},\mathfrak{q}) = \mathsf{h}\left(\frac{\mathfrak{p}\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}}\right) - \mathsf{h}(\mathfrak{p}) - \mathsf{h}(\mathfrak{q}) - 2\sqrt{\mathsf{h}(\mathfrak{p})\mathsf{h}(\mathfrak{q})}$$

for all $\mathfrak{p},\mathfrak{q} \in \mathcal{A}$. In the following results, let $\mathsf{h} : \mathcal{A} \longrightarrow \mathbb{R}$ be a mapping.

**Theorem 4.1.** *Let $\nu = \pm 1$ be fixed. If the mapping $\mathsf{h}$ satisfies*

$$|\mathcal{D}_1\mathsf{h}(\mathfrak{p},\mathfrak{q})| \leq \zeta(\mathfrak{p},\mathfrak{q}) \tag{6}$$

*for all $\mathfrak{p},\mathfrak{q} \in \mathcal{A}$, where $\zeta : \mathcal{A}\times\mathcal{A} \longrightarrow [0,\infty)$ is a function with the condition*

$$\sum_{r=0}^{\infty} \frac{1}{4^{\nu r}}\zeta\left(\frac{\mathfrak{p}}{2^{\nu(r+1)}}, \frac{\mathfrak{q}}{2^{\nu(r+1)}}\right) < +\infty \tag{7}$$

*for all $\mathfrak{p},\mathfrak{q} \in \mathcal{A}$. Then there exists a unique multiplicative inverse quadratic mapping $\mathsf{H} : \mathcal{A} \longrightarrow \mathbb{R}$ satisfying* (1) *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \sum_{r=0}^{\infty} \frac{1}{4^{\nu r}}\zeta\left(\frac{\mathfrak{p}}{2^{\nu(r+1)}}, \frac{\mathfrak{p}}{2^{\nu(r+1)}}\right) \tag{8}$$

*for all $\mathfrak{p} \in \mathcal{A}$.*

*Proof.* Let us first prove this theorem when $\nu = 1$. Considering $(\mathfrak{p},\mathfrak{q})$ as $(\mathfrak{p},\mathfrak{p})$ in (6), we obtain

$$\left|\frac{1}{4^{\nu}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{\nu}}\right) - \mathsf{h}(\mathfrak{p})\right| \leq 4^{\frac{|\nu-1|}{2}}\zeta\left(\frac{\mathfrak{p}}{2^{\frac{\nu+1}{2}}}, \frac{\mathfrak{p}}{2^{\frac{\nu+1}{2}}}\right) \tag{9}$$

for all $\mathfrak{p} \in \mathcal{A}$. Next, replacing $\mathfrak{p}$ by $\frac{\mathfrak{p}}{4^{k\nu}}$ in (9) and then multiplying by $\left|\frac{1}{4}\right|^{k\nu}$, we obtain

$$\left|\frac{1}{4^{k\nu}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{k\nu}}\right) - \frac{1}{4^{(k+1)\nu}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{(k+1)\nu}}\right)\right| \leq \frac{1}{4^{k\nu}}\zeta\left(\frac{\mathfrak{p}}{2^{(k+1)\nu}}, \frac{\mathfrak{p}}{2^{(k+1)\nu}}\right). \tag{10}$$

Now, allowing $k \to \infty$ in (10) and utilizing (7), we find that the sequence $\{\frac{1}{4^{k\nu}}\mathsf{h}(\frac{\mathfrak{p}}{2^{(k+1)\nu}})\}$ turns out to be Cauchy. The completeness of $\mathbb{R}$ indicates that the sequence $\{\frac{1}{4^{k\nu}}\mathfrak{u}(\frac{\mathfrak{p}}{2^{(k+1)\nu}})\}$ converges to a mapping $\mathsf{H} : \mathcal{A} \longrightarrow \mathbb{R}$ defined by

$$\mathsf{H}(\mathfrak{p}) = \lim_{n\to\infty} \frac{1}{4^{k\nu}}\mathsf{h}(2^{(k+1)\nu}\mathfrak{p}). \tag{11}$$

In order to prove that $\mathsf{H}$ is a solution of (1), let us replace $(\mathfrak{p},\mathfrak{q})$ into $\left(\frac{\mathfrak{p}}{2^{k\nu}}, \frac{\mathfrak{q}}{2^{k\nu}}\right)$ in (4) and then multiplying both sides by $\frac{1}{4^{k\nu}}$, we attain

$$\left|\frac{1}{4^{k\nu}}\mathcal{D}_1\mathsf{h}(\mathfrak{p},\mathfrak{q})\left(\frac{\mathfrak{p}}{2^{k\nu}}, \frac{\mathfrak{q}}{2^{k\nu}}\right)\right| \leq \frac{1}{4^{k\nu}}\zeta\left(\frac{\mathfrak{p}}{2^{k\nu}}, \frac{\mathfrak{q}}{2^{k\nu}}\right) \tag{12}$$

for all $\mathfrak{p},\mathfrak{q} \in \mathcal{A}$ and for all positive integer $k$. Now, by utilizing (7) and (11) in (12), we observe that $\mathsf{H}$ satisfies (1) for all $\mathfrak{p},\mathfrak{q} \in \mathcal{A}$. For each $\mathfrak{p} \in \mathcal{A}$ and each integer $k$, we have

$$\begin{aligned}\left|\frac{1}{4^{k\nu}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{k\nu}}\right) - \mathsf{h}(\mathfrak{p})\right| &\leq \sum_{\ell=0}^{k-1}\left|\frac{1}{4^{\ell\nu}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{(\ell+1)\nu}}\right) - \frac{1}{4^{(\ell-1)\nu}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{\ell\nu}}\right)\right| \\ &\leq \sum_{\ell=0}^{k-1}\frac{1}{4^{\ell\nu}}\zeta\left(\frac{\mathfrak{p}}{2^{(\ell+1)\nu}}, \frac{\mathfrak{p}}{2^{(\ell+1)\nu}}\right).\end{aligned}$$

Applying (7) and letting $k \to \infty$, we obtain (8). Next is to show that the mapping $\mathsf{H}$ is unique. To prove this claim, let us assume that $\mathsf{H}' : \mathcal{A} \longrightarrow \mathbb{R}$ is an alternative multiplicative inverse quadratic mapping satisfying (1) and (8). Then, by virtue of (8), we obtain

$$\begin{aligned}
|\mathsf{H}'(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| &= 4^{-k\nu} \left|\mathsf{H}'(2^{-k\nu}\mathfrak{p}) - \mathsf{H}(2^{-k\nu}\mathfrak{p})\right| \\
&\leq 4^{-k\nu} \left|\mathsf{H}'(2^{-k\nu}\mathfrak{p}) - \mathsf{h}(2^{-k\nu}\mathfrak{p})\right| + 4^{-k\nu} \left|\mathsf{h}(2^{-k\nu}\mathfrak{p}) - F(2^{-k\nu}\mathfrak{p})\right| \\
&\leq 2 \sum_{r=0}^{\infty} \frac{1}{4^{(k+r)\nu}} \zeta\left(\frac{\mathfrak{p}}{2^{(k+r+1)\nu}}, \frac{\mathfrak{p}}{2^{(k+r+1)\nu}}\right) \\
&\leq 2 \sum_{r=k}^{\infty} \frac{1}{4^{r\nu}} \zeta\left(\frac{\mathfrak{p}}{2^{(r+1)\nu}}, \frac{\mathfrak{p}}{2^{(r+1)\nu}}\right) \qquad (13)
\end{aligned}$$

for all $\mathfrak{p} \in \mathcal{A}$. When $k$ approaches to $\infty$, the right-hand side of (13) becomes zero which concludes that $\mathsf{H} = \mathsf{H}'$. Hence $\mathsf{H}$ is unique. Thus the proof is complete for $\nu = 1$. When $\nu = -1$, proof follows with the similar arguments as in the case of $\nu = 1$. □

The following stability results are direct outcomes of the above Theorem 4.1. Hence the proof of the results are omitted. The results are obtained by assuming $\alpha \neq -2$, $a + b \neq -2$, $\mathfrak{p}, \mathfrak{q} \in \mathcal{A}$.

The proof of the following corollary is for the case $\nu = -1$ in Theorem 4.1 and taking $\zeta(\mathfrak{p}, \mathfrak{q}) = \mu$.

**Corollary 4.2.** *Let a constant $\mu > 0$ exists so that it does not depend on the values of $\mathfrak{p}$ and $\mathfrak{q}$. Let a mapping $\mathsf{h}$ satisfies the inequality*

$$|\mathcal{D}_1\mathsf{h}(\mathfrak{p}, \mathfrak{q})| \leq \frac{3}{4}\mu$$

*for all $\mathfrak{p}, \mathfrak{q} \in \mathcal{A}$. Then there exists a unique multiplicative inverse quadratic mapping $\mathsf{H} : \mathcal{A} \longrightarrow \mathbb{R}$ satisfying* (1) *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \mu$$

*for all $\mathfrak{p} \in \mathcal{A}$.*

**Corollary 4.3.** *For any fixed $\mu_1 \geq 0$ and $\alpha \neq -2$, if a mapping $\mathsf{h}$ satisfies*

$$|\mathcal{D}_1\mathsf{h}(\mathfrak{p}, \mathfrak{q})| \leq \mu_1(|\mathfrak{p}|^\alpha + |\mathfrak{q}|^\alpha)$$

*for all $\mathfrak{p}, \mathfrak{q} \in \mathcal{A}$, then there exists a unique multiplicative inverse quadratic mapping $\mathsf{H} : \mathcal{A} \longrightarrow \mathbb{R}$ satisfying* (1) *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \begin{cases} \frac{8\mu_1}{|2^{\alpha+2}-1|} |\mathfrak{p}|^\alpha & \text{for } \alpha > -2, \ \nu = 1 \\ \frac{2^{\alpha+1}\mu_1}{|1-2^{\alpha+2}|} |\mathfrak{p}|^\alpha & \text{for } \alpha < -2, \ \nu = -1 \end{cases}$$

*for all $\mathfrak{p} \in \mathcal{A}$.*

**Corollary 4.4.** *Let the constants $a, b$ exist such that $\alpha = a + b \neq -2$ and $\mu_2 \geq 0$. Suppose a mapping $\mathsf{h}$ satisfies the ensuing inequality*

$$|\mathcal{D}_1\mathsf{h}(\mathfrak{p}, \mathfrak{q})| \leq \mu_2 \, |\mathfrak{p}|^a \, |\mathfrak{q}|^b$$

*for all $\mathfrak{p}, \mathfrak{q} \in \mathcal{A}$. Then there exists a unique multiplicative inverse quadratic mapping $\mathsf{H} : \mathcal{A} \longrightarrow \mathbb{R}$ satisfying* (1) *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \begin{cases} \frac{4\mu_2}{|2^{\alpha+2}-1|} |\mathfrak{p}|^\alpha & \text{for } \alpha > -2, \ \nu = 1 \\ \frac{2^{\alpha}\mu_2}{|1-2^{\alpha+2}|} |\mathfrak{p}|^\alpha & \text{for } \alpha < -2, \ \nu = -1 \end{cases}$$

*for all $\mathfrak{p} \in \mathcal{A}$.*

**Remark 4.5.** *The stability results pertinent to various stabilities of equations* (2) *and* (3) *involving different upper bounds are similar to the results of* (1) *and hence, we omit them.*

## 5. Non-Archimedean approximation of (1), (2) and (3)

We elicit the fundamental definition and few basic terminologies in non-Archimedean field.

**Definition 5.1.** *Let $\mathcal{A}$ be a field with a valuation (function) $|\cdot|$ defined on it from $\mathcal{A}$ into non-negative real numbers. Then $\mathcal{A}$ is said to be a non-Archimedean field provided the following conditions hold:*

*(i)* $|\mathfrak{p}| = 0$ *if and only if* $\mathfrak{p} = 0$;
*(ii)* $|\mathfrak{p}\mathfrak{q}| = |\mathfrak{p}||\mathfrak{q}|$ *and*
*(iii)* $|\mathfrak{p} + \mathfrak{q}| \leq max\{|\mathfrak{p}|, |\mathfrak{q}|\}$ *for all* $\mathfrak{p}, \mathfrak{q} \in \mathcal{A}$.

It is easy to verify that $|\pm 1| = 1$ from the above definition and $|k| \leq 1$ for all integers $k > 0$. Moreover, we presume that $|\cdot|$ is not trivial, that is, there exists a $\beta \in \mathbb{M}$ so that $|\beta| \neq 0, 1$. In view of the inequality:

$$|a_t - a_s| \leq \max\{|a_{k+1} - a_k| : s \leq k \leq t-1\} \quad (t > s),$$

a sequence $\{a_t\}$ is Cauchy if and only if $\{a_{t+1} - a_t\}$ tends to 0 in a non-Archimedean field. If each Cauchy sequence converges in a field which does not satisfy Archimedean property, then we say the field to be a complete non-Archimedean field.

The stability problems of various functional equations were solved in the setting of non-Archimedean normed spaces in ([6, 17, 19]).

In this section, we also assume that $\mathfrak{p} + \mathfrak{q} \neq 0$ to avoid singularity in the main results. Also, unless we specify, we assume that $\mathcal{A}$ is a field without Archimedean property and $\mathcal{B}$ is a complete non-Archimedean field. In the following, for a non-Archimdean field $\mathcal{A}$, we symbolize $\mathcal{A}^* = \mathcal{A}\backslash\{0\}$. For a mapping $\mathsf{h} : \mathcal{A}^\star \longrightarrow \mathcal{B}$ and for easy computation, let us define the difference operator $\mathcal{D}_1\mathsf{h} : \mathcal{A}^\star \times \mathcal{A}^\star \longrightarrow \mathcal{B}$ by

$$\mathcal{D}_1\mathsf{h}(\mathfrak{p}, \mathfrak{q}) = \mathsf{h}\left(\frac{\mathfrak{p}\mathfrak{q}}{\mathfrak{p} + \mathfrak{q}}\right) - \mathsf{h}(\mathfrak{p}) - \mathsf{h}(\mathfrak{q}) - 2\sqrt{\mathsf{h}(\mathfrak{p})\mathsf{h}(\mathfrak{q})}$$

for all $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}^\star$.

In the following results, we present the stabilities of equations (1), (2) and (3) in the domain of non-Archimedean fields.

**Theorem 5.2.** *Let $k = \pm 1$ be fixed, and suppose $\Upsilon : \mathcal{A}^* \times \mathcal{A}^* \longrightarrow \infty$ is a function satisfying*

$$\lim_{\ell \to \infty} \left|\frac{1}{4}\right|^{k\ell} \Upsilon\left(\frac{\mathfrak{p}}{2^{k\ell + \frac{k+1}{2}}}, \frac{\mathfrak{q}}{2^{k\ell + \frac{k+1}{2}}}\right) = 0 \tag{14}$$

*for all $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}^*$. Suppose that $\mathsf{h} : \mathbb{A}^* \longrightarrow \mathbb{B}$ is a mapping satisfying the inequality*

$$|\mathcal{D}_1\mathsf{h}(\mathfrak{p}, \mathfrak{q})| \leq \Upsilon(\mathfrak{p}, \mathfrak{q}) \tag{15}$$

*for all $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}^*$. Then, there exists a unique multiplicative inverse quadratic mapping $\mathsf{H} : \mathbb{A}^* \longrightarrow \mathbb{B}$ satisfying* (1) *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \sup\left\{\left|\frac{1}{4}\right|^{ks + \frac{k-1}{2}} \Upsilon\left(\frac{\mathfrak{p}}{2^{ks + \frac{k+1}{2}}}, \frac{\mathfrak{p}}{2^{ks + \frac{k+1}{2}}}\right) : s \in \mathbb{N} \cup \{0\}\right\} \tag{16}$$

*for all $\alpha \in \mathbb{A}^*$.*

*Proof.* First, let us replace $(\mathfrak{p}, \mathfrak{q})$ by $\left(\frac{\mathfrak{p}}{2}, \frac{\mathfrak{p}}{2}\right)$ in (15), we obtain

$$\left|\mathsf{h}(\mathfrak{p}) - \frac{1}{4^k}\mathsf{h}\left(\frac{\mathfrak{p}}{2^k}\right)\right| \leq \left|\frac{1}{4}\right|^{\frac{|k-1|}{2}} \Upsilon\left(\frac{\mathfrak{p}}{2^{\frac{k+1}{2}}}, \frac{\mathfrak{p}}{2^{\frac{k+1}{2}}}\right) \tag{17}$$

for all $\mathfrak{p} \in \mathbb{A}^*$. Then, again replace $\mathfrak{p}$ by $\frac{\mathfrak{p}}{2^{k\ell}}$ in (15) and multiplying by $\left|\frac{1}{4}\right|^{k\ell}$, we have

$$\left|\frac{1}{4^{k\ell}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{k\ell}}\right) - \frac{1}{4^{(\ell+1)k}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{(\ell+1)k}}\right)\right| \leq \left|\frac{1}{4}\right|^{k\ell+\frac{k-1}{2}} \Upsilon\left(\frac{\mathfrak{p}}{2^{k\ell+\frac{k+1}{2}}}, \frac{\mathfrak{p}}{2^{k\ell+\frac{k+1}{2}}}\right) \tag{18}$$

for all $\mathfrak{p} \in \mathbb{A}^*$. Owing to (14) and the inequality (18), we observe that the sequence $\left\{\frac{1}{4^{k\ell}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{k\ell}}\right)\right\}$ becomes Cauchy. The completeness of $\mathbb{B}$ implies that this sequence converges to a mapping $\mathsf{H} : \mathbb{A}^* \longrightarrow \mathbb{B}$ which is defined by

$$\mathsf{H}(\mathfrak{p}) = \lim_{\ell\to\infty} \frac{1}{4^{k\ell}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{k\ell}}\right). \tag{19}$$

Besides, for every $\mathfrak{p} \in \mathbb{A}^*$ and $\ell \geq 0$, we have

$$\begin{aligned}\left|\frac{1}{4^{k\ell}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{k\ell}}\right) - \mathsf{h}(\mathfrak{p})\right| &= \left|\sum_{s=0}^{\ell-1}\left\{\frac{1}{4^{(s+1)k}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{(s+1)k}}\right) - \frac{1}{4^{sk}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{sk}}\right)\right\}\right| \\ &\leq \sup\left\{\left|\frac{1}{4^{(s+1)k}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{(s+1)k}}\right) - \frac{1}{3^{sk}}\mathsf{h}\left(\frac{\mathfrak{p}}{2^{sk}}\right)\right| : 0 \leq s < \ell\right\} \\ &\leq \sup\left\{\left|\frac{1}{4}\right|^{sk+\frac{k-1}{2}} \Upsilon\left(\frac{\mathfrak{p}}{2^{sk+\frac{k+1}{2}}}, \frac{\mathfrak{p}}{2^{sk+\frac{k+1}{2}}}\right) : 0 \leq s < \ell\right\}.\end{aligned}$$

Employing (19) and letting $\ell \to \infty$ in the above inequality, we observe that the inequality (16) is satisfied. Utilizing (14), (15) and (19), for every $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}^*$, we have

$$|\mathcal{D}_1\mathsf{H}(\mathfrak{p}, \mathfrak{q})| = \lim_{\ell\to\infty}\left|\frac{1}{4}\right|^{k\ell}\left|\mathcal{D}_1\mathsf{h}\left(\frac{\mathfrak{p}}{2^{k\ell}}, \frac{\mathfrak{q}}{2^{k\ell}}\right)\right| \leq \lim_{\ell\to\infty}\left|\frac{1}{4}\right|^{k\ell}\Upsilon\left(\frac{\mathfrak{p}}{2^{k\ell}}, \frac{\mathfrak{q}}{2^{k\ell}}\right) = 0.$$

Thus, the mapping $\mathsf{H}$ satisfies (1) and hence it is a multiplicative inverse quadratic mapping. One can easily prove that $\mathsf{H}$ is unique, which completes the proof. □

The subsequent corollaries directly follow from Theorem 5.2 which are relevant to various stability results of (1) and hence we provide only statements.

**Corollary 5.3.** *Assume $\lambda$ is a positive constant. Also, suppose $\mathsf{h} : \mathbb{A}^* \longrightarrow \mathbb{B}$ is a mapping which satisfies*

$$|\mathcal{D}_1\mathsf{h}(\mathfrak{p}, \mathfrak{q})| \leq \lambda$$

*for all $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}^*$, then there exists a unique multiplicative inverse quadratic mapping $\mathsf{H} : \mathbb{A}^* \longrightarrow \mathbb{B}$ satisfying* (1) *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \lambda$$

*for all $\mathfrak{p} \in \mathbb{A}^*$.*

In the following results, let $\lambda$ be a positive constant and $\mu \neq -2$ be a fixed constant. Let $\mathsf{h} : \mathbb{A}^* \longrightarrow \mathbb{B}$ be a mapping.

**Corollary 5.4.** *Suppose $\mathsf{h}$ satisfies*

$$|\mathcal{D}_1\mathsf{h}(\mathfrak{p}, \mathfrak{q})| \leq \lambda\left(|\mathfrak{p}|^{\mu} + |\mathfrak{q}|^{\mu}\right)$$

*for all $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}^*$, then there exists a unique multiplicative inverse quadratic mapping $\mathsf{H} : \mathbb{A}^* \longrightarrow \mathbb{B}$ satisfying* (1) *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \begin{cases} \frac{|2|\lambda}{|2|^{\mu}}\,|\mathfrak{p}|^{\mu}, & \mu > -2 \\ |2|\lambda|2|^{2}\,|\mathfrak{p}|^{\mu}, & \mu < -2 \end{cases}$$

*for all $\mathfrak{p} \in \mathbb{A}^*$.*

**Corollary 5.5.** *Let there exist real numbers* $\gamma, \delta$ *such that* $\mu = \gamma + \delta \neq -2$ *and*

$$|\mathcal{D}_1 \mathsf{h}(\mathfrak{p}, \mathfrak{q})| \leq \lambda\, |\mathfrak{p}|^{\gamma}\, |\mathfrak{q}|^{\delta}$$

*for all* $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}^*$. *Then there exists a unique multiplicative inverse quadratic mapping* $\mathsf{H} : \mathbb{A}^* \longrightarrow \mathbb{B}$ *satisfying* (1) *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \begin{cases} \frac{\lambda}{|2|^{\mu}}\, |\alpha|^{\mu}\,, & \mu > -2 \\ \lambda |2|^2 |\, |\alpha|^{\mu}\,, & \mu < -2 \end{cases}$$

*for every* $\alpha \in \mathbb{A}^*$.

**Remark 5.6.** *Proceeding with the similar arguments applied in the above theorems and corollaries, one can obtain the stability results connected with equations (2) and (3) which are identical with the results of equation (1). Hence we omit the stability results of (2) and (3).*

## 6. Non-stability of equation (1) for singularity

In this section, we illustrate an example to prove that the stability of (1) fails for a singular value. Persuaded by the exceptional example contributed in [13], we present the following counter-example to prove the non-stability for the singular case $\alpha = -2$ in Corollary 4.3.

Define a mapping $\mathsf{h} : \mathcal{A} \longrightarrow \mathbb{R}$ by

$$\mathsf{h}(\mathfrak{p}) = \sum_{k=0}^{\infty} \frac{\phi\left(2^{-k}\mathfrak{p}\right)}{4^k} \tag{20}$$

for all $\mathfrak{p} \in \mathcal{A}$, and $\phi : \mathcal{A} \to \mathbb{R}$ is a function given by

$$\phi(\mathfrak{p}) = \begin{cases} \dfrac{\mu}{\mathfrak{p}^2}, & \text{if } \mathfrak{p} \in (1, \infty) \\ \mu, & \text{otherwise.} \end{cases} \tag{21}$$

Then the function $\mathsf{h}$ turns out to be a suitable counter-example for $\alpha = -2$ which is presented in the following theorem.

**Theorem 6.1.** *The mapping h defined above satisfies*

$$|\mathcal{D}_1 \mathsf{h}(\mathfrak{p}, \mathfrak{q})| \leq \frac{80\mu}{3} \left( \left|\frac{1}{\mathfrak{p}}\right|^2 + \left|\frac{1}{\mathfrak{q}}\right|^2 \right) \tag{22}$$

*for all* $\mathfrak{p}, \mathfrak{q} \in \mathcal{A}$. *Therefore there do not exist a multiplicative inverse quadratic mapping* $\mathsf{H} : \mathcal{A} \longrightarrow \mathbb{R}$ *and a constant* $\beta > 0$ *such that*

$$|\mathsf{h}(\mathfrak{p}) - \mathsf{H}(\mathfrak{p})| \leq \frac{\beta}{|\mathfrak{p}|^2} \tag{23}$$

*for all* $\mathfrak{p} \in \mathcal{A}$.

*Proof.* Firstly, let us show that $\mathsf{h}$ is bounded above. We have, $|\mathsf{h}(\mathfrak{p})| \leq \sum_{k=0}^{\infty} \frac{\left|\phi\left(2^{-k}\mathfrak{p}\right)\right|}{\left|4^k\right|} \leq \sum_{k=0}^{\infty} \frac{\mu}{4^k} = \mu\left(1 - \frac{1}{4}\right)^{-1} = \frac{4}{3}\mu$. If $\left|\frac{1}{\mathfrak{p}}\right|^2 + \left|\frac{1}{\mathfrak{q}}\right|^2 \geq 1$, then the left-hand side of (22) is less than $\frac{4}{3}\mu$. Now, let us consider that $0 < \left|\frac{1}{\mathfrak{p}}\right|^2 + \left|\frac{1}{\mathfrak{q}}\right|^2 < 1$. Then, there exists a positive integer $r$ such that

$$\frac{1}{4^{r+1}} \leq \left|\frac{1}{\mathfrak{p}}\right|^2 + \left|\frac{1}{\mathfrak{q}}\right|^2 < \frac{1}{4^r}. \tag{24}$$

Hence $\left|\frac{1}{\mathfrak{p}}\right|^2 + \left|\frac{1}{\mathfrak{q}}\right|^2 < \frac{1}{4^r}$ implies

$$4^r \left|\frac{1}{\mathfrak{p}}\right|^2 + 4^r \left|\frac{1}{\mathfrak{q}}\right|^2 < 1$$

or

$$4^r \frac{1}{\mathfrak{p}^2} + 4^r \frac{1}{\mathfrak{q}^2} < 1$$

or

$$4^{r-1} \frac{1}{\mathfrak{p}^2} < 1, 4^{r-1} \frac{1}{\mathfrak{q}^2} < 1$$

and consequently $\frac{1}{2^{r-1}}\mathfrak{p}, \frac{1}{2^{r-1}}\mathfrak{q}, \frac{1}{2^{r-1}}\left(\frac{\mathfrak{p}\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}}\right) \in (1,\infty)$. This implies

$$2^{-k}\mathfrak{p}, 2^{-k}\mathfrak{q}, 2^{-k}\left(\frac{\mathfrak{p}\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}}\right) \in (1,\infty)$$

for each value of $k = 0, 1, 2, \dots, r-1$, and

$$\phi\left(2^{-k}\left(\frac{\mathfrak{p}\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}}\right)\right) - \phi\left(2^{-k}\mathfrak{p}\right) - \phi\left(2^{-k}\mathfrak{q}\right) - 2\sqrt{\phi\left(2^{-k}\mathfrak{p}\right)\phi\left(2^{-k}\mathfrak{q}\right)} = 0$$

for $k = 0, 1, 2, \dots, r-1$. In view of (24) and using the definition of $\mathsf{h}$, we obtain

$$\begin{aligned} |\mathcal{D}_1 \mathsf{h}(\mathfrak{p},\mathfrak{q})| &\leq \sum_{k=0}^{\infty} \frac{1}{4^k}\left|\mathcal{D}_1\phi\left(2^{-k}\mathfrak{p}, 2^{-k}\mathfrak{q}\right)\right| \\ &\leq \sum_{k=0}^{\infty} \frac{1}{4^k}\left|\mathcal{D}_1\phi\left(2^{-k}\mathfrak{p}, 2^{-k}\mathfrak{q}\right)\right| \leq \sum_{k=r}^{\infty} \frac{1}{4^k} 5\mu = \frac{80\mu}{3}\left(\left|\frac{1}{\mathfrak{p}}\right|^2 + \left|\frac{1}{\mathfrak{q}}\right|^2\right). \end{aligned}$$

Therefore, the inequality (22) holds true. Now, we claim that (1) fails to be stable for $\alpha = -2$ in Corollary 4.3. For this, let us assume that a multiplicative inverse quadratic mapping $\mathsf{H} : \mathcal{A} \longrightarrow \mathbb{R}$ satisfying (23). Hence, we have

$$|\mathsf{H}(\mathfrak{p})| \leq (\beta + 1)\left|\frac{1}{\mathfrak{p}^2}\right|. \tag{25}$$

However, we can choose an integer $m > 0$ such that $m\mu > \beta + 1$. If $\mathfrak{p} \in (1,\infty)$, then $2^{-k}\mathfrak{p} \in (1,\infty)$ for all $k = 0, 1, 2, \dots, m-1$ and therefore

$$|\mathsf{H}(\mathfrak{p})| = \sum_{k=0}^{\infty} \frac{\phi\left(2^{-k}\mathfrak{p}\right)}{4^k} \geq \sum_{k=0}^{m-1} \frac{1}{4^k}\frac{\mu}{(2^{-k}\mathfrak{p})^2} = m\mu\frac{1}{\mathfrak{p}^2} \geq (\beta+1)\frac{1}{\mathfrak{p}^2}$$

which contradicts (25) and so, (1) is unstable for $\alpha = -2$ in Corollary 4.3. □

## 7. Explication of equations (1), (2) and (3)

Here, we interpret the equations (1), (2) and (3) with the Pythogorean means.

1. Equation (1) indicates that function $\mathsf{h}$ maps into half of harmonic mean of $\mathfrak{p}$ and $\mathfrak{q}$ into sum of half of arithmetic mean of $\mathsf{h}(\mathfrak{p})$ and $\mathsf{h}(\mathfrak{q})$ and twice the geometric mean of $\mathfrak{p}$ and $\mathfrak{q}$.
2. Equation (2) conveys that the sum of the mapping of half of the harmonic mean of $\mathfrak{p}$ and $\mathfrak{q}$ and the mapping of harmonic mean of $\mathfrak{p}$ and $\mathfrak{q}$ is equal to $\frac{5}{2}$ times the sum of arithmetic mean of $\mathsf{h}(\mathfrak{p})$ and $\mathsf{h}(\mathfrak{q})$ and geometric mean of $\mathsf{h}(\mathfrak{p})$ and $\mathsf{h}(\mathfrak{q})$ under the function $\mathsf{h}$.
3. Equation (3) signifies that the difference between the mapping of half of the harmonic mean of $\mathfrak{p}$ and $\mathfrak{q}$ and the mapping of harmonic mean of $\mathfrak{p}$ and $\mathfrak{q}$ is equal to $\frac{3}{2}$ times the sum of arithmetic mean of $\mathsf{h}(\mathfrak{p})$ and $\mathsf{h}(\mathfrak{q})$ and geometric mean of $\mathsf{h}(\mathfrak{p})$ and $\mathsf{h}(\mathfrak{q})$ under the function $\mathsf{h}$.

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