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On harmonic convex functions of three variables and related Hermite-Hadamard inequalities

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Abstract. The main objective of this paper is to extend the notion of harmonic convex functions for three variables. Some associated Hermite-Hadamard type of integral inequalities are also obtained.

1. Introduction and Preliminaries

In this section, we discuss some previously known concepts and results.
Recall that a function $f : \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is harmonic convex on Δ, if

$$f\Big(\frac{ab}{\lambda a + (1-\lambda)b}, \frac{cd}{\lambda c + (1-\lambda)d}\Big) \leq (1-\lambda)f(a, c) + \lambda(b, d),$$

for all (a, b), $(c, d) \in \Delta$ and $\lambda \in [0, 1]$.

The co-ordinated harmonic convex functions is defined as:

Definition 1.1 ([9]). *Consider a rectangle $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \setminus \{0\}$ with $a < b$ and $c < d$. A function $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is said to be harmonic convex function of three variables on the rectangle Δ, if*

$$\begin{aligned} f\Big(\frac{ab}{\lambda a + (1-\lambda)b}, \frac{cd}{tc + (1-t)d}\Big) \quad &\leq \quad (1-\lambda)(1-t)f(a, c) + (1-\lambda)tf(a, d) \\ &\quad +\lambda(1-t)f(b, c) + \lambda t(b, d), \end{aligned}$$

for all (a, b), $(c, d) \in \Delta$ and $\lambda, t \in [0, 1]$.

A function $f : \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ will be called harmonic on the co-ordinates if the partial mappings $f_y : [a, b] \to \mathbb{R}$, defined by $f_y(u) := f(u, y)$, and $f_x : [c, d] \to \mathbb{R}$, defined by $f_x(v) := f(x, v)$, are harmonic convex for all $x \in [a, b]$ and $y \in [c, d]$. Noor et al. [9] introduced the above mentioned concepts and established some related integral inequalities of Hermite-Hadamard type.

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Definition 1.2. *Consider a rectangular box* $\Omega = [a,b]\times[c,d]\times[e,f] \subset \mathbb{R}^3\setminus\{0\}$ *with* $a<b$, $c<d$ *and* $e<f$. *A function* $g:\Omega=[a,b]\times[c,d]\times[e,f]\to\mathbb{R}$ *is said to be harmonic convex function on* Ω, *if*

$$g\Big(\frac{ab}{\lambda a+(1-\lambda)b},\frac{cd}{\lambda c+(1-\lambda)d},\frac{ef}{\lambda e+(1-\lambda)f}\Big)\leq(1-\lambda)g(a,c,e)+\lambda g(b,d,f),$$

for all (a,b), (c,d), $(e,f)\in\Omega$ *and* $\lambda\in[0,1]$.

The harmonic convex functions for three variables on a rectangular box is defined as:

Definition 1.3. *Consider a rectangular box* $\Omega = [a,b]\times[c,d]\times[e,f] \subset \mathbb{R}^3\setminus\{0\}$ *with* $a<b$, $c<d$ *and* $e<f$. *A function* $g:\Omega=[a,b]\times[c,d]\times[e,f]\to\mathbb{R}$ *is said to be harmonic convex function on* Ω, *if*

$$\begin{aligned}
& g\Big(\frac{ab}{\lambda a+(1-\lambda)b},\frac{cd}{tc+(1-t)d},\frac{ef}{re+(1-r)f}\Big)\\
\leq\ & (1-\lambda)(1-t)(1-r)g(a,c,e)+(1-\lambda)t(1-r)g(a,d,e)+(1-\lambda)trg(a,d,f)\\
& +\lambda(1-t)(1-r)g(b,c,e)+\lambda(1-t)rg(b,c,f)+(1-\lambda)(1-t)rg(a,c,f)\\
& +\lambda t(1-r)g(b,d,e)+\lambda trg(b,d,f),
\end{aligned}$$

for all (a,b), (c,d), $(e,f)\in\Omega$ *and* $\lambda,t,r\in[0,1]$.

A function $g:\Omega=[a,b]\times[c,d]\times[e,f]\subset\mathbb{R}^3\setminus\{0\}\to\mathbb{R}$ is said to be harmonic convex function of three variables on a rectangular box Ω if for every $(x,y)\in[a,b]\times[c,d]$, $(x,z)\in[a,b]\times[e,f]$ and $(y,z)\in[c,d]\times[e,f]$, the partial mappings

$$\begin{aligned}
& g_x:[c,d]\times[e,f]\to\mathbb{R}, && g_x(v,w)=g(x,v,w), && x\in[a,b],\\
& g_y:[a,b]\times[e,f]\to\mathbb{R}, && g_y(u,w)=g(u,y,w), && y\in[c,d],\\
& g_z:[a,b]\times[c,d]\to\mathbb{R}, && g_z(u,v)=g(u,v,z), && z\in[e,f],
\end{aligned}$$

are harmonic convex function.
Example. One can easily show that the function g defined by $f(x,y)=\frac{1}{xyz}$ is harmonic convex function of three variables on rectangular box, but it is not harmonic convex function.
The following simple but important result plays an important role in the derivation of our main results.

Remark 1.4. *If* $\Omega=[a,b]\times[c,d]\times[e,f]\subset\mathbb{R}^3\setminus\{0\}$ *and consider the function* $h:\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]\times\left[\frac{1}{f},\frac{1}{e}\right]\to\mathbb{R}$ *defined by* $h(s_1,s_2,s_3)=g\Big(\frac{1}{s_1},\frac{1}{s_2},\frac{1}{s_3}\Big)$, *then* g *is harmonic convex function of three variables on* $[a,b]\times[c,d]\times[e,f]$, *if and only if,* h *is convex function of three variables on* $\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]\times\left[\frac{1}{f},\frac{1}{e}\right]\to\mathbb{R}^3$.

2. Main results

In this section, we discuss our main results.
Before we proceed further let us assume that $\Omega:=[a,b]\times[c,d]\times[e,f]$, $a,b,c,d,e,f\in\mathbb{R}$. In order to prove our main result we first need to prove the following representation theorem.

Theorem 2.1. *Let* $g:\Omega=[a,b]\times[c,d]\times[e,f]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be a Lebesgue integrable function on the rectangle box and* $\lambda,t,r\in[0,1]$, *then*

$$\begin{aligned}
& \int_0^1\int_0^1\int_0^1 g((1-t_1)a+t_1b,(1-t_2)c+t_2d,(1-t_3)e+t_3f)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
=\ & \lambda tr\int_0^1\int_0^1\int_0^1 g\Big((1-t_1)a+t_1\Big((1-\lambda)a+\lambda b\Big),(1-t_2)c+t_2\Big((1-t)c+td\Big),\\
& (1-t_3)e+t_3\Big((1-r)e+rf\Big)\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (1-\lambda)tr\int_0^1\int_0^1\int_0^1 g\Big((1-t_1)\big((1-\lambda)a+\lambda b\big)+t_1b,(1-t_2)c+t_2\big((1-t)c+td\big),\\
&\qquad (1-t_3)e+t_3\big((1-r)e+rf\big)\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&+ \quad \lambda(1-t)r\int_0^1\int_0^1\int_0^1 g\Big((1-t_1)a+t_1\big((1-\lambda)a+\lambda b\big),(1-t_2)\big((1-t)c+td\big)+t_2d,\\
&\qquad (1-t_3)e+t_3\big((1-r)e+rf\big)\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&+ \quad \lambda t(1-r)\int_0^1\int_0^1\int_0^1 g\Big((1-t_1)a+t_1\big((1-\lambda)a+\lambda b\big),(1-t_2)c+t_2\big((1-t)c+td\big),\\
&\qquad (1-t_3)\big((1-r)e+rf\big)+t_3f\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&+ \quad (1-\lambda)(1-t)r\int_0^1\int_0^1\int_0^1 g\Big((1-t_1)\big((1-\lambda)a+\lambda b\big)+t_1b,(1-t_2)\big((1-t)c+td\big)+t_2d,\\
&\qquad (1-t_3)e+t_3\big((1-r)e+rf\big)\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&+ \quad (1-\lambda)t(1-r)\int_0^1\int_0^1\int_0^1 g\Big((1-t_1)\big((1-\lambda)a+\lambda b\big)+t_1b,(1-t_2)c+t_2\big((1-t)c+td\big),\\
&\qquad (1-t_3)\big((1-r)e+rf\big)+t_3f\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&+ \quad \lambda(1-t)(1-r)\int_0^1\int_0^1\int_0^1 g\Big((1-t_1)a+t_1\big((1-\lambda)a+\lambda b\big),(1-t_2)\big((1-t)c+td\big)+t_2d,\\
&\qquad (1-t_3)\big((1-r)e+rf\big)+t_3f\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&+ \quad (1-\lambda)(1-t)(1-r)\int_0^1\int_0^1\int_0^1 g\Big((1-t_1)\big((1-\lambda)a+\lambda b\big)+t_1b,(1-t_2)\big((1-t)c+td\big)+t_2d,\\
&\qquad (1-t_3)\big((1-r)e+rf\big)+t_3f\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 \qquad (1)
\end{aligned}
$$

Proof. One can proof by using the technique used in [10]. □

Now we prove Hermite-Hadamard inequality via harmonic convex functions of three variables on rectangular box.

Theorem 2.2. *Let $g:\Omega=[a,b]\times[c,d]\times[c,d]\subset\mathbb{R}^3\setminus\{0\}\to\mathbb{R}$ be harmonic convex function of three variables on the rectangular box. Then for any $\lambda,t,r\in[0,1]$, we have*

$$
\begin{aligned}
g\Big(\frac{2ab}{a+b},\frac{2cd}{c+d},\frac{2ef}{e+f}\Big) &\leq \phi(\lambda,t,r)\\
&\leq \frac{(ab)(cd)(ef)}{(b-a)(d-c)(f-e)}\int_a^b\int_c^d\int_e^f \frac{g(x,y,z)}{x^2y^2z^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\leq \psi(\lambda,t,r)\\
&\leq \frac{1}{8}\Big[g(a,c,e)+g(a,d,e)+g(a,d,f)+g(b,c,e)+g(b,c,f)\\
&\qquad +g(a,c,f)+g(b,d,e)+g(b,d,f)\Big] \qquad (2)
\end{aligned}
$$

where

$$
\begin{aligned}
&\phi(\lambda,t,r)\\
&=\lambda trg\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(2-r)e+rf}\Big)
\end{aligned}
$$

$$+(1-\lambda)trg\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(2-r)e+rf}\Big)$$
$$+\lambda(1-t)rg\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(1-t)c+(1+t)td},\frac{2ef}{(2-r)e+rf}\Big)$$
$$+\lambda t(1-r)g\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(1-r)e+(1+r)f}\Big)$$
$$+(1-\lambda)(1-t)rg\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(2-r)e+rf}\Big)$$
$$+(1-\lambda)t(1-r)g\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(1-r)e+(1+r)f}\Big)$$
$$+\lambda(1-t)(1-r)g\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(1-t)c+(1+t)td},\frac{2ef}{(1-r)e+(1+r)f}\Big)$$
$$+(1-\lambda)(1-t)(1-r)g\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(1-t)c+(1+t)td},\frac{2ef}{(1-r)e+(1+r)f}\Big).$$

and

$$\begin{aligned}\psi(\lambda,t,r) \;=\; & \frac{1}{8}\Big[\lambda trg(b,d,f)+(1-\lambda)trg(a,d,f)+\lambda(1-t)rg(b,c,f)+\lambda t(1-r)g(b,d,e)\\
&+(1-\lambda)(1-t)rg(a,c,f)+(1-\lambda)t(1-r)g(a,d,e)+\lambda(1-t)(1-r)g(b,c,e)\\
&+(1-\lambda)(1-t)(1-r)g(a,c,e)\Big]+\frac{1}{8}g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)\\
&+\frac{1}{8}\Big[\lambda rg\Big(b,\frac{cd}{(1-t)c+td},f\Big)+trg\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,f\Big)+\lambda tg\Big(b,d,\frac{ef}{(1-r)e+rf}\Big)\\
&+t(1-\lambda)g\Big(a,d,\frac{ef}{(1-r)e+rf}\Big)+r(1-\lambda)g\Big(a,\frac{cd}{(1-t)c+td},f\Big)\\
&+\lambda(1-t)g\Big(b,c,\frac{ef}{(1-r)e+rf}\Big)+r(1-t)g\Big(\frac{ad}{(1-\lambda)a+\lambda b},c,f\Big)\end{aligned}$$

$$+\lambda(1-r)g\Big(b,\frac{cd}{(1-t)c+td},e\Big)+t(1-r)g\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,e\Big)$$
$$+(1-\lambda)(1-t)g\Big(a,c,\frac{ef}{(1-r)e+rf}\Big)+(1-\lambda)(1-r)g\Big(a,\frac{cd}{(1-t)c+td},e\Big)$$
$$+(1-t)(1-r)g\Big(\frac{ab}{(1-\lambda)a+\lambda b},c,e\Big)\Big]+\frac{1}{8}\Big[\lambda g\Big(b,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)$$
$$+tg\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,\frac{ef}{(1-r)e+rf}\Big)+rg\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},f\Big)$$
$$+(1-r)g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},e\Big)+(1-\lambda)g\Big(a,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)$$
$$+(1-t)g\Big(\frac{ab}{(1-\lambda)a+\lambda b},c,\frac{ef}{(1-r)e+rf}\Big)\Big].$$

Proof. Let h be convex function of three variables on the rectangular box $=\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]\times\left[\frac{1}{f},\frac{1}{e}\right]$, defined by $g(s_1,s_2,s_3)=g\Big(\frac{1}{s_1},\frac{1}{s_2},\frac{1}{s_3}\Big)$, $s_1,s_2,s_3\in\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]\times\left[\frac{1}{f},\frac{1}{e}\right]$, then for $\lambda,t,r\in[0,1]$, we have

$$(i).\qquad h\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(2-t)c+td}{2cd},\frac{(2-r)e+rf}{2ef}\Big)$$

$$
\begin{aligned}
&= h\Big(\frac{\frac{1}{b}+(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}}{2},\frac{\frac{1}{d}+(1-t)\frac{1}{d}+t\frac{1}{c}}{2},\frac{\frac{1}{f}+(1-r)\frac{1}{f}+r\frac{1}{e}}{2}\Big)\\
&\leq \int_0^1\int_0^1\int_0^1 h\Big((1-t_1)\frac{1}{b}+t_1\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big),(1-t_2)\frac{1}{d}+t_2\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big),\\
&\quad (1-t_3)\frac{1}{f}+t_3\Big((1-r)\frac{1}{f}+r\frac{1}{e}\Big)\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&\leq \frac{1}{8}\Big[h\Big(\frac{1}{b},\frac{1}{d},\frac{1}{f}\Big)+h\Big(\frac{1}{b},(1-t)\frac{1}{d}+t\frac{1}{c},\frac{1}{f}\Big)h\Big(\frac{1}{b},(1-t)\frac{1}{d}+t\frac{1}{c},(1-r)\frac{1}{f}+r\frac{1}{e}\Big)\\
&\quad +h\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},\frac{1}{d},\frac{1}{f}\Big)+h\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},\frac{1}{d},(1-r)\frac{1}{f}+r\frac{1}{e}\Big)\\
&\quad +h\Big(\frac{1}{b},\frac{1}{d},(1-r)\frac{1}{f}+r\frac{1}{e}\Big)+h\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},(1-t)\frac{1}{d}+t\frac{1}{c},\frac{1}{f}\Big)\\
&\quad +h\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},(1-t)\frac{1}{d}+t\frac{1}{c},(1-r)\frac{1}{f}+r\frac{1}{e}\Big)\Big]\\
&= \frac{1}{8}\Big[h\Big(\frac{1}{b},\frac{1}{d},\frac{1}{f}\Big)+h\Big(\frac{1}{b},\frac{(1-t)c+td}{cd},\frac{1}{f}\Big)h\Big(\frac{1}{b},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)\\
&\quad +h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d},\frac{1}{f}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)\\
&\quad +h\Big(\frac{1}{b},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{1}{f}\Big)\\
&\quad +h\Big(\frac{(1-\lambda)+a\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)\Big] \qquad (3)
\end{aligned}
$$

Similarly

(ii).

$$
\begin{aligned}
&h\Big(\frac{(1-\lambda)a+(\lambda+1)b}{2ab},\frac{(2-t)c+td}{2cd},\frac{(2-r)e+rf}{2ef}\Big)\\
&\leq \int_0^1\int_0^1\int_0^1 h\Big((1-t_1)\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big)+t_1\frac{1}{a},(1-t_2)\frac{1}{d}+t_2\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big),\\
&\quad (1-t_3)\frac{1}{f}+t_3\Big((1-r)\frac{1}{f}+r\frac{1}{e}\Big)\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&\leq \frac{1}{8}\Big[h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d},\frac{1}{f}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{1}{f}\Big)\\
&\quad h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{1}{a},\frac{1}{d},\frac{1}{f}\Big)\\
&\quad +h\Big(\frac{1}{a},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)\\
&\quad +h\Big(\frac{1}{a},\frac{(1-t)c+td}{cd},\frac{1}{f}\Big)+h\Big(\frac{1}{a},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)\Big] \qquad (4)
\end{aligned}
$$

(iii).

$$
\begin{aligned}
&h\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(1-t)c+(1+t)d}{2cd},\frac{(2-r)e+rf}{2ef}\Big)\\
&\leq \int_0^1\int_0^1\int_0^1 h\Big((1-t_1)\frac{1}{b}+t_1\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big),(1-t_2)\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big)+t_2\frac{1}{c},\\
&\quad (1-t_3)\frac{1}{f}+t_3\Big((1-r)\frac{1}{f}+r\frac{1}{e}\Big)\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3
\end{aligned}
$$

$$\begin{aligned}
\leq\ & \frac{1}{8}\Big[h\Big(\frac{1}{b},\frac{(1-t)c+td}{cd},\frac{1}{f}\Big)+h\Big(\frac{1}{b},\frac{1}{c},\frac{1}{f}\Big)+h\Big(\frac{1}{b},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)\\
& h\Big(\frac{1}{b},\frac{1}{c},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{1}{f}\Big)\\
& +h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)\\
& +h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{c},\frac{1}{f}\Big)+h\Big(\frac{(1-\lambda)a\lambda b}{ab},\frac{1}{c},\frac{(1-r)e+rf}{ef}\Big)\Big] \qquad (5)
\end{aligned}$$

(*iv*).

$$\begin{aligned}
& h\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(2-t)c+td}{2cd},\frac{(1-r)e+(1+r)f}{2ef}\Big)\\
\leq\ & \int_0^1\int_0^1\int_0^1 h\Big((1-t_1)\frac{1}{b}+t_1\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big),(1-t_2)\frac{1}{d}+t_2\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big),\\
& (1-t_3)\Big((1-r)\frac{1}{f}+r\frac{1}{e}\Big)+t_3\frac{1}{e}\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
\leq\ & \frac{1}{8}\Big[h\Big(\frac{1}{b},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{1}{b},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)\\
& h\Big(\frac{1}{b},\frac{(1-t)c+td}{cd},\frac{1}{e}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)\\
& +h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d},\frac{1}{e}\Big)+h\Big(\frac{1}{b},\frac{1}{d},\frac{1}{e}\Big)\\
& +h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)\\
& +h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{1}{e}\Big)\Big] \qquad (6)
\end{aligned}$$

(*v*).

$$\begin{aligned}
& h\Big(\frac{(1-\lambda)a+(\lambda+1)b}{2ab},\frac{(1-t)c+(1+t)d}{2cd},\frac{(2-r)e+rf}{2ef}\Big)\\
\leq\ & \int_0^1\int_0^1\int_0^1 h\Big((1-t_1)\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big)+t_1\frac{1}{a},(1-t_2)\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big)+t_2\frac{1}{c},\\
& (1-t_3)\frac{1}{f}+t_3\Big((1-r)\frac{1}{f}+r\frac{1}{e}\Big)\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
\leq\ & \frac{1}{8}\Big[h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{1}{f}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{c},\frac{1}{f}\Big)\\
& h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{c},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{1}{a},\frac{(1-t)c+td}{cd},\frac{1}{f}\Big)\\
& +h\Big(\frac{1}{a},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{1}{a},\frac{1}{c},\frac{(1-r)e+rf}{ef}\Big)\\
& +h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{1}{a},\frac{1}{c},\frac{1}{f}\Big)\Big] \qquad (7)
\end{aligned}$$

(*vi*).

$$\begin{aligned}
& h\Big(\frac{(1-\lambda)a+(\lambda+1)b}{2ab},\frac{(2-t)c+td}{2cd},\frac{(1-r)e+(1+r)f}{2ef}\Big)\\
\leq\ & \int_0^1\int_0^1\int_0^1 h\Big((1-t_1)\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big)+t_1\frac{1}{a},(1-t_2)\frac{1}{d}+t_2\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big),
\end{aligned}$$

$$(1-t_3)\Big((1-r)\frac{1}{f}+r\frac{1}{e}\Big)+t_3\frac{1}{e}\Big)dt_1dt_2dt_3$$

$$\leq \frac{1}{8}\Big[h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)$$

$$+ h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)$$

$$h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{1}{e}\Big)+h\Big(\frac{1}{a},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)$$

$$+h\Big(\frac{1}{a},\frac{1}{d},\frac{1}{e}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d},\frac{(1-r)e+rf}{ef}\Big)$$

$$+h\Big(\frac{1}{a},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{1}{a},\frac{(1-t)c+td}{cd},\frac{1}{e}\Big)\Big] \tag{8}$$

(*vii*).

$$h\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(1-t)c+(1+t)d}{2cd},\frac{(1-r)e+(1+r)f}{2ef}\Big)$$

$$\leq \int_0^1\int_0^1\int_0^1 h\Big((1-t_1)\frac{1}{b}+t_1\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big),(1-t_2)\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big)+t_2\frac{1}{c},$$

$$(1-t_3)\Big((1-r)\frac{1}{f}+r\frac{1}{e}\Big)+t_3\frac{1}{e}\Big)dt_1dt_2dt_3$$

$$\leq \frac{1}{8}\Big[h\Big(\frac{1}{b},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{1}{b},\frac{1}{c},\frac{(1-r)e+rf}{ef}\Big)$$

$$h\Big(\frac{1}{b},\frac{1}{c},\frac{1}{e}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)$$

$$+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{1}{e}\Big)+h\Big(\frac{1}{b},\frac{(1-t)c+td}{cd},\frac{1}{e}\Big)$$

$$+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{c},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{c},\frac{1}{e}\Big)\Big] \tag{9}$$

(*viii*).

$$h\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2ab},\frac{(1-t)c+(1+t)d}{2cd},\frac{(1-r)e+(1+r)f}{2ef}\Big)$$

$$\leq \int_0^1\int_0^1\int_0^1 h\Big((1-t_1)\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big)+t_1\frac{1}{a},(1-t_2)\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big)+t_2\frac{1}{c},$$

$$(1-t_3)\Big((1-r)\frac{1}{f}+r\frac{1}{e}\Big)+t_3\frac{1}{e}\Big)dt_1dt_2dt_3$$

$$\leq \frac{1}{8}\Big[h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)$$

$$+ \Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{c},\frac{(1-r)e+rf}{ef}\Big)+h\Big(\frac{1}{a},\frac{1}{c},\frac{1}{e}\Big)$$

$$h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{c},\frac{1}{e}\Big)+h\Big(\frac{1}{a},\frac{(1-t)c+td}{cd},\frac{(1-r)e+rf}{ef}\Big)$$

$$+h\Big(\frac{1}{a},\frac{(1-t)c+td}{cd},\frac{1}{e}\Big)+h\Big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd},\frac{1}{e}\Big)$$

$$+h\Big(\frac{1}{a},\frac{1}{c},\frac{(1-r)e+rf}{ef}\Big)\Big] \tag{10}$$

Multiply λtr by (3), $(1-\lambda)tr$ by (4), $\lambda(1-t)r$ by (5), $\lambda t(1-r)$ by (6), $(1-\lambda)(1-t)r$ by (7), $(1-\lambda)t(1-r)$ by (8), $\lambda(1-t)(1-r)$ by (9) and $(1-\lambda)(1-t)(1-r)$ by (10) and adding the resultant, we have

$$\begin{aligned}
&\lambda trg\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(2-r)e+rf}\Big)\\
&+(1-\lambda)trg\Big(\frac{2ab}{(1-\lambda)a+(\lambda+1)b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(2-r)e+rf}\Big)\\
&+\lambda(1-t)rg\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(2-r)e+rf}\Big)\\
&+\lambda t(1-r)g\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(1-r)e+(1+r)f}\Big)\\
&+(1-\lambda)(1-t)rg\Big(\frac{2ab}{(1-\lambda)a+(\lambda+1)b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(2-r)e+rf}\Big)\\
&+(1-\lambda)t(1-r)g\Big(\frac{2ab}{(1-\lambda)a+(\lambda+1)b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(1-r)e+(1+r)f}\Big)\\
&+\lambda(1-t)(1-r)g\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(1-r)e+(1+r)f}\Big)\\
&(1-\lambda)(1-t)(1-r)g\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(1-r)e+(1+r)f}\Big)\\
=\;&\phi(\lambda,t)\\
\leq\;&\int_0^1\int_0^1\int_0^1 f\Big(\frac{ab}{(1-t_1)a+t_1b},\frac{cd}{(1-t_2)c+t_2d},\frac{ab}{(1-t_3)e+t_3f}\Big)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
=\;&\frac{(ab)(cd)(ef)}{(b-a)(d-c)(f-e)}\int_a^b\int_c^d\int_e^f\frac{f(x,y,z)}{x^2y^2z^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
\leq\;&\frac{\lambda tr}{8}\Big[g\Big(b,d,f\Big)+g\Big(b,\frac{cd}{(1-t)c+td},f\Big)\\
&g\Big(b,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)+g\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,f\Big)\\
&+g\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,\frac{ef}{(1-r)e+rf}\Big)+g\Big(b,d,\frac{ef}{(1-r)e+rf}\Big)\\
&+g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},f\Big)\\
&+g\Big(\frac{ab}{(1-\lambda)+a\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)\Big]\\
&+\frac{(1-\lambda)tr}{8}\Big[f\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,f\Big)+g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},f\Big)\\
&g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)+g\Big(a,d,f\Big)\\
&+g\Big(a,d,\frac{ef}{(1-r)e+rf}\Big)+g\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,\frac{ef}{(1-r)e+rf}\Big)\\
&+g\Big(a,\frac{cd}{(1-t)c+td},f\Big)+g\Big(a,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)\Big]\\
&+\frac{\lambda(1-t)r}{8}\Big[g\Big(b,\frac{cd}{(1-t)c+td},f\Big)+g\Big(b,c,f\Big)
\end{aligned}$$

$$g\left(b,c,\frac{ef}{(1-r)e+rf}\right)+g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},f\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)+g\left(b,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},c,f\right)+g\left(\frac{ab}{(1-\lambda)a\lambda b},c,\frac{ef}{(1-r)e+rf}\right)\Bigg]$$
$$+\frac{\lambda t(1-r)}{8}\Bigg[g\left(b,d,\frac{ef}{(1-r)e+rf}\right)+g\left(b,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)$$
$$g\left(b,\frac{cd}{(1-t)c+td},e\right)+g\left(\frac{ab}{(1-\lambda)a+\lambda b},d,\frac{ef}{(1-r)e+rf}\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},d,e\right)+g\left(b,d,e\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},e\right)\Bigg]$$
$$+\frac{(1-\lambda)(1-t)r}{8}\Bigg[g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},f\right)+g\left(\frac{ab}{(1-\lambda)a+\lambda b},c,f\right)$$

$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},c,\frac{ef}{(1-r)e+rf}\right)+g\left(a,\frac{cd}{(1-t)c+td},f\right)$$
$$+g\left(a,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)+g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)$$
$$+h\left(a,c,f\right)+g\left(a,c,\frac{ef}{(1-r)e+rf}\right)\Bigg]$$
$$+\frac{(1-\lambda)t(1-r)}{8}\Bigg[g\left(\frac{ab}{(1-\lambda)a+\lambda b},d,\frac{ef}{(1-r)e+rf}\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)$$
$$g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},e\right)+g\left(a,d,\frac{ef}{(1-r)e+rf}\right)$$
$$+g\left(a,d,e\right)+g\left(\frac{ab}{(1-\lambda)a+\lambda b},d,\frac{ef}{(1-r)e+rf}\right)$$
$$+g\left(a,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)+g\left(a,\frac{cd}{(1-t)c+td},e\right)\Bigg]$$
$$+\frac{\lambda(1-t)(1-r)}{8}\Bigg[g\left(b,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)+g\left(b,c,\frac{ef}{(1-r)e+rf}\right)$$
$$g\left(b,c,e\right)+g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},e\right)+g\left(b,\frac{cd}{(1-t)c+td},e\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},c,\frac{ef}{(1-r)e+rf}\right)$$
$$+g\left(\frac{ab}{(1-\lambda)a+\lambda b},c,e\right)\Bigg]$$

$$
\begin{aligned}
&+\frac{(1-\lambda)(1-t)(1-r)}{8}\Big[g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)\\
&+g\Big(\frac{ab}{(1-\lambda)a+\lambda b},c,\frac{ef}{(1-r)e+rf}\Big)\\
&g\Big(\frac{ab}{(1-\lambda)a+\lambda b},c,e\Big)+g\Big(a,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)\\
&+g\Big(a,\frac{cd}{(1-t)c+td},e\Big)+g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},e\Big)\\
&+g\Big(a,c,\frac{ef}{(1-r)e+rf}\Big)+g\Big(a,c,e\Big)\Big]\\
= &\ \frac{1}{8}\Big[\lambda trg(b,d,f)+(1-\lambda)trg(a,d,f)+\lambda(1-t)rg(b,c,f)+\lambda t(1-r)g(b,d,e)\\
&+(1-\lambda)(1-t)rg(a,c,f)+(1-\lambda)t(1-r)g(a,d,e)+\lambda(1-t)(1-r)g(b,c,e)\\
&+(1-\lambda)(1-t)(1-r)g(a,c,e)\Big]+\frac{1}{8}g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)\\
&+\frac{1}{8}\Big[\lambda rg\Big(b,\frac{cd}{(1-t)c+td},f\Big)+trg\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,f\Big)+\lambda tg\Big(b,d,\frac{ef}{(1-r)e+rf}\Big)\\
&+t(1-\lambda)g\Big(a,d,\frac{ef}{(1-r)e+rf}\Big)+r(1-\lambda)g\Big(a,\frac{cd}{(1-t)c+td},f\Big)\\
&+\lambda(1-t)g\Big(b,c,\frac{ef}{(1-r)e+rf}\Big)+r(1-t)g\Big(\frac{ad}{(1-\lambda)a+\lambda b},c,f\Big)\\
&+\lambda(1-r)g\Big(b,\frac{cd}{(1-t)c+td},e\Big)+t(1-r)g\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,e\Big)\\
&+(1-\lambda)(1-t)g\Big(a,c,\frac{ef}{(1-r)e+rf}\Big)+(1-\lambda)(1-r)g\Big(a,\frac{cd}{(1-t)c+td},e\Big)\\
&+(1-t)(1-r)g\Big(\frac{ab}{(1-\lambda)a+\lambda b},c,e\Big)\Big]+\frac{1}{8}\Big[\lambda g\Big(b,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)\\
&+tg\Big(\frac{ab}{(1-\lambda)a+\lambda b},d,\frac{ef}{(1-r)e+rf}\Big)+rg\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},f\Big)\\
&+(1-r)g\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td},e\Big)+(1-\lambda)g\Big(a,\frac{cd}{(1-t)c+td},\frac{ef}{(1-r)e+rf}\Big)\\
&+(1-t)g\Big(\frac{ab}{(1-\lambda)a+\lambda b},c,\frac{ef}{(1-r)e+rf}\Big)\Big]\\
= &\ \psi(\lambda,t,r). \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi(\lambda,t,r) = &\ \frac{1}{8}\Big[\lambda trg(b,d,f)+(1-\lambda)trg(a,d,f)+\lambda(1-t)rg(b,c,f)+\lambda t(1-r)g(b,d,e)\\
&+(1-\lambda)(1-t)rg(a,c,f)+(1-\lambda)t(1-r)g(a,d,e)+\lambda(1-t)(1-r)g(b,c,e)\\
&+(1-\lambda)(1-t)(1-r)g(a,c,e)+\lambda trg(a,c,e)+\lambda(1-t)rg(a,d,e)\\
&+\lambda(1-t)(1-r)g(a,d,f)+(1-\lambda)trg(b,c,e)+(1-\lambda)t(1-r)g(b,c,f)\\
&+\lambda t(1-r)g(a,c,f)+(1-\lambda)(1-t)rg(b,d,e)+(1-\lambda)(1-t)(1-r)g(b,d,f)\\
&+\lambda trg(b,c,f)+\lambda(1-t)rg(b,d,f)+\lambda trg(a,d,f)+(1-\lambda)trg(b,d,f)\\
&+\lambda trg(b,d,e)+\lambda t(1-r)g(b,d,f)+(1-\lambda)trg(a,d,e)
\end{aligned}
$$

$$
\begin{aligned}
&+(1-\lambda)t(1-r)g(a,d,f)+(1-\lambda)trg(a,c,f)+(1-\lambda)(1-t)rg(a,d,f)\\
&+\lambda(1-t)rg(b,c,e)+\lambda(1-t)(1-r)g(b,c,f)+\lambda(1-t)rg(a,c,f)\\
&+(1-\lambda)(1-t)rg(b,c,f)+\lambda t(1-r)g(b,c,e)+\lambda(1-t)(1-r)g(b,d,e)\\
&+\lambda t(1-r)g(a,d,e)+(1-\lambda)t(1-r)g(b,d,e)+(1-\lambda)(1-t)rg(a,c,e)\\
&+(1-\lambda)(1-t)(1-r)g(a,c,f)+(1-\lambda)t(1-r)g(a,c,e)\\
&+(1-\lambda)(1-t)(1-r)g(a,d,e)+\lambda(1-t)(1-r)g(a,c,e)\\
&+(1-\lambda)(1-t)(1-r)g(b,c,e)+\lambda trg(b,c,e)+\lambda t(1-r)g(b,c,f)\\
&+\lambda(1-t)rg(b,d,e)+\lambda(1-t)(1-r)g(b,d,f)+\lambda trg(a,d,e)+\lambda t(1-r)g(a,d,f)\\
&+(1-\lambda)trg(a,d,f)+(1-\lambda)trg(b,d,e)+(1-\lambda)t(1-r)g(b,d,f)+\lambda trg(a,c,f)\\
&+\lambda(1-t)rg(a,d,f)+(1-\lambda)trg(b,c,f)+(1-\lambda)(1-t)rg(b,d,f)\\
&+\lambda t(1-r)g(a,c,e)+\lambda(1-t)(1-r)g(a,d,e)+(1-\lambda)t(1-r)g(b,c,e)\\
&+(1-\lambda)(1-t)(1-r)g(b,d,e)+(1-\lambda)trg(a,c,e)+(1-\lambda)t(1-r)g(a,c,f)\\
&+(1-\lambda)(1-t)rg(a,d,e)+(1-\lambda)(1-t)(1-r)g(a,d,f)+\lambda(1-t)rg(a,c,e)\\
&+\lambda(1-t)(1-r)g(a,c,f)+(1-\lambda)(1-t)rg(b,c,e)+(1-\lambda)(1-t)(1-r)g(b,c,f)\Big]\\
=\ &\frac{g(a,c,e)+g(a,d,e)+g(a,d,f)+g(b,c,e)+g(b,c,f)+g(a,c,f)+g(b,d,e)+g(b,d,f)}{8}.
\end{aligned}
\tag{12}
$$

Also

$$
\begin{aligned}
\phi(\lambda,t,r) =\ &\lambda trg\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(2-r)e+rf}\Big)\\
&+(1-\lambda)trg\Big(\frac{2ab}{(1-\lambda)a+(\lambda+1)b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(2-r)e+rf}\Big)\\
&+\lambda(1-t)rg\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(2-r)e+rf}\Big)\\
&+\lambda t(1-r)g\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(1-r)e+(1+r)f}\Big)\\
&+(1-\lambda)(1-t)rg\Big(\frac{2ab}{(1-\lambda)a+(\lambda+1)b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(2-r)e+rf}\Big)\\
&+(1-\lambda)t(1-r)g\Big(\frac{2ab}{(1-\lambda)a+(\lambda+1)b},\frac{2cd}{(2-t)c+td},\frac{2ef}{(1-r)e+(1+r)f}\Big)\\
&+\lambda(1-t)(1-r)g\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(1-r)e+(1+r)f}\Big)\\
&(1-\lambda)(1-t)(1-r)g\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(1-t)c+(1+t)d},\frac{2ef}{(1-r)e+(1+r)f}\Big)\\
=\ &\lambda trh\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(2-t)c+td}{2cd},\frac{(2-r)e+rf}{2ef}\Big)\\
&+(1-\lambda)trh\Big(\frac{(1-\lambda)a+(\lambda+1)b}{2ab},\frac{(2-t)c+td}{2cd},\frac{(2-r)e+rf}{2ef}\Big)\\
&+\lambda(1-t)rh\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(1-t)c+(1+t)d}{2cd},\frac{(2-r)e+rf}{2ef}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda t(1-r)h\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(2-t)c+td}{2cd},\frac{(1-r)e+(1+r)f}{2ef}\Big)\\
&+(1-\lambda)(1-t)rh\Big(\frac{(1-\lambda)a+(\lambda+1)b}{2ab},\frac{(1-t)c+(1+t)d}{2cd},\frac{(2-r)e+rf}{2ef}\Big)\\
&+(1-\lambda)t(1-r)h\Big(\frac{(1-\lambda)a+(\lambda+1)b}{2ab},\frac{(2-t)c+td}{2cd},\frac{(1-r)e+(1+r)f}{2ef}\Big)\\
&+\lambda(1-t)(1-r)h\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(1-t)c+(1+t)d}{2cd},\frac{(1-r)e+(1+r)f}{2ef}\Big)\\
&(1-\lambda)(1-t)(1-r)h\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2ab},\frac{(1-t)c+(1+t)d}{2cd},\frac{(1-r)e+(1+r)f}{2ef}\Big)\\
\geq\ & h\Big((1-\lambda)\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2ab}\Big)+\lambda\Big(\frac{(2-\lambda)a+\lambda b}{2ab}\Big),\\
&(1-t)\Big(\frac{(1-t)c+(1+t)d}{2cd}\Big)+t\Big(\frac{(2-t)c+td}{2cd}\Big),\\
&(1-r)\Big(\frac{(1-r)e+(1+r)f}{2ef}\Big)+r\Big(\frac{(2-r)e+rf}{2ef}\Big)\Big)\\
=\ & h\Big(\frac{a+b}{2ab},\frac{c+d}{2cd},\frac{e+f}{2ef}\Big)=g\Big(\frac{2ab}{a+b},\frac{2cd}{c+d},\frac{2ef}{e+f}\Big)
\end{aligned}\tag{13}
$$

From (11)-(13), we obtained the desired inequality □

Corollary 2.3. *Let $g:\Delta=[a,b]\times[c,d]\times[c,d]\subset\mathbb{R}^3\setminus\{0\}\rightarrow\mathbb{R}$ be harmonic convex function of three variables on the rectangular box. Then for any $\lambda,t,r\in[0,1]$, we have*

$$
\begin{aligned}
g\Big(\frac{2ab}{a+b},\frac{2cd}{c+d},\frac{2ef}{e+f}\Big) &\leq \phi(\lambda,t,r)\\
&\leq \frac{(ab)(cd)(ef)}{(b-a)(d-c)(f-e)}\int_a^b\int_c^d\int_e^f\frac{g(x,y,z)}{x^2y^2z^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\leq \psi(\lambda,t,r)\\
&\leq \frac{1}{8}\Big[g(a,c,e)+g(a,d,e)+g(a,d,f)+g(b,c,e)+g(b,c,f)\\
&\quad+g(a,c,f)+g(b,d,e)+g(b,d,f)\Big]
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(\lambda,t,r) &= \frac{1}{8}\Big[g\Big(\frac{4ab}{3a+b},\frac{4cd}{3c+d},\frac{4ef}{3e+f}\Big)+g\Big(\frac{4ab}{a+3b},\frac{4cd}{3c+d},\frac{4ef}{3e+f}\Big)\\
&+g\Big(\frac{4ab}{3a+b},\frac{4cd}{c+3d},\frac{4ef}{3e+f}\Big)+g\Big(\frac{4ab}{3a+b},\frac{4cd}{3c+d},\frac{4ef}{e+3f}\Big)\\
&+g\Big(\frac{4ab}{a+3b},\frac{4cd}{c+3d},\frac{4ef}{3e+f}\Big)+g\Big(\frac{4ab}{a+3b},\frac{4cd}{3c+d},\frac{4ef}{e+3f}\Big)\\
&+g\Big(\frac{4ab}{3a+b},\frac{4cd}{c+3d},\frac{4ef}{e+3f}\Big)+g\Big(\frac{4ab}{a+3b},\frac{4cd}{c+3d},\frac{4ef}{e+3f}\Big).
\end{aligned}
$$

and

$$
\psi(\lambda,t,r) = \frac{1}{64}\Big[g(b,d,f)+g(a,d,f)+g(b,c,f)+g(b,d,e)+g(a,c,f)
$$

$$+g(a,d,e)+g(b,c,e)+g(a,c,e)\Big]+\frac{1}{8}g\Big(\frac{2ab}{a+b},\frac{2cd}{c+d},\frac{2ef}{e+f}\Big)$$
$$+\frac{1}{32}\Big[g\Big(b,\frac{2cd}{c+d},f\Big)+g\Big(\frac{2ab}{a+b},d,f\Big)+g\Big(b,d,\frac{2ef}{e+f}\Big)+g\Big(a,d,\frac{2ef}{e+f}\Big)$$
$$+g\Big(a,\frac{2cd}{c+d},f\Big)+g\Big(b,c,\frac{2ef}{e+f}\Big)+g\Big(\frac{2ad}{a+b},c,f\Big)+g\Big(b,\frac{2cd}{c+d},e\Big)$$
$$+g\Big(\frac{2ab}{a+b},d,e\Big)+g\Big(a,c,\frac{2ef}{e+f}\Big)+g\Big(a,\frac{2cd}{c+d},e\Big)+g\Big(\frac{2ab}{a+b},c,e\Big)\Big]$$
$$+\frac{1}{32}\Big[g\Big(b,\frac{2cd}{c+d},\frac{2ef}{e+f}\Big)+g\Big(\frac{2ab}{a+b},d,\frac{2ef}{e+f}\Big)+g\Big(\frac{2ab}{a+b},\frac{2cd}{c+d},f\Big)$$
$$+g\Big(\frac{2ab}{a+b},\frac{2cd}{c+d},e\Big)+g\Big(a,\frac{2cd}{c+d},\frac{2ef}{e+f}\Big)+g\Big(\frac{2ab}{a+b},c,\frac{2ef}{e+f}\Big)\Big].$$

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