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Strong and weak convergence theorems for solutions of equations of Hammerstein-type

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Abstract. In this paper, it is our aim in this paper to introduce a new iterative algorithm for approximation of a solution of an equation of Hammerstein-type. The proposed scheme does not involve computation of inverse of operators under study; it does not involve passing through computation of a certain set that must contain a solution of the equation of Hammerstein-type before convergence takes place. The proposed scheme requires only one parameter satisfying verifiable mild conditions. Moreover, the mappings involved are neither defined on compact subset of the space under study, nor assumed to be angle bounded. Our theorems complement several results that have been obtained in this direction.

1. Introduction.

Let E be a real linear space, and let $F, K : E \to E$ be two mappings such that the range, $R(F)$, of F equals the domain, $D(K)$, of K. For $u \in D(F)$, an equation of Hammerstein-type is of the form

$$u + KFu = h \Leftrightarrow (I + KF)u = h, \tag{1}$$

where I is identity operator. The Operator $I + KF$ are called Hammerstein operator.

Equation (1) has extensively been studied by many authors (see e.g., [1] - [5], [7] - [9], [11], [19], [24], [32], [34] - [36], [41], [46], [47]). Perhaps, due to the fact that several physically significant problems such as problems arising in differential equations (e.g., the problem of forced oscillations of finite amplitude of a pendulum (see [41, Chapter IV])) can be transformed into an operator equations of the form (1). The study of equations of Hammerstein type has been of increasing research interest (see e.g., Dolph [33], Hammerstein [35], Pascali and Sburban [46]).

If in (1), the operator K is the identity operator, then (1) reduces to an equation of the form

$$u + Fu = h. \tag{2}$$

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Existence of solutions of (1) and (2) have been studied by various authors and several existence results have been generated (see for example, [1] - [5], [7] - [9], [24], [32], [34] - [37], [46] - [48], [52]). On the other hand, on assumption that solution of (2) exists, several researchers had dwelled on generation of iterative algorithms for approximation of solutions of the operator equation (2) (see e.g., [5], [12] - [16], [20], [22], [26] - [30], [38], [39], [48] - [53], [54]). In some cases, iteration process of Mann and Ishikawa types and their variants were employed (see e.g., [5], [12] - [16], [19], [20], [26] - [30]).

It is worthy to note, however, that only few authors (see e.g., [4, 11, 22, 26, 41]) have endeavored to address the issue of iterative approximation of solutions of (1) if the solutions are known to exist (or even on assumption that solutions exist). This is, perhaps, because methods of finding solutions of (2) do not easily carry over to (1). Part of the difficulty remains the fact that the composition of two monotone operators need not be monotone (see e.g. Chidume [10]).

In the special case where the operators are defined on subsets D of E which are compact or more generally,*angle bounded* (see e.g. Brežis and Browder [1] or Browder [6] for definition), Brežis and Browder [1] proved strong convergence of a sequence generated by suitably defined *Galerkin approximaiton algorithm* to a solution of (1) (see also Brežis and Browder [3]).

Galerkin approximaiton method is implicit in nature and too difficult to use in real life applications. For this reason, many authors have studied explicit iterative methods for approximation of solutions of equations of Hammerstein type (see for example [17, 18, 23, 31, 42]). In [31], for example, the authors first showed that their iterative algorithms are bounded; then went further to establish existence of a set Ω such that if a solution u^* of equation of Hammerstein-type

$$u + KFu = 0 \tag{3}$$

belongs to Ω, then the sequence generated by their scheme converge to u^*. It is worthy to note that the idea behind this method of proof is not that novel since the assumption that the set Ω contains a solution of (3) to guarantee the convergence of the sequence generated by the algorithm studied in [31] to u^* is rather too strong.

In [43], Ofoedu and Onyi provided an iterative algorithm for aproximation of solution of (3) without construction of a set Ω that must contain its solution(s) before convergence result is achieved. It is worthy to note, at this juncture, that the scheme introduced in [43] is loaded with iterative parameters to enhance convergence. Though introduction of parameters to enhance convergence of a scheme is not bad, having them in excess could be time consuming.

Motivated by the results of the authors mentioned above, it is our aim in this paper to introduce a new iterative algorithm for approximation of a solution of (3). The iterative algorithm does not involve K^{-1}, it does not involve passing through computation of a certain set Ω that must contain a solution of (3) before convergence takes place. The proposed scheme requires only one parameter satisfying verifiable mild conditions. Moreover, the mappings K and F are neither defined on compact subset of H nor assumed to be angle bounded on H. Our theorems complement several results in the literature.

2. Preliminaries.

Let H be a real Hilbert space with inner product $\left\langle .,. \right\rangle_H$ and induced norm $||.||_H$. An operator A with domain $D(A)$, and range $R(A)$, in H is called *monotone* if for all $x, y \in D(A)$, we have that $\left\langle Ax - Ay, x - y \right\rangle_H \geq 0$. The operator A is called *m-strongly monotone* if there exists a constant $m > 0$ such that for all $x, y \in D(A)$, $\left\langle Ax - Ay, x - y \right\rangle_H \geq m||x - y||_H^2$, while the mapping A is said to be *Lipschitz or Lipschitz continuous* if there exists a constant $L \geq 0$ such that for all $x, y \in D(A)$, $||Ax - Ay||_H \leq L||x - y||_H$.

The following Lemmas (whose proofs are immediate) shall be needed in the sequel:

Lemma 2.1. *Let H be a real Hilbert space. Let $F, K : H \to H$ be strongly monotone mappings with monotonicity constants m_1 and m_2 respectively and with $D(F) = D(K) = H$. Let $\tilde{A} : H \times H \to H \times H$ be defined by $\tilde{A}(u, v) = (Fu - v, u + Kv)$ for all $(u, v) \in H \times H$, then $\tilde{A}$ is $\min\{m_1, m_2\}-$strongly monotone.*

Lemma 2.2. *Let H be a real Hilbert space. Let $F, K : H \to H$ be monotone mappings with $D(F) = D(K) = H$. Let $\tilde{A} : H \times H \to H \times H$ be defined by $\tilde{A}(u, v) = (Fu - v, u + Kv)$ for all $(u, v) \in H \times H$, then $\tilde{A}$ is monotone.*

Lemma 2.3. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz mappings with $D(F) = D(K) = H$. Let $\tilde{A} : H \times H \to H \times H$ be defined by $\tilde{A}(u, v) = (Fu - v, u + Kv)$ for all $(u, v) \in H \times H$, then $\tilde{A}$ is Lipschitz.*

Lemma 2.4. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz mappings with $D(F) = D(K) = H$. Let $\tilde{A} : H \times H \to H \times H$ be defined by $\tilde{A}(u, v) = (Fu - v, u + Kv)$ for all $(u, v) \in H \times H$. A point $u^* \in H$ is a solution of the equation Hammerstein-type (3) if and only if $(u^*, Fu^*) \in Z(\tilde{A})$, and $Z(\tilde{A}) = \{z \in H \times H : \tilde{A}z = 0\}$.*

Lemma 2.5. Let H be a real Hilbert space and let $u, v \in H$, then

$$\|2u - v\|^2 = 2\|u\|^2 - \|v\|^2 + 2\|u - v\|^2. \tag{4}$$

We shall also make use of the following Lemmas:

Lemma 2.6. *(Ofoedu* et al. *[44]) Let $\{a_n\}_{n\geq1}$ and $\{b_n\}_{n\geq1}$ be two real sequences such that $a_n \neq 0$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = a^*$, for some $a^* \neq 0$. Suppose $\lim_{n\to\infty} a_n b_n = 0$, then $\lim_{n\to\infty} b_n = 0$.*

Lemma 2.7. *(Ofoedu* et al. *[44])* Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative terms, let $\{\eta_n\}_{n=1}^{\infty}$ be a sequence in $]0, 1[$ and $\beta \in]0, 1[$ and let $\gamma_n = \frac{1}{2}\Big(1 - 2\eta_n + \{1 + 4\eta_n^2\}^{\frac{1}{2}}\Big)$ and $\delta_n = \frac{1}{2}\Big(-1 + 2\eta_n + \{1 + 4\eta_n^2\}^{\frac{1}{2}}\Big)$ $\forall\, n \in \mathbb{N}$, then the following are equivalent:

1. $a_{n+1} + \delta_n a_n + b_{n+1} \leq \gamma_n(a_n + \delta_n a_{n-1}) + \beta b_n$.
2. $a_{n+1} + b_{n+1} \leq (1 - 2\eta_n)a_n + \eta_n a_{n-1} + \beta b_n$

Lemma 2.8. (Opial [45]) Let $\{x_n\}$ be a sequence in H such that $x_n \rightharpoonup x$, then for all $y \neq x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

where $x_n \rightharpoonup x$ as $n \to \infty$ if and only if $\{x_n\}_{n\geq1}$ converges weakly to x.

3. Main Result

For the remaining part of this paper, H is a real Hilbert space. $H \times H$ is endowed with the inner product $\langle z_1, z_2\rangle_{H\times H} = \langle u_1, v_1\rangle_H + \langle u_2, v_2\rangle_H$. Thus for all $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2) \in H \times H$, this inner product clearly induces a norm on $H \times H$ given by $\|z\|^2_{H\times H} = \|u\|^2_H + \|v\|^2_H$ for all $z = (u, v) \in H \times H$.

3.1. Strong convergence theorem for m−strongly monotone mapping

Theorem 3.1. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz strongly monotone mappings with monotonicity constants m_1 and m_2 respectively. Let the sequence $\{(\mu_n, \zeta_n)\}_{n\geq 0}$ in $H\times H$ be generated iteratively from arbitrary $\mu_1, \mu_0, \zeta_1, \zeta_0 \in H$ by*

$$\begin{aligned} \mu_{n+1} &= \mu_n - \lambda_n(Fx_n - y_n) \\ \zeta_{n+1} &= \zeta_n - \lambda_n(Ky_n + x_n), \end{aligned} \tag{5}$$

where, $x_n = 2\mu_n - \mu_{n-1}$, $y_n = 2\zeta_n - \zeta_{n-1}$, $\{\lambda_n\}_{n\geq 1}$ is a decreasing sequence in $]a, b[$ for some $a, b \in \Big]0, \min\left\{\frac{1}{4m^*}, \frac{\sqrt{2}}{4L}\right\}\Big[$, where $m^* \in]0, \min\{m_1, m_2\}]$ is fixed, L is the Lipschitz constant of the mapping $\tilde{A} : H\times H \to H\times H$ given by $\tilde{A}(u, v) = (Fu - v, u + Kv)$. Suppose that the equation of Hammerstein-type (3) *has a solution $u^* \in H$, then $\{\mu_n\}_{n\geq 1}$ converges strongly to u^*.*

Proof. Observe that by Lemma 2.1, the mapping $\tilde{A} : H \times H \to H \times H$ given by $\tilde{A}(u, v) = (Fu - v, u + Kv)$ for all $(u, v) \in H \times H$ is $\min\{m_1, m_2\}$−strongly monotone mapping. Observe that if we let $\overline{\omega}_n := (\mu_n, \zeta_n)$ and $\psi_n := (u_n, v_n)\ \forall\, n \geq 1$, then we obtain that (5) is equivalent to

$$\begin{aligned} \overline{\omega}_{n+1} &= \overline{\omega}_n - \lambda_n \tilde{A}\psi_n, \\ \psi_{n+1} &= 2\overline{\omega}_{n+1} - \overline{\omega}_n, \quad \forall\, n \geq 0, \end{aligned} \tag{6}$$

Let u^* be the solution of (3) and let $\overline{z} = (u^*, Fu^*)$, then by Lemma 2.4, $\overline{z} \in Z(\tilde{A})$; and since $\tilde{A}$ is strongly monotone, $Z(\tilde{A}) = \{\overline{z}\}$. So, using (6), we obtain that

$$\begin{aligned} \|\overline{\omega}_{n+1} - \overline{z}\|^2_{H\times H} &= \|\overline{\omega}_n - \lambda_n\tilde{A}\psi_n - \overline{z}\|^2_{H\times H} - \|\overline{\omega}_n - \lambda_n\tilde{A}\psi_n - \overline{\omega}_{n+1}\|^2_{H\times H} \\ &= \|\overline{\omega}_n - \overline{z}\|^2_{H\times H} + \left\langle \overline{\omega}_n - \overline{z}, -\lambda_n\tilde{A}\psi_n \right\rangle_{H\times H} \\ &\quad - \left\langle \lambda_n\tilde{A}\psi_n, \overline{\omega}_n - \overline{z} \right\rangle_{H\times H} + \left\langle \lambda_n\tilde{A}\psi_n, \lambda_n\tilde{A}\psi_n \right\rangle_{H\times H} \\ &\quad - \|\overline{\omega}_n - \overline{\omega}_{n+1}\|^2_{H\times H} - \left\langle \overline{\omega}_n - \overline{\omega}_{n+1}, -\lambda_n\tilde{A}\psi_n \right\rangle_{H\times H} \\ &\quad + \left\langle \lambda_n\tilde{A}\psi_n, \overline{\omega}_n - \overline{\omega}_{n+1} \right\rangle_{H\times H} - \left\langle \lambda_n\tilde{A}\psi_n, \lambda_n\tilde{A}\psi_n \right\rangle_{H\times H} \\ &= \|\overline{\omega}_n - \overline{z}\|^2_{H\times H} - \|\overline{\omega}_n - \overline{\omega}_{n+1}\|^2_{H\times H} \\ &\quad - 2\lambda_n \left\langle \tilde{A}\psi_n, \overline{\omega}_n - \overline{z} \right\rangle_{H\times H} + 2\lambda_n \left\langle \tilde{A}\psi_n, \overline{\omega}_n - \overline{\omega}_{n+1} \right\rangle_{H\times H} \\ &= \|\overline{\omega}_n - \overline{z}\|^2_{H\times H} - \|\overline{\omega}_n - \overline{\omega}_{n+1}\|^2_{H\times H} - 2\lambda_n \left\langle \tilde{A}\psi_n, \overline{\omega}_{n+1} - \overline{z} \right\rangle_{H\times H}. \end{aligned} \tag{7}$$

Also by Lemma 2.5, we have that

$$\begin{aligned} \|\psi_n - \overline{z}\|^2_{H\times H} &= 2\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} - \|\overline{\omega}_{n-1} - \overline{z}\|^2_{H\times H} + 2\|\overline{\omega}_n - \overline{\omega}_{n-1}\|^2_{H\times H} \\ &\geq 2\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} - \|\overline{\omega}_{n-1} - \overline{z}\|^2_{H\times H}. \end{aligned}$$

From this, $\min m_1, m_2$−strong monotonicity of $\tilde{A}$ and the fact that $\tilde{A}\overline{z} = 0$, we obtain that with $m^* \in (0, \min\{m_1, m_2\}]$,

$$\begin{aligned} &2\lambda_n \left(\left\langle \tilde{A}\psi_n, \psi_n - \overline{z} \right\rangle_{H\times H} - m^*\Big(2\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} - \|\overline{\omega}_{n-1} - \overline{z}\|^2_{H\times H}\Big) \right) \\ &\geq 2\lambda_n \Big(\left\langle \tilde{A}\psi_n, \psi_n - \overline{z} \right\rangle_{H\times H} - \min\{m_1, m_2\}\|\psi_n - \overline{z}\|^2_{H\times H} \Big) \geq 0 \end{aligned} \tag{8}$$

So, using (8) in (7), we have that

$$\begin{aligned}
\|\overline{\omega}_{n+1}-\overline{z}\|^2_{H\times H} &\leq \|\overline{\omega}_n-\overline{z}\|^2_{H\times H}-\|\overline{\omega}_{n+1}-\overline{\omega}_n\|^2_{H\times H}\\
&\quad -2\lambda_n\left\langle \tilde{A}\psi_n,\overline{\omega}_{n+1}-\overline{z}\right\rangle_{H\times H}+2\lambda_n\left\langle \tilde{A}\psi_n,\psi_n-\overline{z}\right\rangle_{H\times H}\\
&\quad -2\lambda_n m^*\Big(2\|\overline{\omega}_n-\overline{z}\|^2_{H\times H}-\|\overline{\omega}_{n-1}-\overline{z}\|^2_{H\times H}\Big)\\
&= \|\overline{\omega}_n-\overline{z}\|^2_{H\times H}-\|\overline{\omega}_{n+1}-\overline{\omega}_n\|^2_{H\times H}+2\lambda_n\left\langle \tilde{A}\psi_n,\psi_n-\overline{\omega}_{n+1}\right\rangle_{H\times H}\\
&\quad -2\lambda_n m^*\Big(2\|\overline{\omega}_n-\overline{z}\|^2_{H\times H}-\|\overline{\omega}_{n-1}-\overline{z}\|^2_{H\times H}\Big)\\
&= (1-4\lambda_n m^*)\|\overline{\omega}_n-\overline{z}\|^2_{H\times H}-\|\overline{\omega}_{n+1}-\overline{\omega}_n\|^2_{H\times H}\\
&\quad +2\lambda_n m^*\|\overline{\omega}_{n-1}-\overline{z}\|^2_{H\times H}+2\lambda_n\left\langle \tilde{A}\psi_n-\tilde{A}\psi_{n-1},\psi_n-\overline{\omega}_{n+1}\right\rangle_{H\times H}\\
&\quad +2\lambda_n\left\langle \tilde{A}\psi_{n-1},\psi_n-\overline{\omega}_{n+1}\right\rangle_{H\times H}. \qquad (9)
\end{aligned}$$

Next, observe that

$$\left\langle \overline{\omega}_n-\overline{\omega}_{n-1}+\lambda_{n-1}\tilde{A}\psi_{n-1},\overline{\omega}_n-\overline{\omega}_{n+1}\right\rangle_{H\times H}=0 \qquad (10)$$

and

$$\left\langle \overline{\omega}_n-\overline{\omega}_{n-1}+\lambda_{n-1}\tilde{A}\psi_{n-1},\overline{\omega}_n-\overline{\omega}_{n-1}\right\rangle_{H\times H}=0. \qquad (11)$$

Adding (10) and (11) gives

$$\left\langle \overline{\omega}_n-\overline{\omega}_{n-1}+\lambda_{n-1}\tilde{A}\psi_{n-1},\psi_n-\overline{\omega}_{n+1}\right\rangle_{H\times H}=0. \qquad (12)$$

It immediately follows from (12) that

$$\begin{aligned}
2\lambda_{n-1}\left\langle \tilde{A}\psi_{n-1},\psi_n-\overline{\omega}_{n+1}\right\rangle_{H\times H} &= 2\left\langle \overline{\omega}_n-\overline{\omega}_{n-1},\overline{\omega}_{n+1}-\psi_n\right\rangle_{H\times H}\\
&= 2\left\langle \psi_n-\overline{\omega}_n,\overline{\omega}_{n+1}-\psi_n\right\rangle_{H\times H}\\
&= \|\overline{\omega}_{n+1}-\overline{\omega}_n\|^2_{H\times H}\\
&\quad -\|\overline{\omega}_n-\psi_n\|^2-\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}. \qquad (13)
\end{aligned}$$

It is easy to see that

$$2\lambda_n\left\langle \tilde{A}\psi_{n-1},\psi_n-\overline{\omega}_{n+1}\right\rangle_{H\times H}=\frac{2\lambda_n\lambda_{n-1}}{\lambda_{n-1}}\left\langle \tilde{A}\psi_{n-1},\psi_n-\overline{\omega}_{n+1}\right\rangle_{H\times H} \qquad (14)$$

Thus, we obtain from (13) and inequality (14) that

$$\begin{aligned}
2\lambda_n\left\langle \tilde{A}\psi_{n-1},\psi_n-\overline{\omega}_{n+1}\right\rangle_{H\times H} &= \frac{\lambda_n}{\lambda_{n-1}}\Big[\|\overline{\omega}_{n+1}-\overline{\omega}_n\|^2_{H\times H}\\
&\quad -\|\overline{\omega}_n-\psi_n\|^2_{H\times H}-\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\Big]. \qquad (15)
\end{aligned}$$

Furthermore, observe that since A is L-Lipschitz continuous mapping, we have that

$$\begin{aligned}
2\lambda_n \left\langle \tilde{A}\psi_n - \tilde{A}\psi_{n-1}, \psi_n - \overline{\omega}_{n+1} \right\rangle_{H\times H} &\leq 2\lambda_n L\|\psi_n - \psi_{n-1}\|_{H\times H}\|\overline{\omega}_{n+1} - \psi_n\|_{H\times H} \\
&\leq \lambda_n L\left(\frac{1}{\sqrt{2}}\|\psi_n - \psi_{n-1}\|^2_{H\times H} + \sqrt{2}\|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H}\right) \\
&\leq \lambda_n L\Big[\frac{1}{\sqrt{2}}(\|\psi_n - \overline{\omega}_n\|^2_{H\times H} + 2\|\psi_n - \overline{\omega}_n\|_{H\times H}\|\overline{\omega}_n - \psi_{n-1}\|_{H\times H} \\
&\quad +\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H}) + \sqrt{2}\|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H}\Big] \\
&\leq \lambda_n L\Big[\frac{1}{\sqrt{2}}\Big(\|\psi_n - \overline{\omega}_n\|^2_{H\times H} + (1+\sqrt{2})\|\psi_n - \overline{\omega}_n\|^2_{H\times H} \\
&\quad +(\sqrt{2}-1)\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H} + \|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H}\Big) \\
&\quad + \sqrt{2}\|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H}\Big] \\
&= \lambda_n L(1+\sqrt{2})\|\psi_n - \overline{\omega}_n\|^2_{H\times H} + \lambda_n L\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H} \\
&\quad + \sqrt{2}\lambda_n L\|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H} \qquad (16)
\end{aligned}$$

Using (15) and (16), we deduce from (9) that

$$\begin{aligned}
\|\overline{\omega}_{n+1} - \overline{z}\|^2_{H\times H} \leq\ & (1 - 4\lambda_n m^*)\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} - \|\overline{\omega}_{n+1} - \overline{\omega}_n\|^2_{H\times H} \\
&+2\lambda_n m^*\|\overline{\omega}_{n-1} - \overline{z}\|^2_{H\times H} + \lambda_n L(1+\sqrt{2})\|\psi_n - \overline{\omega}_n\|^2_{H\times H} \\
&+\lambda_n L\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H} + \sqrt{2}\lambda_n L\|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H} \\
&+\frac{\lambda_n}{\lambda_{n-1}}\Big[\|\overline{\omega}_{n+1} - \overline{\omega}_n\|^2_{H\times H} - \|\overline{\omega}_n - \psi_n\|^2_{H\times H} - \|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H}\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\overline{\omega}_{n+1} - \overline{z}\|^2_{H\times H} \leq\ & (1 - 4\lambda_n m^*)\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} + \left(\frac{\lambda_n}{\lambda_{n-1}} - 1\right)\|\overline{\omega}_{n+1} - \overline{\omega}_n\|^2_{H\times H} \\
&+2\lambda_n m^*\|\overline{\omega}_{n-1} - z\|^2_{H\times H} + \left(\lambda_n L(1+\sqrt{2}) - \frac{\lambda_n}{\lambda_{n-1}}\right)\|\psi_n - \overline{\omega}_n\|^2_{H\times H} \\
&+\left(\sqrt{2}\lambda_n L - \frac{\lambda_n}{\lambda_{n-1}}\right)\|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H} + \lambda_n L\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H} \\
\leq\ & (1 - 4\lambda_n m^*)\|\overline{\omega}_n - z\|^2_{H\times H} + 2\lambda_n m^*\|\overline{\omega}_{n-1} - \overline{z}\|^2_{H\times H} \\
&+\left(\lambda_n L(1+\sqrt{2}) - \frac{\lambda_n}{\lambda_{n-1}}\right)\|\psi_n - \overline{\omega}_n\|^2_{H\times H} \\
&+\left(\sqrt{2}\lambda_n L - \frac{\lambda_n}{\lambda_{n-1}}\right)\|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H} + \lambda_n L\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H}.
\end{aligned}$$

But $\lambda_n L(1+\sqrt{2}) - \frac{\lambda_n}{\lambda_{n-1}} \leq 0$. Therefore,

$$\begin{aligned}
\|\overline{\omega}_{n+1} - \overline{z}\|^2_{H\times H} &+ \left(\frac{\lambda_n - \sqrt{2}\lambda_n\lambda_{n-1}L}{\lambda_{n-1}}\right)\|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H} \\
&\leq (1 - 4\lambda_n m^*)\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} + 2\lambda_n m^*\|\overline{\omega}_{n-1} - \overline{z}\|^2_{H\times H} + \lambda_n L\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H} \\
&= (1 - 4\lambda_n m^*)\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} + 2\lambda_n m^*\|\overline{\omega}_{n-1} - \overline{z}\|^2_{H\times H} + \lambda_n L\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H}
\end{aligned}$$

$$
\begin{aligned}
&\le (1-4\lambda_n m^*)\|\overline{\omega}_n-\overline{z}\|^2_{H\times H}+2\lambda_n m^*\|\overline{\omega}_{n-1}-\overline{z}\|^2_{H\times H}+\lambda_{n-1}L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}\\
&\le (1-4\lambda_n m^*)\|\overline{\omega}_n-\overline{z}\|^2_{H\times H}+2\lambda_n m^*\|\overline{\omega}_{n-1}-\overline{z}\|^2_{H\times H}\\
&\quad+\max\left\{\frac{\lambda_{n-1}\lambda_{n-2}L}{\lambda_{n-1}-\sqrt{2}\lambda_{n-1}\lambda_{n-2}L},\frac{1}{2}\right\}\\
&\quad\times\left(\frac{\lambda_{n-1}-\sqrt{2}\lambda_{n-1}\lambda_{n-2}L}{\lambda_{n-2}}\right)\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H} \qquad (17)
\end{aligned}
$$

Since $\lambda_n < \frac{\sqrt{2}}{4L}$, it follows from (17) that

$$
\begin{aligned}
\|\overline{\omega}_{n+1}-\overline{z}\|^2_{H\times H} &+ \left(\frac{\lambda_n-\sqrt{2}\lambda_n\lambda_{n-1}L}{\lambda_{n-1}}\right)\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\\
&\le (1-4\lambda_n m^*)\|\overline{\omega}_n-\overline{z}\|^2_{H\times H}+2\lambda_n m^*\|\overline{\omega}_{n-1}-\overline{z}\|^2_{H\times H}\\
&\quad+\frac{\sqrt{2}}{2}\left(\frac{\lambda_{n-1}-\sqrt{2}\lambda_{n-1}\lambda_{n-2}L}{\lambda_{n-2}}\right)\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H} \qquad (18)
\end{aligned}
$$

Now, set

$$
\begin{aligned}
a_n &= \|\overline{\omega}_n-\overline{z}\|^2_{H\times H},\;\; \beta=\frac{\sqrt{2}}{2},\;\; \eta_n=2\lambda_n m_1,\;\; \text{and}\\
b_n &= \left(\frac{\lambda_{n-1}-\sqrt{2}\lambda_{n-1}\lambda_{n-2}L}{\lambda_{n-2}}\right)\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H},
\end{aligned}
$$

then (18) becomes

$$
a_{n+1}+b_{n+1}\le(1-2\eta_n)a_n+\eta_n a_{n-1}+\beta b_n,\quad \forall\, n\in\mathbb{N}. \qquad (19)
$$

Thus, if we set

$$
\gamma_n=\frac{1-4\lambda_n m^*+\sqrt{1+16\lambda_n^2 m^{*2}}}{2}=\frac{1-2\eta_n+\sqrt{1+4\eta_n^2}}{2}.
$$

and

$$
\delta_n=\frac{-1+4\lambda_n m^*+\sqrt{1+16\lambda_n^2 m^{*2}}}{2}=\frac{-1+2\eta_n+\sqrt{1+4\eta_n^2}}{2},
$$

then by Lemma 2.7, (19) is equivalent to

$$
a_{n+1}+\delta_n a_n+b_{n+1}\le\gamma_n(a_n+\delta_n a_{n-1})+\beta b_n. \qquad (20)
$$

Moreover, observe that $\lambda_n \in \left]0,\min\left\{\frac{1}{4m^*},\frac{\sqrt{2}}{4L},\right\}\right[$ implies that

$$
\lambda_n<\frac{1}{4m_1}\iff\frac{\sqrt{2}}{2}<\frac{1-4\lambda_n m^*+\sqrt{1+16\lambda_n^2 m^{*2}}}{2},
$$

that is, $\gamma_n > \beta$, so that (20) gives

$$
\begin{aligned}
a_{n+1}+\delta_n a_n+b_{n+1} &\le \gamma_n(a_n+\delta_n a_{n-1}+b_n)\\
&= \gamma_n(a_n+\delta_{n-1}a_{n-1}+b_n)+\gamma_n(\delta_n-\delta_{n-1})a_{n-1}. \qquad (21)
\end{aligned}
$$

Since $\{\lambda_n\}$ is decreasing, we have that $\gamma_n(\delta_n-\delta_{n-1})a_{n-1} \le 0$, and since the function f given by $f(x) = \frac{1-2x+\sqrt{1+4x^2}}{2}$ is strictly decreasing on $[0,1]$ with $f(0) = 1$, we obtain that since $0 < 2am_1 < 2\lambda_n m_1 < 1$,

$$\gamma_n < \frac{1-4am^* + \sqrt{1+16a^2m^{*2}}}{2} := \kappa_0 < 1.$$

So we obtain from (21) that

$$\begin{aligned} a_{n+1} + \delta_n a_n + b_{n+1} &\le \gamma_n(a_n + \delta_{n-1}a_{n-1} + b_n) \\ &\le \kappa_0(a_n + \delta_{n-1}a_{n-1} + b_n) \\ &\vdots \\ &\le \kappa_0^n(a_1 + \delta_0 a_0 + b_1). \end{aligned}$$

This implies that $a_{n+1} \le \kappa_0^n M$, where $M = a_1 + \delta_0 a_0 + b_1 > 0$. Hence, $\{\overline{\omega}_n\}_{n=1}^{\infty}$ converges strongly to $\overline{z}$. But $\overline{\omega}_n = (\mu_n, \zeta_n)\forall n \in \mathbb{N}$ and $\overline{z} = (u^*, Fu^*)$. Hence, $\mu_n \to u^*$ and $\zeta_n \to Fu^*$ as $n \to \infty$. This completes the proof.

Remark 3.2. *We note that Theorem 3.1 (which gives a strong convergence result) holds for the class of* $\min\{m_1, m_2\}$*-strongly monotone mappings which is a proper subclass of class of monotone mappings.*

Question: *Does Theorem 3.1 hold for larger class of monotone mappings?*

Answer: *For larger class of monotone mappings, weak convergence result suffices. The next theorem affirms this position.*

Theorem 3.3. *Let H be a real Hilbert space, H. Let $F, K : H \to H$ be Lipschitz monotone mappings. Let the sequence $\{(\mu_n, \zeta_n)\}_{n\ge1}$ in $H \times H$ be generated iteratively from arbitrary $\mu_1, \mu_0, \zeta_1, \zeta_0 \in H$ by* (5) *but with $\{\lambda_n\}_{n=1}^{\infty}$ being a monotone decreasing sequence in $[a,b]$, for some $a, b \in \left]0, \frac{\sqrt{2}-1}{L}\right[$, where L is the Lipschitz constant of the mapping $\tilde{A} : H \times H \to H \times H$ given by $\tilde{A}(u,v) = (Fu - v, u + Kv)$. Suppose the solution set, S, of* (3) *is nonempty, then $\{\mu_n\}_{n\ge1}$ converges weakly to some $u^* \in S$.*

Proof. By Lemma 2.2 the mapping $\tilde{A} : H\times H \to H\times H$ given by $\tilde{A}(u,v) = (Fu-v, u+Kv)$ for all $(u,v) \in H\times H$ is monotone mapping and by Lemma 2.4, $u^* \in S$ if and only if $(u^*, Fu^*) \in Z(\tilde{A})$. We note immediately that in the case at hand, S is not necessarily a singleton. Thus, $Z(\tilde{A})$ may contain more than one point.

Fix $u^* \in S$, then with $\overline{z} = (u^*, Fu^*) \in Z(\tilde{A})$ and using the same transformation as in the proof of Theorem 3.1, we obtain from (7) that

$$\begin{aligned} \|\overline{\omega}_{n+1} - \overline{z}\|_{H\times H}^2 &\le \|\overline{\omega}_n - \overline{z}\|_{H\times H}^2 - \|\overline{\omega}_n - \overline{\omega}_{n+1}\|_{H\times H}^2 \\ &\quad -2\lambda_n\left\langle \tilde{A}\psi_n, \overline{\omega}_{n+1} - \overline{z}\right\rangle_{H\times H}. \end{aligned} \tag{22}$$

Since $\tilde{A}$ is monotone, we have that $2\lambda_n\left\langle \tilde{A}\psi_n, \psi_n - \overline{z}\right\rangle \ge 0$. Thus, adding $2\lambda_n\left\langle \tilde{A}\psi_n, \psi_n - \overline{z}\right\rangle$ to the right hand side of (22) yields

$$\begin{aligned} \|\overline{\omega}_{n+1} - \overline{z}\|_{H\times H}^2 &\le \|\overline{\omega}_n - \overline{z}\|_{H\times H}^2 - \|\overline{\omega}_n - \overline{\omega}_{n+1}\|_{H\times H}^2 \\ &\quad +2\lambda_n\left\langle \tilde{A}\psi_n, \psi_n - \overline{z}\right\rangle_{H\times H} - 2\lambda_n\left\langle \tilde{A}\psi_n, \overline{\omega}_{n+1} - \overline{z}\right\rangle_{H\times H} \\ &= \|\overline{\omega}_n - \overline{z}\|_{H\times H}^2 - \|\overline{\omega}_n - \overline{\omega}_{n+1}\|_{H\times H}^2 \\ &\quad +2\lambda_n\left\langle \tilde{A}\psi_n, \psi_n - \overline{\omega}_{n+1}\right\rangle_{H\times H} \\ &= \|\overline{\omega}_n - \overline{z}\|_{H\times H}^2 - \|\overline{\omega}_n - \overline{\omega}_{n+1}\|_{H\times H}^2 \\ &\quad +2\lambda_n\left\langle \tilde{A}\psi_n - \tilde{A}\psi_{n-1}, \psi_n - \overline{\omega}_{n+1}\right\rangle_{H\times H} \\ &\quad +2\lambda_n\left\langle \tilde{A}\psi_{n-1}, \psi_n - \overline{\omega}_{n+1}\right\rangle_{H\times H}. \end{aligned} \tag{23}$$

Using (15) and (16) in (23), we obtain that

$$\begin{aligned}
\|\overline{\omega}_{n+1}-\overline{z}\|^2_{H\times H} &\leq \|\overline{\omega}_n-\overline{z}\|^2_{H\times H}-\|\overline{\omega}_{n+1}-\overline{\omega}_n\|^2_{H\times H}\\
&+\lambda_n L(1+\sqrt{2})\|\psi_n-\overline{\omega}_n\|^2_{H\times H}+\lambda_n L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}\\
&+\sqrt{2}\lambda_n L\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}+\frac{\lambda_n}{\lambda_{n-1}}\|\overline{\omega}_{n+1}-\overline{\omega}_n\|^2_{H\times H}\\
&-\frac{\lambda_n}{\lambda_{n-1}}\|\overline{\omega}_n-\psi_n\|^2_{H\times H}-\frac{\lambda_n}{\lambda_{n-1}}\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\\
&\leq \|\overline{\omega}_n-\overline{z}\|^2_{H\times H}-\Big(\frac{\lambda_n}{\lambda_{n-1}}-\lambda_n L(1+\sqrt{2})\Big)\|\overline{\omega}_n-\psi_n\|^2_{H\times H}\\
&+\lambda_n L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}-\Big(\frac{\lambda_n}{\lambda_{n-1}}-\sqrt{2}\lambda_n L\Big)\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}. \qquad (24)
\end{aligned}$$

It is easy to see from (24) that

$$\begin{aligned}
\|\overline{\omega}_{n+1}-\overline{z}\|^2_{H\times H} &+ \lambda_n L\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\\
&\leq \|\overline{\omega}_n-\overline{z}\|^2_{H\times H}-\Big(\frac{\lambda_n}{\lambda_{n-1}}-\lambda_n L(1+\sqrt{2})\Big)\|\overline{\omega}_n-\psi_n\|^2_{H\times H}\\
&+\lambda_n L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}-\Big(\frac{\lambda_n}{\lambda_{n-1}}-\sqrt{2}\lambda_n L\Big)\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\\
&+\lambda_n L\|\overline{\omega}_{n+1}-\overline{\omega}_n\|^2_{H\times H}\\
&= \|x_n-z\|^2_{H\times H}-\Big(\frac{\lambda_n}{\lambda_{n-1}}-\lambda_n L(1+\sqrt{2})\Big)\|\overline{\omega}_n-\psi_n\|^2_{H\times H}\\
&+\lambda_n L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}-\Big(\frac{\lambda_n}{\lambda_{n-1}}-\lambda_n L(1+\sqrt{2})\Big)\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\\
&\leq \|\overline{\omega}_n-\overline{z}\|^2_{H\times H}+\lambda_n L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}\\
&-\Big(\frac{\lambda_n}{\lambda_{n-1}}-\lambda_n L(1+\sqrt{2})\Big)\Big(\|\overline{\omega}_n-\psi_n\|^2_{H\times H}+\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\Big). \qquad (25)
\end{aligned}$$

Now, using the fact that $\|\overline{\omega}_n-\psi_n\|_{H\times H}=\|\overline{\omega}_n-\overline{\omega}_{n-1}\|_{H\times H}$, we obtain from (25) that

$$\begin{aligned}
\frac{\|\overline{\omega}_n-\overline{\omega}_{n-1}\|^2_{H\times H}+\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}}{\Big(\frac{\lambda_n}{\lambda_{n-1}}-\lambda_n L(1+\sqrt{2})\Big)^{-1}} &\leq \|x_n-z\|^2_{H\times H}+\lambda_n L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}\\
&-\Big(\|\overline{\omega}_{n+1}-\overline{z}\|^2_{H\times H}+\lambda_n L\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\Big).
\end{aligned}$$

So that

$$\begin{aligned}
\frac{\|\overline{\omega}_n-\overline{\omega}_{n-1}\|^2_{H\times H}+\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}}{\Big(\frac{\lambda_n}{\lambda_{n-1}}-\lambda_n L(1+\sqrt{2})\Big)^{-1}} &\leq \|\overline{\omega}_n-\overline{z}\|^2_{H\times H}+\lambda_n L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}\\
&-\|\overline{\omega}_{n+1}-\overline{z}\|^2_{H\times H}-\lambda_{n+1} L\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\\
&+\|\overline{\omega}_{n+1}-\overline{z}\|^2_{H\times H}+\lambda_{n+1} L\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}\\
&-\|\overline{\omega}_{n+1}-\overline{z}\|^2_{H\times H}-\lambda_n L\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}.
\end{aligned}$$

Since $\{\lambda_n\}_{n\geq 1}$ is monotone decreasing, we have that

$$\begin{aligned}
\frac{\|\overline{\omega}_n-\overline{\omega}_{n-1}\|^2_{H\times H}+\|\overline{\omega}_{n+1}-\psi_n\|^2_{H\times H}}{\Big(\frac{\lambda_n}{\lambda_{n-1}}-\lambda_n L(1+\sqrt{2})\Big)^{-1}} &\leq \|\overline{\omega}_n-\overline{z}\|^2_{H\times H}+\lambda_n L\|\overline{\omega}_n-\psi_{n-1}\|^2_{H\times H}\\
&-\|\overline{\omega}_{n+1}-\overline{z}\|^2-\lambda_{n+1} L\|\overline{\omega}_{n+1}-\psi_n\|^2. \qquad (26)
\end{aligned}$$

It therefore follows from (26) that for any $p \in \mathbb{N}$,

$$\sum_{n=1}^{p}\left(\frac{\|\overline{\omega}_n - \overline{\omega}_{n-1}\|^2_{H\times H} + \|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H}}{\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)^{-1}}\right) \leq \|\overline{\omega}_1 - \overline{z}\|^2_{H\times H} + \lambda_1 L\|\overline{\omega}_1 - \psi_0\|^2_{H\times H}$$

so that as $p \to \infty$, we have that

$$\sum_{n=1}^{\infty}\left(\frac{\|\overline{\omega}_n - \overline{\omega}_{n-1}\|^2_{H\times H} + \|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H}}{\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)^{-1}}\right) < +\infty.$$

This implies that $\left(\|\overline{\omega}_n - \overline{\omega}_{n-1}\|^2_{H\times H} + \|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H}\right)\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right) \to 0$ as $n \to \infty$. But $\lim_{n\to\infty}\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)$ exists and it is not equal to zero. Thus, by Lemma 2.6, we have that $\lim_{n\to\infty}(\|\overline{\omega}_n - \overline{\omega}_{n-1}\|^2 + \|\overline{\omega}_{n+1} - \psi_n\|^2_{H\times H}) = 0$. This implies that $\lim_{n\to\infty}\|\overline{\omega}_n - \overline{\omega}_{n-1}\|_{H\times H} = 0 \iff \lim_{n\to\infty}\|\overline{\omega}_{n+1} - \overline{\omega}_n\|_{H\times H} = 0$ and $\lim_{n\to\infty}\|\overline{\omega}_{n+1} - \psi_n\|_{H\times H} = 0$.

We note here that the sequence $\{\overline{\omega}_n\}_{n\geq 1}$ is bounded. This follows from inequality (26) which gives

$$\begin{aligned} b_{n+1} &:= \|\overline{\omega}_{n+1} - \overline{z}\|^2_{H\times H} + \lambda_{n+1} L\|\overline{\omega}_{n+1} - \psi_n\|^2 \\ &\leq \|\overline{\omega}_n - \overline{z}\|^2_{H\times H} + \lambda_n L\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H} =: b_n \ \forall\, n \in \mathbb{N}. \end{aligned} \tag{27}$$

Thus, the sequence $\{b_n\}_{n\geq 1}$ is monotone decreasing sequence of non-negative real numbers which is bounded above by b_1. It is easy to see that

$$\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} \leq b_n \leq b_1 \ \forall\, n \in \mathbb{N}. \tag{28}$$

Hence, the sequence $\{\|\overline{\omega}_n - \overline{z}\|_{H\times H}\}_{n\geq 1}$ is bounded. Boundednes of $\{\overline{\omega}_n\}_{n\geq 1}$ thus follows.

Since $\{\overline{\omega}_n\}$ is bounded, there exists a subsequence $\{\overline{\omega}_{n_i}\}_{i=1}^{\infty}$ of $\{\overline{\omega}_n\}$ which converges weakly to some $\overline{z}^* \in H\times H$. Since $\lim_{n\to\infty}\|\overline{\omega}_{n+1} - \psi_n\|_{H\times H} = 0$, it is easy to see that $\{\psi_{n_i}\}_{i=1}^{\infty}$ also converges weakly to $\overline{z}^*$.

We show that $\overline{z}^* \in Z(\tilde{A})$. Observe that for any $\psi \in H \times H$,

$$\left\langle \overline{\omega}_{n+1} - \overline{\omega}_n + \lambda_n \tilde{A}\psi_n, \psi - \overline{\omega}_{n+1}\right\rangle_{H\times H} = 0. \tag{29}$$

So, using the fact that $\tilde{A}$ is monotone, we obtain that for all $\psi \in H \times H$,

$$\begin{aligned} 0 &= \langle \overline{\omega}_{n+1} - \overline{\omega}_n, \psi - \overline{\omega}_{n+1}\rangle_{H\times H} + \lambda_n\left\langle \tilde{A}\psi_n, \psi - \psi_n\right\rangle_{H\times H} \\ &\quad +\lambda_n\left\langle \tilde{A}\psi_n, \psi_n - \overline{\omega}_{n+1}\right\rangle_{H\times H} \\ &\leq \langle \overline{\omega}_{n+1} - \overline{\omega}_n, \psi - \overline{\omega}_{n+1}\rangle_{H\times H} + \lambda_n\left\langle \tilde{A}\psi, \psi - \psi_n\right\rangle_{H\times H} \\ &\quad +\lambda_n\left\langle \tilde{A}\psi_n, \psi_n - \overline{\omega}_{n+1}\right\rangle_{H\times H} \\ &\leq \|\overline{\omega}_{n+1} - \overline{\omega}_n\|_{H\times H}(\|\psi\|_{H\times H} + M) + \lambda_n\left\langle \tilde{A}\psi, \psi - \psi_n\right\rangle_{H\times H} \\ &\quad +M\|\psi_n - \overline{\omega}_{n+1}\|_{H\times H} \\ &= \|\overline{\omega}_{n+1} - \overline{\omega}_n\|_{H\times H}(\|\psi\|_{H\times H} + M) + \lambda_n\left\langle \tilde{A}\psi, \psi - \overline{z}^*\right\rangle_{H\times H} \\ &\quad +\lambda_n\left\langle \tilde{A}\psi, \overline{z}^* - \psi_n\right\rangle_{H\times H} + M\|\psi_n - \overline{\omega}_{n+1}\|_{H\times H}, \end{aligned} \tag{30}$$

for some $M > 0$. Taking limit as $i \to \infty$ in (30) and using the fact that $\lim_{n\to\infty}\|\overline{\omega}_{n+1} - \overline{\omega}_n\|_{H\times H} = \lim_{n\to\infty}\|\psi_{n+1} - \psi_n\|_{H\times H} = 0$, $\lim_{n\to\infty}\lambda_n > 0$ and $\{\psi_n\}_{i=1}^{\infty}$ converges weakly to $\overline{z}^*$ we obtain from (30) that for any $\psi \in H \times H$,

$$0 \leq \left\langle \tilde{A}\psi, \psi - \overline{z}^*\right\rangle_{H\times H}. \tag{31}$$

Now, let $\varepsilon \in (0,1)$ be given, then we obtain from (31) that for any $\psi \in H \times H$,

$$\begin{aligned} 0 &\leq \left\langle \tilde{A}\left(\overline{z}^* + \varepsilon(\psi - \overline{z}^*)\right), \varepsilon(\psi - \overline{z}^*)\right\rangle_{H\times H} \\ &= \left\langle \tilde{A}\left(\overline{z}^* + \varepsilon(\psi - \overline{z}^*)\right) - \tilde{A}\overline{z}^*, \varepsilon(\psi - \overline{z}^*)\right\rangle_{H\times H} \\ &\quad + \left\langle \tilde{A}\overline{z}^*, \varepsilon(\psi - \overline{z}^*)\right\rangle_{H\times H}. \end{aligned} \tag{32}$$

So, we obtain using (32) that

$$\begin{aligned} 0 &\leq \left\langle \tilde{A}\left(\overline{z}^* + \varepsilon(\psi - \overline{z}^*)\right) - \tilde{A}\overline{z}^*, \psi - \overline{z}^*\right\rangle_{H\times H} \\ &\quad + \left\langle \tilde{A}\overline{z}^*, \psi - \overline{z}^*\right\rangle_{H\times H}. \end{aligned} \tag{33}$$

Inequality (33) implies that

$$\begin{aligned} 0 &\leq \|\tilde{A}\left(\overline{z}^* + \varepsilon(\psi - \overline{z}^*)\right) - \tilde{A}\overline{z}^*\|_{H\times H}\|\psi - \overline{z}^*\|_{H\times H} \\ &\quad + \left\langle \tilde{A}\overline{z}^*, \psi - \overline{z}^*\right\rangle_{H\times H}. \end{aligned} \tag{34}$$

So, using (34) and the fact that $\tilde{A}$ is continuous, we obtain for all $\psi \in H \times H$ that as $\varepsilon \to 0$,

$$0 \leq \left\langle \tilde{A}\overline{z}^*, \psi - \overline{z}^*\right\rangle_{H\times H}. \tag{35}$$

In particular, for $\psi = -\tilde{A}\overline{z}^* + \overline{z}^* \in H \times H$, we obtain from (35) that

$$0 \leq -\|\tilde{A}\overline{z}^*\|_{H\times H}.^2.$$

This implies that $\tilde{A}\overline{z}^* = 0$. Thus, $\overline{z}^* \in Z(\tilde{A})$.

Next, observe that since the sequence

$$\left\{\|\overline{\omega}_n - \overline{z}\|^2_{H\times H} + \lambda_n L\|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H}\right\}_{n\geq 1}$$

is monotone nonincreasing (see (27))and bounded below by 0, then it converges. But

$$\lim_{n\to\infty} \|\overline{\omega}_n - \psi_{n-1}\|^2_{H\times H} = 0,$$

Thus,

$$\lim_{n\to\infty} \|\overline{\omega}_n - \overline{z}\|^2_{H\times H} \text{ exists } \forall\ \overline{z} \in Z(\tilde{A}). \tag{36}$$

We now show that $\{\overline{\omega}_n\}$ converges weakly to $\overline{z}^*$. Suppose for contradiction that $\{\overline{\omega}_n\}$ does not converge weakly to $\overline{z}^*$. Let $\bar{x}^* \in H \times H$ be a weak cluster point of $\{\overline{\omega}_n\}_{n\geq 1}$ such that $\bar{x}^* \neq \overline{z}^*$, then the same line of argument which led to obtaining that $\overline{z}^* \in Z(\tilde{A})$ gives that $x^* \in Z(\tilde{A})$. Thus, we obtain from (36) that $\lim_{n\to\infty} \|\overline{\omega}_n - \overline{z}^*\|^2_{H\times H}$ and $\lim_{n\to\infty} \|\overline{\omega}_n - x^*\|^2_{H\times H}$ both exist. Let $\{\overline{\omega}_{n_k}\}_{k\geq 1}$ be a subsequence of $\{\overline{\omega}_n\}_{n\geq 1}$ such that $\overline{\omega}_{n_k} \rightharpoonup \bar{x}^*$ as $k \to \infty$. Then by Lemma 2.8 we have that

$$\begin{aligned} \lim_{n\to\infty} \|\overline{\omega}_n - \bar{x}^*\|^2_{H\times H} &= \lim_{k\to\infty}(\|\overline{\omega}_{n_k} - \bar{x}^*\|^2_{H\times H} = \liminf_{k\to\infty} \|\overline{\omega}_{n_k} - \bar{x}^*\|^2_{H\times H} \\ &< \liminf_{k\to\infty} \|\overline{\omega}_{n_k} - \overline{z}^*\|^2_{H\times H} = \lim_{k\to\infty}(\|\overline{\omega}_{n_k} - \overline{z}^*\|^2_{H\times H} \\ &= \lim_{n\to\infty} \|\overline{\omega}_n - \overline{z}^*\|^2_{H\times H}. \end{aligned}$$

Similarly, we can deduce that

$$\lim_{n\to\infty} \|\overline{\omega}_n - \overline{z}^*\|^2_{H\times H} < \lim_{n\to\infty} \|\overline{\omega}_n - \bar{x}^*\|^2_{H\times H},$$

a contradiction. Hence, $\{\overline{\omega}_n\}_{n=1}^{\infty}$ converges weakly to $\overline{z}^*$, where $\overline{z}^* = (u^*, v^*) \in Z(\tilde{A})$ for some $u^* \in S$ with $v^* = Fu^*$. This completes the proof. □

Remark 3.4. *It is well known that in finite dimensional space, weak and strong convergences coincides. Immediate consequence of this is the following theorem:*

Theorem 3.5. *Let $F, K : \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz monotone mappings. Let the sequence $\{(\mu_n, \zeta_n)\}_{n\geq 1}$ in $\mathbb{R}^N \times \mathbb{R}^N$ be generated iteratively from arbitrary $\mu_1, \mu_0, \zeta_1, \zeta_0 \in \mathbb{R}^N$ by* (5) *but with $\{\lambda_n\}_{n=1}^{\infty}$ as a monotone decreasing sequence in $[a, b]$, for some $a, b \in \left]0, \frac{\sqrt{2}-1}{L}\right[$, where L is the Lipschitz constant of the mapping $\tilde{A} : H \times H \to H \times H$ given by $\tilde{A}(u, v) = (Fu - v, u + Kv)$. Suppose the solution set, S, of* (3) *is nonempty, then $\{\mu_n\}_{n\geq 1}$ converges weakly to some $u^* \in S$.*

4. Numerical Example

Example 4.1. *Let $F, K : \mathbb{R} \to \mathbb{R}$ be defined by $Fu = 2u + 1, u \in \mathbb{R}$ and $Kv = 2v, v \in \mathbb{R}$, then F and K are clearly both strongly monotone mappings. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined for $(u, v) \in \mathbb{R}^2$ by*

$$A(u, v) = (Fu - v, u + Kv).$$

It could be easily shown that he mapping A is Lipschtz and strongly monotone. To see this, let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, then

$$\begin{aligned} \|Ax - Ay\|^2 &= [2(x_1 - y_1) - (x_2 - y_2)]^2 \\ &\quad + [(x_1 - y_1) - 2(x_2 - y_2)]^2 \\ &= 5[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ \|Ax - Ay\| &= \sqrt{5}\|x - y\|, \end{aligned}$$

showing that A is Lipschitz. Moreover,

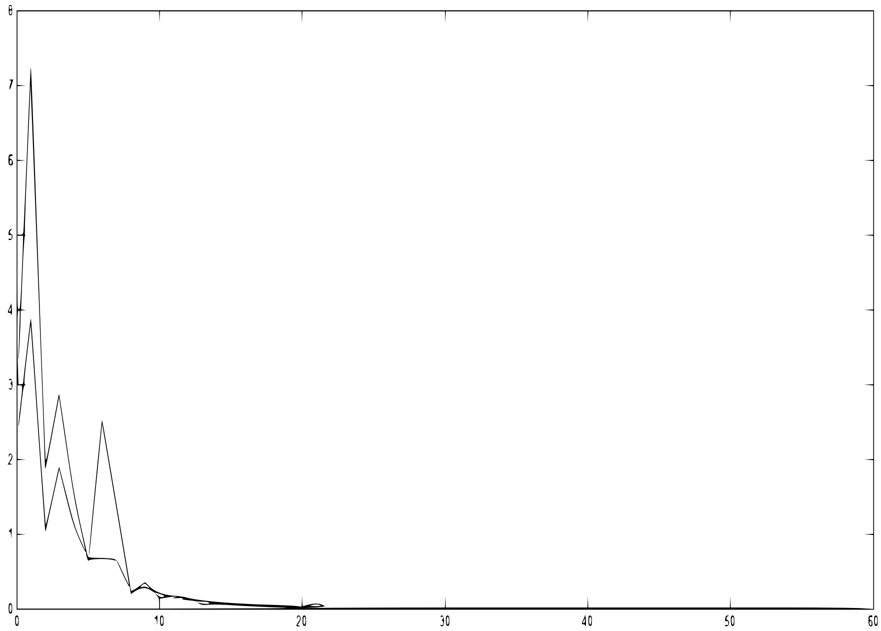
$$\begin{aligned} \langle x - y, Ax - Ay\rangle &= \langle (x_1 - y_1, x_2 - y_2), (2(x_1 - y_1) - (x_2 - y_2), (x_1 - y_1) + 2(x_2 - y_2))\rangle \\ &= 2[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &= 2\|x - y\|^2, \end{aligned}$$

showing that A is $m-$strongly monotone with $m = 2$. Observe that $(-0.4, 0.2)$ is a zero of the operator A. Now, fix $m_1 = 1 \in]0, m[$ and let $\lambda_n = \frac{1}{2n} + \frac{1}{4\sqrt{10}}$. Observe that $\{\lambda_n\}_{n\geq 1}$ is a decreasing sequence $0 < a < \lambda_n < \min\left\{\frac{1}{4m_1}, \frac{\sqrt{2}}{4L}\right\} = \min\left\{\frac{1}{4}, \frac{1}{2\sqrt{10}}\right\} = \frac{1}{2\sqrt{10}}$ for all $n \geq 7$, where $a = \frac{1}{4\sqrt{10}}$.

From $\overline{\omega}_0 = (1, 2)$ and $\psi_0 = (-1, 3) \in \mathbb{R}^2$, let $\{\overline{\omega}_n\}_{n\geq 0}$ be iteratively generated by

$$\overline{\omega}_{n+1} = x_n - \lambda_n A\psi_n, \ \psi_{n+1} = 2\overline{\omega}_{n+1} - \overline{\omega}_n, \tag{37}$$

then with $x^* = (-0.4, 0.2) \in A^{-1}(0)$, the following graph shows the behaviour of $\|\overline{\omega}_n - x^*\|$ and $\|\psi_n - x^*\|$:



Remark 4.2. *The above figure is drawn with the aid of MATLAB R2008b. Values of $n \in \mathbb{N}$ are plotted on the horizontal axis, while the values of $\|\overline{\omega}_n - x^*\|$ and $\|\psi_n - x^*\|$ are plotted on the vertical axis. The blue curve represents the graph of $\|\overline{\omega}_n - x^*\|$ while the green curve denotes the graph of $\|\psi_n - x^*\|$.*

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