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Existence of coupled systems for impulsive of Hilfer fractional stochastic equations with the measure of noncompactness

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Abstract. The present research is concern to the solution of a class of mild solutions linked to a class of impulsive Hilfer fractional differential equations driven Brownian motion with non-compact semi group in Hilbert spaces. All the more obviously the Hausdorff measure of noncompactness has been utilized to get these new results, in like manner, the arguments were scarred by following tools such as the Darbo-Sadovskii fixed point theorem principle associated with vector-valued metrics technique as well as convergent to zero matrices. An illustrated example has been provided for demonstrating efficiency and accuracy.

1. Introduction

In preset time, numerous evolution processes have been so for described that they take shapes by a change of state in an abrupt manner in a form of shocks such as harvesting natural disasters and so. From this starting point, it has been shown that these phenomena include short term perturbation resulting from smooth and continuous dynamics which make their term insignificant when compared with that of whole evolution. That is the reason another part of the theory of ordinary differential equations called impulsive differential equations has drawn a lot of consideration recently (see [1, 2] and the references therein).
The impulsive systems with the compactness presumption on related operators just as the nonlinear function to be either completely continuous or Lipschitz function [3–5], Not with standing, these conditions are stronger restrictions, they are not of the fulfilled as it can be seen in various practical problems [6, 7]. A measure of non-compactness can be utilized to evacuate the suppositions for both the Lipschitz continuity of the nonlinear item and compactness of the operator, to the extent we worried there are no important reports linked to impulsive stochastic differential equations with non-instantaneous impulses towards non-compactness measure techniques, and this over which our results stand. Besides, concerning stochastic differential equations, basic theory can be found in [8, 9].

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Fractional differential equations (FDEs) have known much consideration recently, in different research areas and in a variety of aspects since they have features for real-world problems (see [10–12], and references in that).

Furthermore, Hilfer presented a generalized Riemann-Liouville fractional derivative for short, Hilfer fractional (HF) derivative, which incorporates Caputo as well as Riemann-Liouville fractional derivatives (see [13–15]). Likewise, several authors discussed the existence of the solution for FDEs having HF derivatives (see [16–18]). Stochastic differential equations are today of a significant use they play a down-to-earth in mathematical modelling in a major variety phenomenon when noises are non-insignificant. Add to this, noise or stochastic perturbation bother is both unavoidable and inescapable in nature and wide range in man-made frameworks [19]. Along these lines, stochastic effects need to be incorporated in any fractional differential systems related investigation [20, 21].

In Hamdy et al. [22] examined the existence of mild solutions of HF stochastic integro-differential equations (IDEs) with nonlocal conditions by method of the fixed-point theorems due to Sadovskii. Very recently Yan and Jia [23] discussed the existence of mild solutions for a new class of impulsive stochastic partial neutral functional IDEs. A new result on impulsive HF stochastic differential system has been proved by Saravanakumar et al. [24].

This paper is concern to studying the existence of mild solutions of coupled systems for impulsive Hilfer fractional stochastic IDEs of the following form

$$\begin{cases} D_0^{p_1,q_1}[w(t) - g_1(t, w(t), v(t))] = [A_1 w(t) + f_1(t, w(t), v(t))] \\ + \displaystyle\int_0^t \sigma_1(s, w(s), v(s)) dB(s), \quad t \in J := [0, b], \\ D_0^{p_1,q_1}[v(t) - g_2(t, w(t), v(t))] = [A_2 v(t) + f_2(t, w(t), v(t))] \\ + \displaystyle\int_0^t \sigma_2(s, w(s), v(s)) dB(s), \quad t \in J := [0, b], \\ I_{t_k}^{1-\gamma} w(t_k^+) = w(t_k) + I_k(w(t_k), v(t_k)), \ k = 1, \dots, m, \\ I_{t_k}^{1-\gamma} v(t_k^+) = v(t_k) + \bar{I}_k(w(t_k), v(t_k)), \ k = 1, \dots, m, \\ I_{0^+}^{1-\gamma}(w(0) - h_1(0, w(0), v(0)) = w_0, \qquad \gamma = p_1 + q_1 - p_1 q_1 \\ I_{0^+}^{1-\gamma}(v(0) - h_2(0, w(0), v(0)) = v_0, \qquad \gamma = p_1 + q_1 - p_1 q_1 \end{cases} \tag{1}$$

where $D_0^{p_1,q_1}$ is the Hilfer derivative of order $0 < q_1 < 1$, A_i and type $0 \le p_1 \le 1$, $i = 1, 2$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\mathcal{T}_i(t)$, $t > 0$, with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$ on a separable Hilbert space $\mathcal{X}$. The state $w(\cdot), v(\cdot)$ takes the value in $\mathcal{X}$ with $\langle \cdot, \cdot \rangle$ prompted by $\| \cdot \|$, and as for a complete probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$ furnished with a family of right continuous and increasing $\mathcal{F}$-algebras $\{\mathcal{F}_t, \quad t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Also $\sigma_i : J \times \mathcal{X} \times \mathcal{X} \to \mathcal{L}_Q(\mathcal{Y}, \mathcal{X})$, where $\mathcal{L}_Q(\mathcal{Y}, \mathcal{X}) = \mathcal{L}^0(\mathcal{Y}, \mathcal{X}) = L_2(Q^{1/2}\mathcal{Y}, \mathcal{X})$ be a separable Hilbert space with respect to the Hilbert-Schmidt norm $\| \cdot \|_{L^0}$ and Q-Wiener process on $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$ with a linear bounded covariance operator Q such that $Tr(Q) < \infty$. Let $\{B(t), t \in \mathbb{R}\}$ be a standard cylindrical Wiener process with values in $\mathcal{Y}$ for more details (see[26, 32]). Further $g_i : J \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $f_i : J \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}, i = 1, 2$, which will be also defined in the next section. In addition, the initial condition w_0, v_0 is an $\mathcal{F}_0$ measurable $\mathcal{X}$-valued stochastic process independent of Wiener process B with finite second moment.

This paper is presented as below. Section 1 is introductory. Basic definition, some notations and necessary preliminaries are provided in section 2. In section 3, we give our main results, while in section 4 results are discussed and illustrated by conducting numerical experiments solving an example.

2. Preliminaries

In this section, some notations and necessary preliminaries as well as some lemmas, which are used throughout the current work, the reader may refer to [27–29].

Let $\mathcal{PC}((0,b],L_2(\Omega,\mathcal{X}))$ be the space of mean square valued bounded functions from $(0,b]$ into $L_2(\Omega,\mathcal{X})$, which satisfies the condition $\mathbb{E}|w|^2_{\mathcal{PC}} = \sup_{t\in(0,b]} \mathbb{E}|w(t)|^2_{\mathcal{X}} < \infty$ such that $w(t_k^+)$ exists for any $k = 1,\dots,m$ and $w(t)$ is continuous on $J_k = (t_k, t_{k+1})$ with $0 < t_1 < t_2 < \dots < t_m < b < \infty$. Let us introduce the spaces

$$\mathcal{H}_1 = \mathcal{PC}_{1-\gamma} = \Big\{u:\ (t-t_k)^{(1-\gamma)}w(t) \in \mathcal{PC}((0,b],L_2(\Omega,\mathcal{X}))\Big\},$$

endowed with the norm

$$\|w\|_{\mathcal{H}_1} = \sqrt{(\max_{k=0,1,\dots,m}\sup_{t\in J_k}\mathbb{E}|(t-t_k)^{(1-\gamma)}w(t,\cdot)|^2_{\mathcal{X}})}.$$

where $\mathbb{E}$ is given by $\mathbb{E}(w) = \int_\Omega w(w)dP$.

Definition 2.1. *[11] The left-sided mixed Riemann-Liouville integral operator of order $q_1 > 0$ for a function w can be defined as the following*

$$I_1^q w(t) = \frac{1}{\Gamma(q_1)}\int_0^t (t-s)^{q-1}w(s)ds \qquad \textit{for a.e.} \qquad t \geq 0$$

where the $\Gamma(\cdot)$ is (Euler's) Gamma function.

Definition 2.2. *[11] The Riemann-Liouville fractional derivative of order q_1 is defined as follows:*

$$\begin{aligned} D_0^\alpha w(t) &= \frac{d}{dt}I^{1-q_1}w(t) \\ &= \frac{1}{\Gamma(1-q_1)}\frac{d}{dt}\left(\int_0^t (t-s)^{-q_1}w(s)ds\right) \qquad \textit{for a.e.} \qquad t\geq 0 \end{aligned}$$

Definition 2.3. *[18](Hilfer Derivative) Let $0 \leq p_1 \leq 1$ and order $0 < q_1 < 1$. The Hilfer fractional derivative of order p_1 and type q_1 of the function w is defined as follows*

$$D_0^{p_1,q_1}w(t) = I_0^{p_1(1-q_1)}\frac{d}{dt}I_0^{(1-p_1)(1-q_1)}w(t) \qquad \textit{for all} \qquad t\geq 0. \tag{2}$$

Properties 2.1. *For $p_1 = 0,\ 0 < q_1 < 1$, generalization (2) coincides with the Riemann-Liouville derivative:*

$$D_0^{0,q_1}w(t) = I_0\frac{d}{dt}I_0^{(1-q_1)}w(t) = ^L D_0^{q_1}w(t).$$

For $p_1 = 1,\ 0 < q_1 < 1$, with the Caputo derivative, namely,

$$D_0^{1,q_1}w(t) = I_0^{(1-q_1)}\frac{d}{dt}w(t) = ^C D_0^{q_1}w(t)$$

For $x, y \in X$, two families of operators are defined $\{\mathcal{S}_{p_1,q_1}(t),\quad t\geq 0\}$ and $\{\mathcal{T}_{q_1}(t),\quad t\geq 0\}$ by

$$\mathcal{S}_{p_1,q_1}(t) = I_{0^+}^{p_1(1-q_1)}\mathcal{P}_{q_1}(t),\quad \mathcal{P}_{q_1}(t) = t^{q_1-1}\mathcal{T}_{q_1}(t),\quad \mathcal{T}_{q_1}(t) = \int_0^\infty q_1\theta\psi_{q_1}(\theta)\mathcal{T}(t^{q_1}\theta)d\theta,$$

where

$$\psi_{q_1}(\theta) = \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\alpha q_1)},\ 0 < q_1 < 1,\ \theta\in(0,\infty)$$

is function of Wright-type satisfying

$$\int_0^\infty \theta^\varsigma\psi_{q_1}(\theta)d\theta = \frac{\Gamma(1+\varsigma)}{\Gamma(1+q_1\varsigma)} \qquad \text{for} \qquad \theta\geq 0$$

Lemma 2.1. *[11] Let $\alpha, q_1 \in (0,1)$ and $\forall w \in \mathcal{D}((A)^{\alpha})$,*

$$A\mathcal{T}_{q_1}(t)w = A^{(1-\alpha)}\mathcal{T}_{q_1}(t)A^{\alpha}w,$$

and

$$\|A^{\alpha}\mathcal{T}_{q_1}(t)\| \leq \frac{q_1 C_{\alpha}\Gamma(2-\alpha)}{t^{q_1\alpha}\Gamma(1+q_1(1-\alpha))}$$

where $t \in [0,b]$ and $C_{\alpha} > 0$.

Lemma 2.2. *[10] The operators $\mathcal{S}_{p_1,q_1}$ and $\mathcal{P}_{q_1}$ are as follows*

(A) *$\{\mathcal{P}_{q_1}(t),\ t > 0\}$ is continuous in the uniform operator topology*

(B) *For any fixed $t > 0$, $\mathcal{S}_{p_1,q_1}(t)$ and $\mathcal{P}_{q_1}(t)$ are bounded and linear operators, and*

$$\|\mathcal{P}_{q_1}(t)w\| \leq \frac{M_T t^{q_1-1}}{\Gamma(q_1)}\|w\|.$$

$$\|\mathcal{S}_{p_1,q_1}(t)w\| \leq \frac{M_T t^{\gamma-1}}{\Gamma(q_1)}\|w\|,\ \gamma = (1-p_1)(1-q_1).$$

(C) *$\{\mathcal{P}_{q_1}(t),\ t > 0\}$ and $\{\mathcal{S}_{p_1,q_1}(t),\ t > 0\}$ are strongly continuous.*

Now some useful definitions and results are recalled.

Lemma 2.3. *[30] Let $D_0, D_1 \subset \mathcal{X}$ be bounded , a measure of non-compactness χ is called*

(i) Monotone if $D_0, D_1 \subset \mathcal{X}$, $D_0 \subset D_1$ implies $\chi(D_0) \leq \chi(D_1)$

(ii) Nonsingular if $\chi(\{a\} \cup D) = \chi(D)$ for every $a \in \mathcal{X}$, $D \subset \mathcal{X}$;

(iii) Invariant with respect to union with compact sets, if $\chi(\{K\} \cup D) = \chi(D)$ for every $K \subset \mathcal{X}$ and $D \in \mathcal{X}$.

(iv) Regular if the condition $\chi(D) = 0$ is equivalent to the relative compactness of D.

(v) Lower-additive if $\chi(D_0 + D_1) \leq \chi(D_0) + \chi(D_1)$ for each $D_0, D_1 \subset \mathcal{X}$.

A typical example of an m.n.c. is the Hausdorff measure of noncompactness χ defined for all $D_3 \subset X$ by where

$$\chi(D_3) = \{\epsilon \in \mathbb{R}_+^n : \text{there exists} \quad n \in \mathbb{N} \quad \text{such that} \quad D_3 \subseteq \cup_{i=1}^{n} B(x_i, \epsilon)\}$$

3. Mild Solutions

Definition 3.1. *An $\mathcal{X}-$ adapted stochastic process $w(t) = (w(t), v(t)) \in \mathcal{H}_1 \times \mathcal{H}_1$ is said to be a mild solution of the problem* (1) *if.* 1) *the function $A_i\mathcal{P}_{q_1}(t-s)h_i(s, w_s)$, $i = 1,2$, is integrable*

2) $I_{0^+}^{1-\gamma}\Big(w(t) - h_1(t, w(t), v(t)\Big)_{t=0} = w_0$ *and* $I_{0^+}^{1-\gamma}\Big(v(t) - h_2(t, w(t), v(t)\Big)_{t=0} = v_0$, *a $\mathcal{F}_0$ adapted stochastic process* 3) *the process w satisfying the beneath integral equation:*

$$\begin{cases} w(t) &= \mathcal{S}_{p_1,q_1}(t)w_0 + h_1(t, w(t), v(t)) + \displaystyle\int_0^t \mathcal{P}_{q_1}(t-s)\Big(A_1 h_1(s, w(s), v(s)) \\ &+ f_1(s, w(s), v(s)) + \displaystyle\int_0^s \sigma_1(\zeta, w(\zeta), v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \ \ t \in [0, t_1] \\ w(t) &= \mathcal{S}_{p_1,q_1}(t-t_k)(w(t_k^-) + I_k((w(t_k^-), v(t_k^-)) + h_1(t, x(t), y(t)) \\ &+ \displaystyle\int_{t_k}^t \mathcal{P}_q(t-s)\Big(A_1 h_1(s, w(s), v(s)) + f_1(s, w(s), v(s))ds \\ &+ \displaystyle\int_0^s \sigma_1(\zeta, w(\zeta), v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \ \ t \in (t_k, t_{k+1}], \ \gamma = p_1 + q_1 - p_1 q_1 \end{cases} \tag{3}$$

and

$$\begin{cases} v(t) &= \mathcal{S}_{p_1,q_1}(t)y_0 + h_2(t,x(t),y(t)) + \int_0^t \mathcal{P}_{q_1}(t-s)\Big(A_2h_2(s,w(s),v(s)) \\ &+ f_1(s,w(s),v(s)) + \int_0^s \sigma_2(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \ \ t\in[0,t_1] \\ v(t) &= \mathcal{S}_{p_1,q_1}(t-t_k)(v(t_k^-) + \bar{I}_k((w(t_k^-),v(t_k^-)) + h_2(t,w(t),v(t)) \\ &+ \int_{t_k}^t \mathcal{P}_{q_1}(t-s)\Big(A_2h_2(s,w(s),v(s)) + f_2(s,w(s),v(s))ds \\ &+ \int_0^s \sigma_2(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \ \ t\in(t_k,t_{k+1}], \ \ \gamma = p_1+q_1-p_1q_1 \end{cases} \tag{4}$$

To prove the principle results, the accompanying hypotheses are required:

(H_1) $\mathcal{T}_i(t)$ of bounded linear operators on $\mathcal{X}$, we suppose that $0\in\rho(A_i)$, and $\exists$ a constant M_T such that

$$\|\mathcal{T}_i(t)\|^2 \le M_T \text{ for all } \quad t\ge 0,\ i=1,2$$

(H_2) **(i)** $\exists$ a positive constant $\beta,\alpha\in(0,1)$, the functions $h_i: J\times\mathcal{X}\times\mathcal{X}\to\mathcal{X}, i=1,2$ are a continuous function and satisfies with $\alpha\beta>\frac{1}{2}$ such that $h_i\in\mathcal{D}(A_i^\alpha)$ and for any $w,v\in\mathcal{X}$, $t\in J$,

(ii) $\exists$ constants $c_{h^i},\alpha^i_{h_i}>0$ and $\bar{c}_{h^i},\bar{\alpha}^i_{h_i}>0$, $A_i^\alpha h_i$ satisfies the inequality

$$\mathbb{E}|A_i^\alpha h_i(t,w,v) - A_i^\alpha h_i(t,\bar{w},\bar{v})|^2_{\mathcal{X}} \le c_h^i\|w-\bar{w}\|^2_{\mathcal{H}_1} + \bar{c}_h^i\|v-\bar{v}\|^2_{\mathcal{H}_1}, \ \ t\in J,$$

$$\mathbb{E}|A_i^\alpha h_i(t,w,v)|^2_{\mathcal{X}} \le \alpha^i_{h_i}(\|w\|^2_{\mathcal{H}_1} + \|v\|^2_{\mathcal{H}_1}) + \bar{\alpha}^i_{h_i}, \ \ t\in J,$$

for all $w,v,\overline{w},\overline{v}\in X$

(iii) The function $s\longmapsto A\mathcal{P}_{q_1}(t-s)h_i(s,w(s),v(s)),\ i=1,2$ is measurable.

(H_3) **(i)** f_i is a L^2-Carathéodory map and for every $t\in[0,b]$ for each $i=1,2$ the function $t\longmapsto f_i(t,w(t),v(t))$ and $t\longmapsto f_i(t,w(t),v(t)), u,v\in\mathcal{X}$ are measurable.

(ii) $\exists$ constants $c^i_{f_i},\bar{c}^i_{f_i}$ and a function $\psi_i\in L^{\frac{1}{\beta}}(J,\mathbb{R}^+)$, $0<\beta<q_1$ such that for any $w,v\in\mathcal{X}$ and each $t\in J$;

$$\mathbb{E}|f_i(t,w,v)|^2_{\mathcal{X}} \le \psi_i(t) + c^i_{f_i}\|w\|^2_{\mathcal{H}_1} + \bar{c}^i_{f_i}\|v\|^2_{\mathcal{H}_1}.$$

(H_4) The function $\sigma_i:[0,b]\times\mathcal{X}\times\mathcal{X}\to\mathcal{L}_Q(\mathcal{Y},\mathcal{X})$ satisfying the beneath conditions:

(i) The function $\sigma_i(t,\cdot,\cdot):\mathcal{X}\times\mathcal{X}\to\mathcal{L}_Q(\mathcal{Y},\mathcal{X})$ is a continuous function for each $w,v\in\mathcal{X}$, $t\in J$

(ii) The function $\sigma_i(\cdot,w(\cdot),v(\cdot)):[0,b]\to\mathcal{L}_Q(\mathcal{Y},\mathcal{X})$ is strongly measurable function

(iii) $\exists$ a function $\phi_i\in L^{\frac{1}{\beta}}(J,\mathbb{R}^+)$, $0<\beta<q_1$ and a positive constant $c^i_{\sigma_i},\lambda^i_{\sigma_i},\bar{c}^i_{\sigma_i},\lambda^i_{\sigma_i},R$ such that

$$\sup_{\|w\|^2,\|v\|^2\le R}\int_0^t \mathbb{E}|\sigma_i(s,w,v)|^2_{L^0} \le \phi_i(t) + c^i_{\sigma_i}\|w\|^2_{\mathcal{H}_1} + \bar{c}^i_{\sigma_i}\|v\|^2_{\mathcal{H}_1},$$

and

$$\sup_{\|x\|^2,\|y\|^2\le R}\int_0^t \mathbb{E}|\sigma_i(s,w,v)-\sigma_i(t,\bar{w},\bar{v})|^2_{L^0} \le \lambda^i_{\sigma_i}\|w-\bar{w}\|^2_{\mathcal{H}_1} + \bar{\lambda}^i_{\sigma_i}\|v-\bar{v}\|^2_{\mathcal{H}_1}.$$

for any $w,v,\bar{w},\bar{v}\in\mathcal{X}$ and each $t\in J$.

(H_5) $\exists$ constants $d_k^i, \bar{d}_k^i > 0, k = 1, \dots, m, \dots$ for each

$$\mathbb{E}|I_k(w,v) - I_k(\bar{w},\bar{v})|_{\mathcal{X}}^2 \leq d_k^1\|w-\bar{w}\|_{\mathcal{H}_1}^2 + \bar{d}_k^1\|v-\bar{v}\|_{\mathcal{H}_1}^2,$$

and

$$\mathbb{E}|\bar{I}_k(w,v) - I_k(\bar{w},\bar{v})|_{\mathcal{X}}^2 \leq d_k^2\|w-\bar{w}\|_{\mathcal{H}_1}^2 + \bar{d}_k^2\|v-\bar{v}\|_{\mathcal{H}_1}^2,$$

for all $w, v, \bar{w}, \bar{v} \in \mathcal{X}$, and $\sum_{k=1}^{m} d_k^i < \infty$, $\sum_{k=1}^{m} \bar{d}_k^i < \infty$

For suitability, we start introduce some notation

$$M_\alpha = \|A_i^\alpha\|, \quad \Lambda_5 = \max\{\Lambda_2, \bar{\Lambda}_2, \bar{\Lambda}_4, \Lambda_4\} \neq 1$$

and

$$\Lambda_6 = \max\{\Lambda_1, \bar{\Lambda}_1, \bar{\Lambda}_3, \Lambda_3\}$$

$$\begin{aligned}
\Lambda_1 &= 5\Big[t_1^{2(1-p_1)(1-q_1)}\Big(\Big(\frac{M_T t_1^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\mathbb{E}|w_0|_{\mathcal{X}}^2 + M_\alpha^2\bar{\alpha}_{h_1}^1\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}_{h_1}^1 \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{q_1}}{q_1}\Big(\frac{(1-\beta)}{(q-\beta)}\Big)^{(q_1-\beta)} t_1^{(q_1-\beta)}\|\psi_1\|_{L^{\frac{1}{\beta}}} \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{q_1}}{q_1}Tr(Q)\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)} t_1^{(q_1-\beta)}\|\phi_1\|_{L^{\frac{1}{\beta}}}\Big],
\end{aligned}$$

$$\begin{aligned}
\Lambda_2 &= 5\Big[t_1^{2(1-p_1)(1-q_1)}\Big(2M_\alpha^2\alpha_{h_1}^1 + 2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha_{h_1}^1 \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{t_1^{q_1}}{q_1}\Big)^2(c_{f_1}^1 + \bar{c}_{f_1}^1) + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{q_1}}{q_1}Tr(Q)\Big(\frac{t_1^{q_1}}{q_1}(c_{\sigma_1}^1 + \bar{c}_{\sigma_1}^1)\Big)\Big)\Big]
\end{aligned}$$

and

$$\begin{aligned}
\bar{\Lambda}_1 &= 5\Big[t_1^{2(1-p_1)(1-q_1)}\Big(\Big(\frac{M_T t_1^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\mathbb{E}|y_0|_{\mathcal{X}}^2 + M_\alpha^2\bar{\alpha}_{h_2}^2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}_{h_2}^2 \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{q_1}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q-\beta)} t_1^{(q-\beta)}\|\psi_2\|_{L^{\frac{1}{\beta}}} \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{q_1}}{q}Tr(Q)\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)} t_1^{(q_1-\beta)}\|\phi_2\|_{L^{\frac{1}{\beta}}}\Big],
\end{aligned}$$

$$\begin{aligned}
\bar{\Lambda}_2 &= 5\Big[t_1^{2(1-p_1)(1-q_1)}\Big(2M_\alpha^2\alpha_{h_1}^1 + 2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha_{h_2}^2 \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{t_1^{q_1}}{q_1}\Big)^2(c_{f_2}^2 + \bar{c}_{f_2}^2) + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{q_1}}{q_1}Tr(Q)\Big(\frac{t_1^{q_1}}{q_1}(c_{\sigma_2}^2 + \bar{c}_{\sigma_2}^2)\Big)\Big)\Big]
\end{aligned}$$

$$\begin{aligned}
\Lambda_3 &= 6b^{2(1-\gamma)}\Big(M_\alpha^2\bar{\alpha}_{h_1}^1 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}_{h_1}^1 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1-\beta}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\psi_1\|_{L^{\frac{1}{\beta}}} \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1-\beta}}{q_1}Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\phi_1\|_{L^{\frac{1}{\beta}}}\Big),
\end{aligned}$$

$$\begin{aligned}\Lambda_4 &= 6R\Big(\Big(\frac{M_T b^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\Big(1+d_k^1+\bar{d}_k^1\Big)+2M_\alpha^2\alpha_{h_1}^1+2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha_{h_1}^1\Big)\\ &+ \Big(\frac{M_T}{\sqrt{q_1}\Gamma(q_1)}\Big)^2\frac{b^{q_1-\beta}}{q_1}Tr(Q)(c_{\sigma_1}^1+\bar{c}_{\sigma_1}^1)+\Big(\frac{M_T b^{q_1}}{\Gamma(q_1)q_1}\Big)^2(c_{f_1}^1+\bar{c}_{f_1}^1)\Big)\end{aligned}$$

and

$$\begin{aligned}\bar{\Lambda}_3 &= 6b^{2(1-\gamma)}\Big(M_\alpha^2\bar{\alpha}_{h_2}^2+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}_{h_2}^2+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1-\beta}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\psi_2\|_{L^{\frac{1}{\beta}}}\\ &+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1-\beta}}{q_1}Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\phi_2\|_{L^{\frac{1}{\beta}}}\Big),\end{aligned}$$

$$\begin{aligned}\bar{\Lambda}_4 &= 6R\Big(\Big(\frac{M_T b^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\Big(1+d_k^2+\bar{d}_k^2\Big)+2M_\alpha^2\alpha_{h_2}^2+2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha_{h_2}^2\Big)\\ &+ \Big(\frac{M_T}{\sqrt{q_1}\Gamma(q_1)}\Big)^2\frac{b^{q_1-\beta}}{q_1}Tr(Q)(c_{\sigma_2}^2+\bar{c}_{\sigma_2}^2)+\Big(\frac{M_T b^{q_1}}{\Gamma(q_1)q_1}\Big)^2(c_{f_2}^1+\bar{c}_{f_2}^1)\Big).\end{aligned}$$

We have the beneath results:

Theorem 3.1. *Assume that* (H_1)-(H_5) *are satisfied and the matrix*

$$M_{trix}=\begin{pmatrix}\mu_1(b) & \mu_2(b)\\ \bar{\mu}_1(b) & \bar{\mu}_2(b)\end{pmatrix}$$

where

$$\mu_1(b)=3b^{2(1-\gamma)}\Big(M_\alpha^2c_h^1+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2c_h^1+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1}}{q_1^2}Tr(Q)\lambda_{\sigma_1}^1\Big),$$

$$\mu_2(b)=3b^{2(1-\gamma)}\Big(M_\alpha^2\bar{c}_h^1+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{c}_h^1+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1}}{q_1^2}Tr(Q)\bar{\lambda}_{\sigma_1}^1\Big)$$

and

$$\bar{\mu}_1(b)=3b^{2(1-\gamma)}\Big(M_\alpha^2c_h^2+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2c_h^2+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1}}{q_1^2}Tr(Q)\lambda_{\sigma_2}^2\Big)$$

$$\bar{\mu}_2(b)=3b^{2(1-\gamma)}\Big(M_\alpha^2\bar{c}_h^2+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{c}_h^1+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1}}{q_1^2}Tr(Q)\bar{\lambda}_{\sigma_2}^2\Big)$$

If M_{trix} *converges to zero, Therefore, the problem* (1) *possesses a unique mild solution on* $[0,b]$.

Proof. Consider the operator $\Phi=(\Phi_1,\Phi_2):\mathcal{H}_1\times\mathcal{H}_1\to\mathcal{H}_1\times\mathcal{H}_1$ associated with the problem (1) defined by

$$\Phi(w,v)=(\Phi_1(w,v),\Phi_2(w,v)),\ (w,v)\in\mathcal{H}_1\times\mathcal{H}_1$$

where

$$\Phi_1(w,v)(t)=\begin{cases}\quad\mathcal{S}_{p_1,q_1}(t)w_0+h_1(t,w(t),v(t))+\int_0^t\mathcal{P}_{q_1}(t-s)\Big(A_1h_1(s,w(s),v(s))\\ +\quad f_1(s,w(s),v(s))+\int_0^s\sigma_1(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds,\quad \mathbb{P}-a.s,\ \ t\in[0,t_1]\\ \quad\mathcal{S}_{p_1,q_1}(t-t_k)(w(t_k^-)+I_k((w(t_k^-),v(t_k^-))+h_1(t,w(t),v(t))\\ +\quad\int_{t_k}^t\mathcal{P}_{q_1}(t-s)\Big(A_1h_1(s,w(s),v(s))+f_1(s,w(s),v(s))ds\\ +\quad\int_0^s\sigma_1(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds,\quad\mathbb{P}-a.s,\ \ t\in(t_k,t_{k+1}],\ \gamma=p_1+q_1-p_1q_1\end{cases}$$

and

$$\Phi_2(w,v)(t) = \begin{cases} \mathcal{S}_{p_1,q_1}(t)y_0 + h_2(t,w(t),v(t)) + \int_0^t \mathcal{P}_{q_1}(t-s)\Big(A_2h_2(s,w(s),v(s)) \\ + \quad f_1(s,w(s),v(s)) + \int_0^s \sigma_2(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \ \ t\in[0,t_1] \\ \\ \mathcal{S}_{p_1,q_1}(t-t_k)(v(t_k^-) + \bar{I}_k((x(t_k^-),y(t_k^-)) + h_2(t,w(t),v(t)) \\ + \quad \int_{t_k}^t \mathcal{P}_{q_1}(t-s)\Big(A_2h_2(s,w(s),v(s)) + f_2(s,x(s),y(s))ds \\ + \quad \int_0^s \sigma_2(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \ \ t\in(t_k,t_{k+1}], \ \gamma = p_1+q_1-p_1q_1 \end{cases}$$

Let

$$\mathcal{B}_R = \Big\{u \in \mathcal{H}_1\times\mathcal{H}_1, \quad \|w\|^2_{\mathcal{H}_1} \le R, \quad R \ge \frac{\Lambda_6}{1-\Lambda_5}\Big\}$$

and

$$\mathcal{B}_R = \Big\{v \in \mathcal{H}_1\times\mathcal{H}_1, \quad \|v\|^2_{\mathcal{H}_1} \le R, \quad R \ge \frac{\Lambda_6}{1-\Lambda_5}\Big\}$$

is obviously a bounded closed convex set in $\mathcal{H}_1\times\mathcal{H}_1$. Noting that any fixed point of the operator Φ corresponds to the classical solution of the system (1), we break the proof into a sequence of steps.

The first step. $\Phi(\mathcal{B}_R\times\mathcal{B}_R)\subseteq\mathcal{B}_R\times\mathcal{B}_R$.

$$\|\Phi_i(w,v)\|^2_{\mathcal{H}_1} = \sup_{t\in[0,t_1]} t^{2(1-p_1)(1-q_1))}\mathbb{E}|\Phi_i(w(t),v(t))|^2_{\mathcal{X}}, \ i=1,2$$

for each $t\in[0,t_1]$, $(w,v)\in\mathcal{B}_R\times\mathcal{B}_R$, we get

$$\begin{aligned} \mathbb{E}|\Phi_1(w,v)(t)|^2_{\mathcal{X}} &= 5\Big(\mathbb{E}|\mathcal{S}_{p_1,q_1}(t)x_0|^2 + \mathbb{E}|h_1(t,w(t),v(t)|^2 + \mathbb{E}\Big|\int_0^t A_1\mathcal{P}_{q_1}(t-s)h_1(s,w(s),v(s))ds\Big|^2 \\ &+ \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)f_1(s,w(s),v(s))ds\Big|^2 \\ &+ \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)\Big(\int_0^s \sigma_1(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds\Big|^2\Big) \\ &= 5\sum_{l=1}^5 J_l \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{E}|\Phi_2(w,v)(t)|^2_{\mathcal{X}} &= 5\Big(\mathbb{E}|\mathcal{S}_{p_1,q_1}(t)w_0|^2 + \mathbb{E}|h_2(t,w(t),v(t))|^2 + \mathbb{E}\Big|\int_0^t A_2\mathcal{P}_{q_1}(t-s)h_2(s,w(s),v(s))ds\Big|^2 \\ &+ \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)f_2(s,w(s),v(s))ds\Big|^2 \\ &+ \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)\Big(\int_0^s \sigma_2(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds\Big|^2\Big) \\ &= 5\sum_{l=1}^5 \bar{J}_l \end{aligned}$$

by utilizing the Lemma 2.2, one can have

$$J_1 = \mathbb{E}|\mathcal{S}_{p_1,q_1}(t)w_0|^2_{\mathcal{X}} \leq \Big(\frac{M_T t_1^{1-\gamma}}{\Gamma(\gamma)}\Big)^2 \mathbb{E}|w_0|^2_{\mathcal{X}}.$$

From $(H_2)(ii)$, we obtain

$$J_2 = \mathbb{E}|h_2(t, w(t), v(t)|^2_{\mathcal{X}} \leq \|A_1^{-\alpha}\|^2 \mathbb{E}|A_1^{\alpha} h_1(t, w(t), v(t)|^2_{\mathcal{X}} \leq M_\alpha^2(2\alpha^1_{h_1} R + \bar{\alpha}^1_{h_1})).$$

Using Hölder inequality and Lemma 2.1, yields

$$\begin{aligned}
J_3 &= \mathbb{E}\Big|\int_0^t A_1 \mathcal{P}_{q_1}(t-s) h_1(s, w(s), v(s)) ds\Big|^2_{\mathcal{X}} \\
&= \mathbb{E}\Big|\int_0^t (t-s)^{q_1-1} A_1^{1-\alpha} \mathcal{T}_q(t-s) A_1^{\alpha} h_1(s, w(s), v(s)) ds\Big|^2_{\mathcal{X}} \\
&\leq \mathbb{E}\Big|\int_0^t (t-s)^{q_1-1} \frac{q_1 C_{1-\alpha}\Gamma(1+\alpha)}{t^{q_1\alpha}\Gamma(1+q_1\alpha)} (t-s)^{q_1\alpha - q_1} A_1^{\alpha} h_1(s, w(s), v(s)) ds\Big|^2_{\mathcal{X}} \\
&= \mathbb{E}\Big|\int_0^t \frac{q_1 C_{1-\beta}\Gamma(1+\alpha)}{t^{q_1\alpha}\Gamma(1+q_1\alpha)} (t-s)^{q_1\alpha-1} A_1^{\alpha} h_1(s, w(s), v(s)) ds\Big|^2_{\mathcal{X}} \\
&\leq \Big(\frac{q_1 C_{1-\alpha}\Gamma(1+\alpha)}{q_1\alpha\Gamma(1+q_1\alpha)}\Big)^2 \Big(\int_0^t (t-s)^{q_1\alpha-1} ds\Big)^2 (2\alpha^1_{h_1} R + \bar{\alpha}^1_{h_1}) \\
&= \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2 (2\alpha^1_{h_1} R + \bar{\alpha}^1_{h_1})
\end{aligned}$$

By Bochner's theorem [31] and $A_1\mathcal{P}_{q_1}(t-s)h_1(s, w(s), v(s))$ is integrable on J , so Φ_1 is well defined on $\mathcal{B}_R$. Applying (H_3) (ii) and Lemma 2.2 together with the Hölder inequality, we find that

$$\begin{aligned}
J_4 &= \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s) f_1(s, w(s), v(s)) ds\Big|^2_{\mathcal{X}} \\
&\leq \Big(\frac{M_T}{\Gamma(q)}\Big)^2 \Big(\int_0^{t_1} (t_1-s)^{q_1-1} ds\Big)\Big(\int_0^{t_1} (t_1-s)^{q_1-1}\Big(\psi_1(s) + c^1_{f_1}\mathbb{E}|w(s)|^2_{\mathcal{X}} + \bar{c}^1_{f_1}\mathbb{E}|v(s)|^2_{\mathcal{X}}\Big) ds \\
&\leq \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2 \Big(\int_0^{t_1} (t_1-s)^{q_1-1} ds\Big)\Big(\int_0^{t_1} (t_1-s)^{q_1-1}\psi_1(s) ds \\
&+ c^1_{f_1}\int_0^{t_1} (t_1-s)^{q_1-1}\mathbb{E}|w(s)|^2_{\mathcal{X}} ds + \bar{c}^1_{f_1}\int_0^{t_1} (t_1-s)^{q_1-1}\mathbb{E}|w(s)|^2_{\mathcal{X}} ds\Big) \\
&\leq \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2 \frac{t_1^{q_1}}{q_1}\Big(\int_0^{t_1} (t_1-s)^{\frac{q_1-1}{1-\beta}}\Big)^{1-\beta}\Big(\int_0^{t_1} |\psi_1(s)|^{\frac{1}{\beta}} ds\Big)^{\beta} + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{t_1^{q_1}}{q_1}\Big)^2 R(c^1_{f_1} + \bar{c}^1_{f_1}) \\
&= \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2 \frac{t_1^{q_1}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)} t_1^{(q_1-\beta)}\|\psi_1\|_{L^{\frac{1}{\beta}}} + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{t_1^{q_1}}{q_1}\Big)^2 R(c^1_{f_1} + \bar{c}^1_{f_1})
\end{aligned}$$

By assumption (H_4) (iii), Lemma 2.2 and Burkholder Gundy's inequality,

$$\begin{aligned}
J_5 &= \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)\Big(\int_0^s \sigma_1(\zeta, w(\zeta), v(\zeta)) dB(\zeta)\Big) ds\Big|^2 \\
&\leq \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2 \frac{t_1^{q_1}}{q_1} Tr(Q)\Big(\int_0^{t_1} (t_1-s)^{q_1-1}\mathbb{E}|\sigma_1(s, w(s), v(s))|^2_{L^0} ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2 \frac{t_1^{q_1}}{q_1} Tr(Q)\Big(\int_0^{t_1}(t_1-s)\phi_1(s)ds + \frac{t_1^{q_1}}{q_1}R(c^1_{\sigma_1}+\bar{c}^1_{\sigma_1})\Big)\\
&\leq \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2 \frac{t_1^{q_1}}{q_1} Tr(Q)\Big(\Big(\int_0^{t_1}(t_1-s)^{\frac{q_1-1}{1-\beta}}\Big)^{1-\beta}\Big(\int_0^{t_1}|\phi_1(s)|^{\frac{1}{\beta}}ds\Big)^{\beta} + \frac{t_1^{q_1}}{q_1}R(c^1_{\sigma_1}+\bar{c}^1_{\sigma_1})\Big)\\
&= \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2 \frac{t_1^{q_1}}{q_1} Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)} t_1^{(q_1-\beta)}\|\phi_1\|_{L^{\frac{1}{\beta}}} + \frac{t_1^{q_1}}{q_1}R(c^1_{\sigma_1}+\bar{c}^1_{\sigma_1})\Big)
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
\mathbb{E}|\Phi_1(w,v)|^2_{\mathcal{X}} &= 5\Big[\Big(\frac{M_T t_1^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\mathbb{E}|w_0|^2_{\mathcal{X}} + M^2_{\alpha}(2\alpha^1_{h_1}R+\bar{\alpha}^1_{h_1})) + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2(2\alpha^1_{h_1}R+\bar{\alpha}^1_{h_1})\\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1-\beta}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\psi_1\|_{L^{\frac{1}{\beta}}} + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{t_1^{q_1}}{q_1}\Big)^2R(c^1_{f_1}+\bar{c}^1_{f_1})\\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1-\beta}}{q_1}Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\phi_1\|_{L^{\frac{1}{\beta}}} + \frac{t_1^{q_1}}{q_1}R(c^1_{\sigma_1}+\bar{c}^1_{\sigma_1})\Big)\Big]
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\sup_{t\in[0,t_1]} t^{2(1-p_1)(1-q_1))}\mathbb{E}|\Phi_1(w,v)|^2_{\mathcal{X}} = 5\Big[\sup_{t\in[0,t_1]} t^{2(1-p_1)(1-q_1)}\Big(\Big(\frac{M_T t_1^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\mathbb{E}|w_0|^2_{\mathcal{X}} + M^2_{\alpha}\bar{\alpha}^1_{h_1}\\
&+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}^1_{h_1} + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\frac{t^{(2q_1-\beta)}}{q_1}\|\psi_1\|_{L^{\frac{1}{\beta}}}\\
&+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2Tr(Q)\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\frac{t^{(2q_1-\beta)}}{q_1}\|\phi_1\|_{L^{\frac{1}{\beta}}}\Big]\\
&+5R\Big[\sup_{t\in[0,t_1]} t^{2(1-p_1)(1-q_1)}\Big(2M^2_{\alpha}\alpha^1_{h_1} + 2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha^1_{h_1}\\
&+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{t^{q_1}}{q_1}\Big)^2(c^1_{f_1}+\bar{c}^1_{f_1}) + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{q_1}}{q_1}Tr(Q)\Big(\frac{t^{q_1}}{q_1}(c^1_{\sigma_1}+\bar{c}^1_{\sigma_1})\Big)\Big)\Big]
\end{aligned}
$$

Thus,

$$
\|\Phi_1(w,v)\|^2_{\mathcal{H}_1} \leq \Lambda_1 + \Lambda_2 R
$$

By similar technique, we get

$$
\begin{aligned}
\|\Phi_2(w,v)\|^2_{\mathcal{H}_1} &= 5\Big[\sup_{t\in[0,t_1]} t^{2(1-p_1)(1-q_1)}\Big(\Big(\frac{M_T t_1^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\mathbb{E}|v_0|^2_{\mathcal{X}} + M^2_{\alpha}\bar{\alpha}^2_{h_2}\\
&+ \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q\alpha)}\Big)^2\bar{\alpha}^2_{h_2} + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t^{q_1}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q-\beta)}t^{(q_1-\beta)}\|\psi_2\|_{L^{\frac{1}{\beta}}}\\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t^{q_1}}{q_1}Tr(Q)\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}t^{(q_1-\beta)}\|\phi_2\|_{L^{\frac{1}{\beta}}}\Big]\\
&+ 5R\Big[\sup_{t\in[0,t_1]} t^{2(1-p_1)(1-q_1)}\Big(2M^2_{\alpha}\alpha^2_{h_2} + 2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha^2_{h_2}\\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{t^{q_1}}{q_1}\Big)^2(c^2_{f_2}+\bar{c}^2_{f_2}) + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{q_1}}{q_1}Tr(Q)\Big(\frac{t^{q_1}}{q_1}(c^2_{\sigma_2}+\bar{c}^2_{\sigma_2})\Big)\Big)\Big]\\
&= \bar{\Lambda}_1 + \bar{\Lambda}_2 R
\end{aligned}
$$

In the similar way, for any $t \in J_k, k = 1, \dots, m$:

$$\begin{aligned}
\mathbb{E}|\Phi_1(w,v)(t)|^2 &\leq 6\Big(\mathbb{E}|\mathcal{S}_{p_1,q_1}(t-t_k)(w(t_k^-))|^2 + \mathbb{E}|\mathcal{S}_{p_1,q_1}(t-t_k)I_k((w(t_k^-), v(t_k^-))|^2 \\
&+ \mathbb{E}|h_1(t,x(t),y(t))| + \mathbb{E}\Big|\int_{t_k}^t A_1\mathcal{P}_{q_1}(t-s)h_1(s,w(s),v(s))ds\Big|^2 \\
&+ \mathbb{E}\Big|\int_{t_k}^t f_1(s,x(s),y(s))ds\Big|^2 + \mathbb{E}\Big|\int_{t_k}^t \mathcal{P}_{q_1}(t-s)\Big(\int_0^s \sigma_1(\zeta,x(\zeta),y(\zeta))dB(\zeta)\Big)ds\Big|^2\Big) \\
&\leq 6\Big(\Big(\frac{M_T(t_{k+1}-t_k)^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\Big(\mathbb{E}|(w(t_k^-))|^2_{\mathcal{X}} + d_k^1\mathbb{E}|w(t_k^-)|^2_{\mathcal{X}} + \bar{d}_k^1\mathbb{E}|v(t_k^-)|^2\Big) \\
&+ M_\alpha^2(\alpha_{h_1}^1(\mathbb{E}|w(t)|^2_{\mathcal{X}} + \mathbb{E}|v(t)|^2_{\mathcal{X}}) + \bar{\alpha}_{h_1}^1) \\
&+ \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q\alpha)}\Big)^2(\alpha_{h_1}^1(\mathbb{E}|w(t)|^2_{\mathcal{X}} + \mathbb{E}|v(t)|^2_{\mathcal{X}}) + \bar{\alpha}_{h_1}^1) \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{(t_{k+1}-t_k)^{2q_1-\beta}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\psi_1\|_{L^{\frac{1}{\beta}}} \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\Big(\frac{(t_{k+1}-t_k)^{q_1}}{q_1}\Big)^2(c_{f_1}^1\mathbb{E}|w(t)|^2_{\mathcal{X}} + \bar{c}_{f_1}^1\mathbb{E}|v(t)|^2_{\mathcal{X}}) \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{(t_{k+1}-t_k)^{2q_1-\beta}}{q_1}Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\phi_1\|_{L^{\frac{1}{\beta}}} \\
&+ \frac{(t_{k+1}-t_k)^{q_1}}{q_1}(c_{\sigma_1}^1\mathbb{E}|w(t)|^2_{\mathcal{X}} + \bar{c}_{\sigma_1}^1\mathbb{E}|v(t)|^2_{\mathcal{X}})\Big)
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\Phi_1(w,v)\|^2_{\mathcal{H}_1} &\leq 6R\Big(\Big(\frac{M_Tb^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\Big(1 + d_k^1 + \bar{d}_k^1\Big) + 2M_\alpha^2\alpha_{h_1}^1 + 2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha_{h_1}^1\Big) \\
&+ \Big(\frac{M_T}{\sqrt{q_1}\Gamma(q_1)}\Big)^2\frac{b^{q_1-\beta}}{q_1}Tr(Q)(c_{\sigma_1}^1 + \bar{c}_{\sigma_1}^1) + \Big(\frac{M_Tb^{q_1}}{\Gamma(q_1)q_1}\Big)^2(c_{f_1}^1 + \bar{c}_{f_1}^1)\Big) \\
&+ 6b^{2(1-\gamma)}\Big(M_\alpha^2\bar{\alpha}_{h_1}^1 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}_{h_1}^1 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1-\beta}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\psi_1\|_{L^{\frac{1}{\beta}}} \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1-\beta}}{q_1}Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\phi_1\|_{L^{\frac{1}{\beta}}}\Big) \\
&= \Lambda_3 + \Lambda_4R
\end{aligned}$$

Similarly

$$\begin{aligned}
\|\Phi_2(w,v)\|^2_{\mathcal{H}_1} &\leq 6R\Big(\Big(\frac{M_Tb^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\Big(1 + d_k^2 + \bar{d}_k^2\Big) + 2M_\alpha^2\alpha_{h_2}^2 + 2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha_{h_2}^2\Big) \\
&+ \Big(\frac{M_T}{\sqrt{q_1}\Gamma(q_1)}\Big)^2\frac{b^{q_1-\beta}}{q_1}Tr(Q)(c_{\sigma_2}^2 + \bar{c}_{\sigma_2}^2) + \Big(\frac{M_Tb^{q_1}}{\Gamma(q_1)q_1}\Big)^2(c_{f_2}^2 + \bar{c}_{f_2}^2)\Big) \\
&+ 6b^{2(1-\gamma)}\Big(M_\alpha^2\bar{\alpha}_{h_2}^2 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}_{h_2}^2 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1-\beta}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\psi_2\|_{L^{\frac{1}{\beta}}} \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1-\beta}}{q_1}Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\phi_2\|_{L^{\frac{1}{\beta}}}\Big) \\
&= \bar{\Lambda}_3 + \bar{\Lambda}_4R
\end{aligned}$$

This yields that $\Phi_1(\mathcal{B}_R \times \mathcal{B}_R) \subseteq \mathcal{B}_R$ and $\Phi_2(\mathcal{B}_R \times \mathcal{B}_R) \subseteq \mathcal{B}_R$.

The second step: Φ is continuous.

Let $\{(w^n, v^n)\}_{n=1}^{\infty} \subseteq \mathcal{H}_1 \times \mathcal{H}_1$ be a sequence such that $(w^n, v^n) \to (w, v)$, $(n \to \infty)$ in $\mathcal{H}_1 \times \mathcal{H}_1$. Then there is a number $R > 0$ such that $\|w^n\|^2_{\mathcal{H}_1} \leq R$, $\|v^n\|^2_{\mathcal{H}_1} \leq R$ for all n and a.e. $t \in J$, so $w^n, v^n \in \mathcal{B}_R$ and $w^n, v^n \in \mathcal{B}_R$. Thus, by (H_3)(i) and (H_4)(i),

$$f_i(t, w^n(t), v^n(t)) \to f_i(t, w(t), v(t)),\ i = 1, 2, \quad \text{as } n \to \infty$$

and

$$\sigma_i(t, w^n(t), v^n(t)) \to \sigma_i(t, w(t), v(t)),\ i = 1, 2, \quad \text{as } n \to \infty$$

Hence, by $(H_2)(ii)$ and $(H_4)(iii)$, we see that, for $t \in [0, t_1]$,

$$(t-s)^{q_1-1}\mathbb{E}\Big|(f_i(s, w^n(s), v^n(s)) - f_i(s, w(s), v(s)))\Big|^2_{\mathcal{X}} \leq 2(t-s)^{q_1-1}(\psi_i(t) + c^i_{f_i}R + \bar{c}^i_{f_i}R)$$

and

$$(t-s)^{q_1-1}\mathbb{E}\Big|\int_0^s (\sigma_i(\zeta, w^n(\zeta), v^n(\zeta)) - \sigma_i(\zeta, w(\zeta), v(\zeta)))dB\Big|^2_{\mathcal{X}} \leq 2Tr(Q)(t-s)^{q_1-1}(\phi_i(s) + c^i_{\sigma_i}R + \bar{c}^i_{\sigma_i}R).$$

Lebesgue dominated the convergence theorem, we need the uniform convergence of the preceding relationships to

$$\begin{aligned}
&\mathbb{E}|\Phi_i(w^n, v^n)(t) - \Phi_i(w, v)(t)|^2_{\mathcal{X}}\\
&\quad \leq 4\Big(\mathbb{E}|h_i(t, w^n(t), v^n(t)) - h_1(t, w(t), v(t)))|^2_{\mathcal{X}}\\
&\qquad + \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)(A_i h_i(s, w^n(s), v^n(s)) - A_i h_i(s, w(s), v(s)))ds\Big|^2_{\mathcal{X}}\\
&\qquad + \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)(f_i(s, w^n(s), v^n(s)) - f_i(s, w(s), v(s)))ds\Big|^2_{\mathcal{X}}\\
&\qquad + \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)\Big(\int_0^s (\sigma_i(\zeta, w^n(\zeta), v^n(\zeta)) - \sigma_i(\zeta, w(\zeta), v(\zeta)))dB(\zeta)\Big)ds\Big|^2_{\mathcal{X}}\Big)
\end{aligned}$$

for each $t \in J_k, k = 1, \ldots, m$, by a similar argument we obtain

$$\begin{aligned}
&\mathbb{E}|\Phi_1(w^n, v^n)(t) - \Phi_1(w, v)(t)|^2_{\mathcal{X}}\\
&\quad \leq 6\Big(\mathbb{E}|\mathcal{S}_{p_1,q_1}(t-t_k)((w^n(t_k^-)) - (w(t_k^-))|^2_{\mathcal{X}}\\
&\qquad + \mathbb{E}|\mathcal{S}_{p_1,q_1}(t-t_k)(I_k(x^n(t_k^-), v^n(t_k^-)) - I_k(w^n(t_k^-), v^n(t_k^-))|^2_{\mathcal{X}}\\
&\qquad + \mathbb{E}|h_1(t, w^n(t), v^n(t)) - h_1(t, x(t), v(t)))|^2_{\mathcal{X}}\\
&\qquad + \mathbb{E}\Big|\int_{t_k}^t \mathcal{P}_{q_1}(t-s)(A_1 h_1(s, w^n(s), v^n(s)) - A_1 h_1(s, w(s), v(s))ds\Big|^2_{\mathcal{X}}\\
&\qquad + \mathbb{E}\Big|\int_{t_k}^t \mathcal{P}_{q_1}(t-s)(f_1(s, w^n(s), v^n(s)) - f_1(s, w(s), v(s)))ds\Big|^2_{\mathcal{X}}\\
&\qquad + \mathbb{E}\Big|\int_{t_k}^t \mathcal{P}_{q_1}(t-s)\Big(\int_0^s (\sigma_1(\zeta, w^n(\zeta), v^n(\zeta)) - \sigma_1(\zeta, w(\zeta), y(\zeta)))dB(\zeta)\Big)ds\Big|^2_{\mathcal{X}}\Big)
\end{aligned}$$

similarly, using the same way, we find

$$\mathbb{E}|\Phi_2(w^n, v^n)(t) - \Phi_2(w, v)(t)|^2_{\mathcal{X}}$$

$$\leq 6\Big(\mathbb{E}|\mathcal{S}_{p_1,q_1}(t-t_k)(v^n(t_k^-)) - (v(t_k^-))|^2_{\mathcal{X}}$$
$$+\mathbb{E}|\mathcal{S}_{p_1,q_1}(t-t_k)(\bar{I}_k(w^n(t_k^-), v^n(t_k^-)) - \bar{I}_k(w^n(t_k^-), v^n(t_k^-)|^2_{\mathcal{X}}$$
$$+\mathbb{E}|h_2(t, w^n(t), y^n(t)) - h_2(t, w(t), v(t))|^2_{\mathcal{X}}$$
$$+\mathbb{E}\Big|\int_{t_k}^{t}\mathcal{P}_{q_1}(t-s)(A_2h_2(s, w^n(s), v^n(s)) - A_2h_2(s, w(s), v(s)))ds\Big|^2_{\mathcal{X}}$$
$$+\mathbb{E}\Big|\int_{t_k}^{t}\mathcal{P}_{q_1}(t-s)(f_2(s, w^n(s), v^n(s)) - f_2(s, w(s), v(s)))ds\Big|^2_{\mathcal{X}}$$
$$+\mathbb{E}\Big|\int_{t_k}^{t}\mathcal{P}_{q_1}(t-s)\Big(\int_0^s(\sigma_2(\zeta, w^n(\zeta), v^n(\zeta)) - \sigma_2(\zeta, w(\zeta), v(\zeta)))dB(\zeta)\Big)ds\Big|^2_{\mathcal{X}}\Big)$$

Clearly, the right-hand side of tends to zero as $n \to \infty$. This implies that Φ is continuous in $\mathcal{B}_R \times \mathcal{B}_R$.

The third step: Φ maps bounded sets into bounded sets in $\mathcal{H}_1 \times \mathcal{H}_1$. Indeed, it is sufficient to show that for any $R > 0$, $\exists$ a positive constant κ such that for each $z \in \mathcal{B}_R = \{(z \in \mathcal{H}_1,\ \|z\|_{\mathcal{H}_1} \leq R\}$, we have

$$\|\Phi(w, v)\|_{\mathcal{H}_1} \leq \kappa = (\kappa_1, \kappa_2).$$

Then, for each $t \in J$ and thanks to Lemma 2.1 and 2.2, for each $t \in [0, t_1]$

$$\|\Phi_1(w, v)\|^2_{\mathcal{H}_1} \leq \Lambda_1 + \Lambda_2 R = \kappa_1^1$$

In the same fashion, one has

$$\|\Phi_2(w, v)\|^2_{\mathcal{H}_1} \leq \bar{\Lambda}_1 + \bar{\Lambda}_2 R = \kappa_1^2$$

for each $t \in J_k, k = 1, \ldots, m$, which gives

$$\begin{aligned}
\|\Phi_1(w, v)\|^2_{\mathcal{H}_1} &\leq 6R\Big(\Big(\frac{M_T(t-t_k)^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\Big(1 + d_k^1 + \bar{d}_k^1\Big) + 2M_\alpha^2\alpha_{h_1}^1 + 2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha_{h_1}^1\Big) \\
&+ \Big(\frac{M_T}{\sqrt{q_1}\Gamma(q_1)}\Big)^2\frac{(t-t_k)^{q_1-\beta}}{q_1}Tr(Q)(c_{\sigma_1}^1 + \bar{c}_{\sigma_1}^1) + \Big(\frac{M_T(t-t_k)^{q_1}}{\Gamma(q_1)q_1}\Big)^2(c_{f_1}^1 + \bar{c}_{f_1}^1)\Big) \\
&+ 6(t-t_k)^{2(1-\gamma)}\Big(M_\alpha^2\bar{\alpha}_{h_1}^1 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}_{h_1}^1 \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{(t-t_k)^{2q_1-\beta}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\psi_1\|_{L^{\frac{1}{\beta}}} \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{(t-t_k)^{2q_1-\beta}}{q_1}Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\phi_1\|_{L^{\frac{1}{\beta}}}\Big) \\
&= \kappa_1^k
\end{aligned}$$

Similarly, we can also get

$$\begin{aligned}
\|\Phi_2(w, v)\|^2_{\mathcal{H}_1} &\leq 6R\Big(\Big(\frac{M_T(t-t_k)^{1-\gamma}}{\Gamma(\gamma)}\Big)^2\Big(1 + d_k^2 + \bar{d}_k^2\Big) + 2M_\alpha^2\alpha_{h_2}^2 + 2\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\alpha_{h_2}^2\Big) \\
&+ \Big(\frac{M_T}{\sqrt{q_1}\Gamma(q_1)}\Big)^2\frac{(t-t_k)^{q_1-\beta}}{q_1}Tr(Q)(c_{\sigma_2}^2 + \bar{c}_{\sigma_1}^1) + \Big(\frac{M_T(t-t_k)^{q_1}}{\Gamma(q_1)q_1}\Big)^2(c_{f_2}^1 + \bar{c}_{f_2}^1)\Big) \\
&+ 6(t-t_k)^{2(1-\gamma)}\Big(M_\alpha^2\bar{\alpha}_{h_2}^2 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{\alpha}_{h_2}^2 \\
&+ \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{(t-t_k)^{2q_1-\beta}}{q_1}\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)}\|\psi_2\|_{L^{\frac{1}{\beta}}}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2 \frac{(t-t_k)^{2q_1-\beta}}{q_1} Tr(Q)\Big(\Big(\frac{(1-\beta)}{(q_1-\beta)}\Big)^{(q_1-\beta)} \|\phi_2\|_{L^{\frac{1}{\beta}}}\Big)\\
&= \quad \kappa_2^k
\end{aligned}
$$

Take $\kappa_1 = \max\limits_{k=1,\dots,\cdot,m}\{\kappa_1^1, \kappa_1^k\}$ and $\kappa_2 = \max\limits_{k=1,\dots,\cdot,m}\{\kappa_1^2, \kappa_2^k\}$

Further, we obtain

$$
\begin{pmatrix} \|\Phi_1(w,v)\|^2_{\mathcal{H}_1} \\ \|\Phi_2(w,v)\|^2_{\mathcal{H}_1} \end{pmatrix} \leq \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}
$$

The fourth step: $\Phi = (\Phi_1, \Phi_2)$ is a $\chi-$ contraction

We decompose Φ as $\Phi_1 = \Phi_1^1 + \Phi_1^2$ and $\Phi_2 = \Phi_2^1 + \Phi_2^2$ where the operators Φ_1^1, Φ_1^2 and Φ_2^1, Φ_2^2 are defined on $\mathcal{H}_1 \times \mathcal{H}_1$, respectively, by

$$
\Phi_1^1(w,v)(t) = \begin{cases} \quad \mathcal{S}_{p_1,q_1}(t)w_0 + \displaystyle\int_0^t \mathcal{P}_{q_1}(t-s)f_1(s,w(s),v(s)), \quad \mathbb{P}-a.s, \;\; t\in[0,t_1] \\[2ex] \quad \mathcal{S}_{p_1,q_1}(t-t_k)((t_k^-) + I_k((w(t_k^-),v(t_k^-)) \\ + \;\; \displaystyle\int_{t_k}^t \mathcal{P}_{q_1}(t-s)f_1(s,x(s),y(s))ds, \quad \mathbb{P}-a.s, \;\; t\in(t_k,t_{k+1}] \end{cases}
$$

$$
\Phi_1^2(w,v)(t) = \begin{cases} \quad h_1(t,w(t),v(t)) + \displaystyle\int_0^t \mathcal{P}_{q_1}(t-s)\Big(A_1h_1(s,w(s),v(s))ds \\ + \;\; \displaystyle\int_0^s \sigma_1(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \;\; t\in[0,t_1] \\[2ex] \quad h_1(t,w(t),v(t)) + \displaystyle\int_{t_k}^t \mathcal{P}_{q_1}(t-s)\Big(A_1h_1(s,w(s),v(s)) \\ + \;\; \displaystyle\int_0^s \sigma_1(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \;\; t\in(t_k,t_{k+1}], \; \gamma = p_1+q_1-p_1q_1 \end{cases}
$$

and

$$
\Phi_2^1(x,y)(t) = \begin{cases} \quad \mathcal{S}_{p_1,q_1}(t)y_0 + \displaystyle\int_0^t \mathcal{P}_{q_1}(t-s)f_2(s,w(s),v(s)), \quad \mathbb{P}-a.s, \;\; t\in[0,t_1] \\[2ex] \quad \mathcal{S}_{p_1,q_1}(t-t_k)(w(t_k^-) + \bar{I}_k((w(t_k^-),v(t_k^-)) \\ + \;\; \displaystyle\int_{t_k}^t \mathcal{P}_{q_1}(t-s)f_2(s,w(s),v(s))ds, \quad \mathbb{P}-a.s, \;\; t\in(t_k,t_{k+1}] \end{cases}
$$

$$
\Phi_2^2(w,v)(t) = \begin{cases} \quad h_2(t,w(t),v(t)) + \displaystyle\int_0^t \mathcal{P}_{q_1}(t-s)\Big(A_2h_2(s,w(s),v(s))ds \\ + \;\; \displaystyle\int_0^s \sigma_2(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \;\; t\in[0,t_1] \\[2ex] \quad h_2(t,w(t),v(t)) + \displaystyle\int_{t_k}^t \mathcal{P}_{q_1}(t-s)\Big(A_2h_2(s,w(s),v(s)) \\ + \;\; \displaystyle\int_0^s \sigma_2(\zeta,w(\zeta),v(\zeta))dB(\zeta)\Big)ds, \quad \mathbb{P}-a.s, \;\; t\in(t_k,t_{k+1}], \; \gamma = p_1+q_1-p_1q_1 \end{cases}
$$

We will show that Φ_1^1, Φ_2^1 is a compact operator, while the Φ_1^2, Φ_2^2 verifies a contraction condition. We take $(w,v) \in \mathcal{B}_R \times \mathcal{B}_R$, to prove Φ_1^2, Φ_2^2 satisfies a contraction condition. We require the beneath three claims.

The first claim: Φ_1^2, Φ_2^2 is Lipschitz continuous.
Let $t \in [0, t_1]$ and $(w, v), (\bar{w}, \bar{v}) \in \mathcal{B}_R \times \mathcal{B}_R$. From (H_2) (ii),(H_4)(iii) and Lemma 2.2, we have

$$\begin{aligned}
&\mathbb{E}|\Phi_1^2(w,v)(t) - \Phi_1^2(\bar{w},\bar{v})(t)|_{\mathcal{X}}^2 \\
&\leq 3\Big(\mathbb{E}|h_1(t,w(t),v(t)) - h_1(t,\bar{w}(t),\bar{v}(t)))|_{\mathcal{X}}^2 \\
&\quad + \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)(A_1h_1(s,w(s),v(s)) - A_1h_1(s,\bar{w}(s),\bar{v}(s)))ds\Big|_{\mathcal{X}}^2 \\
&\quad + \mathbb{E}\Big|\int_0^t \mathcal{P}_{q_1}(t-s)\Big(\int_0^s (\sigma_1(\zeta,x(\zeta),y(\zeta)) - \sigma_1(\zeta,\bar{x}(\zeta),\bar{y}(\zeta)))dB(\zeta)\Big)ds\Big|_{\mathcal{X}}^2\Big) \\
&\leq M_\alpha^2\Big(c_h^1\mathbb{E}|w(t)-\bar{w}(t)|_{\mathcal{X}}^2 + \bar{c}_h^1\mathbb{E}|v(t)-\bar{v}(t)|_{\mathcal{X}}^2\Big) \\
&\quad + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q\alpha)}\Big)^2\Big(c_h^1\mathbb{E}|w(t)-\bar{w}(t)|_X^2 + \bar{c}_h^1\mathbb{E}|v(t)-\bar{v}(t)|_{\mathcal{X}}^2\Big) \\
&\quad + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\Big(\lambda_{\sigma_1}^1\mathbb{E}|v(t)-\bar{v}(t)|_{\mathcal{X}}^2 + \bar{\lambda}_{\sigma_1}^1\mathbb{E}|v(t)-\bar{v}(t)|_{\mathcal{X}}^2)\Big) \\
&\leq 3\Big(M_\alpha^2c_h^1 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2c_h^1 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\lambda_{\sigma_1}^1\Big)\mathbb{E}|w(t)-\bar{w}(t)|_{\mathcal{X}}^2 \\
&\quad + 3\Big(M_\alpha^2\bar{c}_h^1 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{c}_h^1 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\bar{\lambda}_{\sigma_1}^1\Big)\mathbb{E}|v(t)-\bar{v}(t)|_{\mathcal{X}}^2
\end{aligned}$$

Thus,

$$||\Phi_1^2(w,v) - \Phi_1^2(\bar{w},\bar{v})||_{\mathcal{H}_1}^2 \leq \mu_1(t_1)||w-\bar{w}||_{\mathcal{H}_1}^2 + \mu_2(t_1)||v-\bar{v}||_{\mathcal{H}_1}^2$$

Similarly, using the same way, we find

$$||\Phi_2^2(w,v) - \Phi_2^2(\bar{w},\bar{v})||_{\mathcal{H}_1}^2 \leq \bar{\mu}_1(t_1)||w-\bar{w}||_{\mathcal{H}_1}^2 + \bar{\mu}_2(t_1)||v-\bar{v}||_{\mathcal{H}_1}^2$$

where

$$\mu_1(t_1) = 3t_1^{2(1-\gamma)}\Big(M_\alpha^2c_h^1 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2c_h^1 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\lambda_{\sigma_1}^1\Big),$$

$$\mu_2(t_1) = 3t_1^{2(1-\gamma)}\Big(M_\alpha^2\bar{c}_h^1 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{c}_h^1 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\bar{\lambda}_{\sigma_1}^1\Big)$$

and

$$\bar{\mu}_1(t_1) = 3t_1^{2(1-\gamma)}\Big(M_\alpha^2c_h^2 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2c_h^2 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\lambda_{\sigma_2}^2\Big)$$

$$\bar{\mu}_2(b) = 3t_1^{2(1-\gamma)}\Big(M_\alpha^2\bar{c}_h^2 + \Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{c}_h^1 + \Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\bar{\lambda}_{\sigma_2}^2\Big)$$

Hence

$$\begin{aligned}
||\Phi^2(w,v) - \Phi^2(\overline{w},\overline{v})||_{\mathcal{H}_1}^2 &= \begin{pmatrix} ||\Phi_1^2(w,v) - \Phi_1^2(\bar{w},\bar{v})||_{\mathcal{H}_1}^2 \\ ||\Phi_2^2(w,v) - \Phi_2^2(\bar{w},\bar{v})||_{\mathcal{H}_1}^2 \end{pmatrix} \\
&\leq 3\begin{pmatrix} \mu_1(t_1) & \mu_2(t_1) \\ \bar{\mu}_1(t_1) & \bar{\mu}_2(t_1) \end{pmatrix}\begin{pmatrix} ||w-\bar{w}||_{\mathcal{H}_1}^2 \\ ||v-\bar{v}||_{\mathcal{H}_1}^2 \end{pmatrix}.
\end{aligned}$$

Therefore

$$\|\Phi^2(w,v)-\Phi^2(\overline{w},\overline{v})\|^2_{\mathcal{H}_1} \le M_{trix}(t_1)\begin{pmatrix}\|w-\bar{w}\|^2_{\mathcal{H}_1}\\ \|v-\bar{v}\|^2_{\mathcal{H}_1}\end{pmatrix}, \text{ for all}, (x,y),(\overline{x},\overline{y}) \in \mathcal{H}_1\times\mathcal{H}_1.$$

In the similar way, for any $t\in J_k, k=1,\dots,m$, we have

$$\|\Phi_1^2(x,y)-\Phi_1^2(\bar{x},\bar{y})\|^2_{\mathcal{H}_1} \le \mu_1(b)\|w-\bar{w}\|^2_{\mathcal{H}_1}+\mu_2(b)\|v-\bar{v}\|^2_{\mathcal{H}_1}$$

Similarly, we have

$$\|\Phi_2^2(w,v)-\Phi_2^2(\bar{w},\bar{v})\|^2_{\mathcal{H}_1} \le \bar{\mu}_1(b)\|v-\bar{v}\|^2_{\mathcal{H}_1}+\bar{\mu}_2(b)\|v-\bar{v}\|^2_{\mathcal{H}_1}$$

Then, for all $t\in[0,b]$, we have

$$\|\Phi^2(w,v)-\Phi^2(\overline{w},\overline{v})\|^2_{\mathcal{H}_1} \le M_{trix}(b)\begin{pmatrix}\|w-\bar{w}\|^2_{\mathcal{H}_1}\\ \|v-\bar{v}\|^2_{\mathcal{H}_1}\end{pmatrix}, \text{ for all}, (w,v),(\overline{u},\overline{v}) \in \mathcal{H}_1\times\mathcal{H}_1.$$

Since the spectral radius $\rho(M_{trix})<1$. Hence, Φ_1^2,Φ_2^2 is Lipschitz continuous.

The second claim: Φ_1^1,Φ_2^1 maps bounded sets into equicontinuous sets of $\mathcal{H}_1\times\mathcal{H}_1$.
Let B_q be a bounded set in $\mathcal{H}_1\times\mathcal{H}_1$ as in Step 3. Let $\zeta_1,\zeta_2\in[0,t_1],\zeta_1<\zeta_2$ and $w,v\in\mathcal{B}_R$. Then, for i=1,2, we get

$$\begin{aligned}
&\mathbb{E}|\Phi_1^1(w(\zeta_2),v(\zeta_2))-\Phi_1^1(w(\zeta_1),v(\zeta_1))|^2_{\mathcal{X}}\\
&\le 2\Big(\mathbb{E}|(\mathcal{S}_{p_1,q_1}(\zeta_2)-\mathcal{S}_{p_1,q_1}(\zeta_2))w_0|^2_{\mathcal{X}}\\
&\quad+\mathbb{E}\Big|\int_0^{\zeta_2}\mathcal{P}_q(\zeta_2-s)f_1(s,w(s),v(s))ds-\int_0^{\zeta_1}\mathcal{P}_{q_1}(\zeta_2-s)f_1(s,w(s),v(s))ds\Big|^2_{\mathcal{X}}\Big)\\
&\le 2\Big(\|(\mathcal{S}_{p_1,q_1}(\zeta_2)-\mathcal{S}_{p_1,q_1}(\zeta_2))\|^2\mathbb{E}|w_0|^2_{\mathcal{X}}+\mathbb{E}\Big|\int_{\zeta_1}^{\zeta_2}\mathcal{P}_{q_1}(\zeta_2-s)f_1(s,w(s),v(s))ds\\
&\quad+\int_0^{\zeta_1}(\mathcal{P}_{q_1}(\zeta_2-s)-\mathcal{P}_{q_1}(\zeta_1-s))f_1(s,w(s),v(s))ds\Big|^2_{\mathcal{X}}\Big)\\
&\le 2\Big(\|(\mathcal{S}_{p_1,q_1}(\zeta_2)-\mathcal{S}_{p_1,q_1}(\zeta_2))\|^2\mathbb{E}|w_0|^2_{\mathcal{X}}+(\zeta_2-\zeta_1)\int_{\zeta_1}^{\zeta_2}\mathbb{E}\Big|\mathcal{P}_q(\zeta_2-s)f_1(s,w(s),v(s))\Big|^2_{\mathcal{X}}ds\\
&\quad+\mathbb{E}\Big|\int_0^{\zeta_1-\epsilon}(\mathcal{P}_q(\zeta_2-s)-\mathcal{P}_{q_1}(\zeta_1-s))f_1(s,w(s),v(s))ds\Big|^2_{\mathcal{X}}\\
&\quad+\mathbb{E}\Big|\int_{\zeta_1-\epsilon}^{\zeta_1}(\mathcal{P}_{q_1}(\zeta_2-s)-\mathcal{P}_{q_1}(\zeta_1-s))f_1(s,w(s),v(s))ds\Big|^2_{\mathcal{X}}\\
&\le 2\Big(\|(\mathcal{S}_{p,q}(\zeta_2)-\mathcal{S}_{p,q}(\zeta_2))\|^2\mathbb{E}|w_0|^2_{\mathcal{X}}+(\zeta_2-\zeta_1)\int_{\zeta_1}^{\zeta_2}\mathbb{E}\Big|\mathcal{P}_{q_1}(\zeta_2-s)f_1(s,w(s),v(s))\Big|^2_{\mathcal{X}}ds\\
&\quad+\zeta_1\int_0^{\zeta_1-\epsilon}(\mathcal{P}_{q_1}(\zeta_2-s)-\mathcal{P}_q(\zeta_1-s))\mathbb{E}\Big|f_1(s,w(s),v(s))ds\Big|^2_{X}\\
&\quad+\epsilon\int_{\zeta_1-\epsilon}^{\zeta_1}(\mathcal{P}_{q_1}(\zeta_2-s)-\mathcal{P}_{q_1}(\zeta_1-s))\mathbb{E}\Big|f_1(s,w(s),v(s))ds\Big|^2_{\mathcal{X}}.
\end{aligned}$$

Similarly, for any $\zeta_1,\zeta_2\in[0,t_1],\zeta_1<\zeta_2$

$$\begin{aligned}
&\mathbb{E}|\Phi_2^1(w(\zeta_2),v(\zeta_2))-\Phi_2^1(w(\zeta_1),v(\zeta_1))|^2_{\mathcal{X}}\\
&\le 2\Big(\|(\mathcal{S}_{p_1,q_1}(\zeta_2)-\mathcal{S}_{p,q}(\zeta_2))\|^2\mathbb{E}|v_0|^2_{\mathcal{X}}+(\zeta_2-\zeta_1)\int_{\zeta_1}^{\zeta_2}\mathbb{E}\Big|\mathcal{P}_{q_1}(\zeta_2-s)f_2(s,w(s),v(s))\Big|^2_{\mathcal{X}}ds
\end{aligned}$$

$$+\zeta_1\int_0^{\zeta_1-\epsilon}(\mathcal{P}_{q_1}(\zeta_2-s)-\mathcal{P}_{q_1}(\zeta_1-s))\mathbb{E}\Big|f_2(s,w(s),v(s))ds\Big|^2_{\mathcal{X}}$$
$$+\epsilon\int_{\zeta_1-\epsilon}^{\zeta_1}(\mathcal{P}_{q_1}(\zeta_2-s)-\mathcal{P}_{q_1}(\zeta_1-s))\mathbb{E}\Big|f_2(s,w(s),v(s))ds\Big|^2_{\mathcal{X}}.$$

Observe that from $\|\Phi_i^1(w(\zeta_2),v(\zeta_2))-\Phi_i^1(w(\zeta_1),y(\zeta_1))\|^2_{\mathcal{H}_1}$ tends to zero independently of $w,v\in\mathcal{B}_R, i=1,2$ as $\zeta_1\to\zeta_2$, with ϵ sufficiently small since the compactness of $\mathcal{S}_{p_1,q_1}(t)$ for $t>0$ (see [33]) implies the continuity in the uniform operator topology. Similar result can be found when $t_k<\zeta_1<\zeta_2<t_k,\ k=1,2,\dots,m$. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem Φ_1^1,Φ_2^1 is relatively compact.

The third claim: Φ_1^2,Φ_2^2 is a χ-contraction.
Let $t\in[0,t_1]$ and $w,v,\bar{w},\bar{v}\in\mathcal{B}_R$.

$$\|\Phi_1^2(w,v)-\Phi_1^2(\bar{w},\bar{v})\|^2_{\mathcal{H}_1}$$
$$\leq 3\sup_{t\in[0,t_1]}t^{2(1-\gamma)}\Big(\mathbb{E}|h_1(t,w(t),v(t))-h_1(t,\bar{w}(t),\bar{v}(t)))|^2_{\mathcal{X}}$$
$$+\mathbb{E}\Big|\int_0^t\mathcal{P}_{q_1}(t-s)(A_1h_1(s,w(s),v(s))-Ah_1(s,\bar{w}(s),\bar{v}(s)))ds\Big|^2_{\mathcal{X}}$$
$$+\mathbb{E}\Big|\int_0^t\mathcal{P}_{q_1}(t-s)\Big(\int_0^s(\sigma_1(\zeta,w(\zeta),v(\zeta))-\sigma_1(\zeta,\bar{w}(\zeta),\bar{v}(\zeta)))dB(\zeta)\Big)ds\Big|^2_{\mathcal{X}}\Big).$$

From $(H_2)(ii),(H_4)(iii)$ and Lemma 2.1, we have

$$|\Phi_1^2(w,v)-\Phi_1^2(\bar{w},\bar{v})\|^2_{\mathcal{H}_1}$$
$$\leq 3t_1^{2(1-\gamma)}\Big(M_\alpha^2c_h^1+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2c_h^1+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\lambda^1_{\sigma_1}\Big)\|w-\bar{w}\|^2_{\mathcal{H}_1}$$
$$+3t_1^{2(1-\gamma)}\Big(M_\alpha^2\bar{c}_h^1+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{c}_h^1+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{t_1^{2q_1}}{q_1^2}Tr(Q)\bar{\lambda}^1_{\sigma_1}\Big)\|v-\bar{v}\|^2_{\mathcal{H}_1}$$

which implies that

$$\|\Phi_1^2(w,v)-\Phi_1^2(\bar{w},\bar{v})\|^2_{\mathcal{H}_1}\leq\mu_1(t_1)\|w-\bar{w}\|^2_{\mathcal{H}_1}+\mu_2(t_1)\|v-\bar{v}\|^2_{\mathcal{H}_1}$$

Similarly, for any $t\in[0,t_1]$, we see that

$$\|\Phi_2^2(w,v)-\Phi_2^2(\bar{w},\bar{v})\|^2_{\mathcal{H}_1}\leq\bar{\mu}_1(t_1)\|w-\bar{w}\|^2_{\mathcal{H}_1}+\bar{\mu}_2(t_1)\|v-\bar{v}\|^2_{\mathcal{H}_1}$$

For any $t\in J_k,k=1,2,...,m$ and $w,v,\bar{w},\bar{v}\in\mathcal{B}_R$, we have

$$\|\Phi_1^2(w,v)-\Phi_1^2(\bar{w},\bar{v})\|^2_{\mathcal{H}_1}$$
$$\leq 3b^{2(1-\gamma)}\Big(M_\alpha^2c_h^1+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2c_h^1+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1}}{q_1^2}Tr(Q)\lambda^1_{\sigma_1}\Big)\|w-\bar{w}\|^2_{\mathcal{H}_1}$$
$$+3b^{2(1-\gamma)}\Big(M_\alpha^2\bar{c}_h^1+\Big(\frac{C_{1-\alpha}\Gamma(1+\alpha)}{\alpha\Gamma(1+q_1\alpha)}\Big)^2\bar{c}_h^1+\Big(\frac{M_T}{\Gamma(q_1)}\Big)^2\frac{b^{2q_1}}{q_1^2}Tr(Q)\bar{\lambda}^1_{\sigma_1}\Big)\|v-\bar{v}\|^2_{\mathcal{H}_1}$$

Hence, for all $t\in[0,b]$:

$$\|\Phi_1^2(w,v)-\Phi_1^2(\bar{w},\bar{v})\|^2_{\mathcal{H}_1}\leq\mu_1(b)\|w-\bar{w}\|^2_{\mathcal{H}_1}+\mu_2(b)\|v-\bar{v}\|^2_{\mathcal{H}_1}$$

Similarly,

$$\|\Phi_2^2(w,v)-\Phi_2^2(\bar{w},\bar{v})\|^2_{\mathcal{H}_1}\leq\bar{\mu}_1(b)\|w-\bar{w}\|^2_{\mathcal{H}_1}+\bar{\mu}_2(b)\|v-\bar{v}\|^2_{\mathcal{H}_1}$$

We give now an upper estimate for $\chi(N_1^2((\mathcal{B}_R,\mathcal{B}_R))$. we get the inequalities given bellow

$$\chi(\Phi_1^2((\mathcal{B}_R,\mathcal{B}_R)) \leq \mu_1(b)\chi(\mathcal{B}_R) + \mu_2(b)\chi(\mathcal{B}_R)$$

Similarly, we have

$$\chi(\Phi_2^2((\mathcal{B}_R,\mathcal{B}_R)) \leq \bar{\mu}_1(b)\chi(\mathcal{B}_R) + \bar{\mu}_2(b)\chi(\mathcal{B}_R)$$

Then

$$\begin{pmatrix}\chi(\Phi_1((\mathcal{B}_R,\mathcal{B}_R))\\ \chi(\Phi_2((\mathcal{B}_R,\mathcal{B}_R))\end{pmatrix} \leq \begin{pmatrix}\chi(\Phi_1^1((\mathcal{B}_R,\mathcal{B}_R)) + \chi(N_1^2((\mathcal{B}_R,\mathcal{B}_R))\\ \chi(\Phi_2^1((\mathcal{B}_R,\mathcal{B}_R)) + \chi(N_2^2((\mathcal{B}_R,\mathcal{B}_R))\end{pmatrix}$$

Since $N_1^1(\mathcal{B}_R,\mathcal{B}_R)$ and $\Phi_2^1(\mathcal{B}_R,\mathcal{B}_R)$ is relatively compact in $\mathcal{X}$,

$$\begin{pmatrix}\chi(\Phi_1^1((\mathcal{B}_R,\mathcal{B}_R))\\ \chi(\Phi_2^1((\mathcal{B}_R,\mathcal{B}_R))\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Hence,

$$\begin{aligned}\chi(\Phi^2(\mathcal{B}_R,\mathcal{B}_R)) &= \begin{pmatrix}\chi(\Phi_1^2((\mathcal{B}_R,\mathcal{B}_R))\\ \chi(\Phi_2^2((\mathcal{B}_R,\mathcal{B}_R))\end{pmatrix}\\ &\leq \begin{pmatrix}\mu_1(b) & \mu_2(b)\\ \bar{\mu}_1(b) & \bar{\mu}_2(b)\end{pmatrix}\begin{pmatrix}\chi(\mathcal{B}_R)\\ \chi(\mathcal{B}_R)\end{pmatrix}.\end{aligned}$$

where

$$M_{trix}(b) = \begin{pmatrix}\mu_1(b) & \mu_2(b)\\ \bar{\mu}_1(b) & \bar{\mu}_2(b)\end{pmatrix}$$

As a consequence of Lemma **??**, the mapping Φ has at least one fixed point on $\mathcal{B}_R \times \mathcal{B}_R$. □

4. Example

Consider the following Hilfer fractional stochastic partial differential equation with impulsive effects

$$\begin{cases} D_{0^+}^{p_1,\frac{3}{4}}\Big(w(t,\varsigma) - H_1(t,w(t,\varsigma),v(t,\varsigma))\Big) = \frac{\partial^2}{\partial\varsigma^2}w(t,\varsigma) + F(t,w(t,\varsigma),v(t,\varsigma)) \\ + \displaystyle\int_0^t \sigma_1(\zeta,w(\zeta,\varsigma),v(\zeta,\varsigma)dB(\zeta), \quad t\geq 0, \quad t\neq t_k, \quad 0\leq\varsigma\leq\pi, \\ D_{0^+}^{p_1,\frac{3}{4}}\Big(v(t,\varsigma) - H_2(t,w(t,\varsigma),v(t,\varsigma))\Big) = \frac{\partial^2}{\partial\varsigma^2}v(t,\varsigma) + G(t,w(t,\varsigma),v(t,\varsigma)) \\ + \displaystyle\int_0^t \sigma_2(\zeta,w(\zeta,\varsigma),v(\zeta,\varsigma)dB(\zeta), \quad t\geq 0, \quad t\neq t_k, \quad 0\leq\varsigma\leq\pi, \\ w(t_k^+,\varsigma) - w(t_k^-,\varsigma) = \alpha_k w(t_k^-,\varsigma), \quad k=1,\cdots,m, \\ v(t_k^+,\varsigma) - v(t_k^-,\varsigma) = \bar{\alpha}_k v(t_k^-,\varsigma), \quad k=1,\cdots,m, \\ w(t,0) = w(t,\pi) = 0, t\geq 0, \\ v(t,0) = v(t,\pi) = 0, t\geq 0, \\ I_0^{(1-p_1)\frac{1}{4}}(w(0,\varsigma)) + \int_0^\pi \mathcal{K}_1(\varsigma,z)w(t,\varsigma) = w_0(\varsigma),\ 0\leq\varsigma\leq\pi, \\ I_0^{(1-p_1)\frac{1}{4}}(v(0,\varsigma)) + \int_0^\pi \mathcal{K}_2(\varsigma,z)v(t,\varsigma) = v_0(\varsigma),\ 0\leq\varsigma\leq\pi, \end{cases} \tag{5}$$

where $D_{0^+}^{p_1,\frac{3}{4}}$ is the Hilfer fractional derivative of order $p_1 \in [0,1]$, $q_1 = \frac{3}{4}$, $I_0^{\frac{1}{4}(1-p_1)}$ is the Riemann-Liouville integral of order $\frac{1}{4}(1-p_1)$ and $\alpha_k > 0$, B represents a one-dimensional standard Brownian motion.

Let

$$x(t)(\varsigma) = w(t,\varsigma), y(t)(\varsigma) = v(t,\varsigma) \ \ t\in J, \ \ \varsigma\in[0,\pi],$$

$$I_k(x(t_k))(\varsigma) = \alpha_k w(t_k^-,\varsigma), \bar{I}_k(y(t_k))(\varsigma) = \alpha_k v(t_k^-,\varsigma) \ \ \varsigma\in[0,\pi], \ \ k=1,\cdots,m,$$

$$f(t,x(t),y(t))(\varsigma) = F(t,w(t,\varsigma),v(t,\varsigma)), \ \ , \ \varsigma \in [0,\pi],$$
$$g(t,x(t),y(t))(\varsigma) = G(t,w(t,\varsigma),v(t,\varsigma)), \ \ , \ \varsigma \in [0,\pi].$$
$$w_0(\varsigma) = w(0,\varsigma), \ v_0(\varsigma) = v(0,\varsigma) \ \ , \ \varsigma \in [0,\pi],$$

Take $\mathcal{Y} = \mathcal{X} = L^2([0,\pi]), w_0(\varsigma), v_0(\varsigma) \in L^2([0,\pi]), \mathcal{K}_i(\varsigma,z) \in L^2([0,\pi]\times[0,\pi])$. With domain $D(A) = \{u \in \mathcal{X}, u', u'' \in \mathcal{X} \quad \text{and} \quad w(0) = w(\pi) = 0\}$, the operator A is defined byy $Au = u''$, .

So, it is clear that

$$Az = \sum_{n=1}^{\infty} -n^2 t\langle z, e_n\rangle e_n, z \in \mathcal{X},$$

and A is the infinitesimal generator of an analytic semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on $\mathcal{X}$, which is written as

$\mathcal{T}(t)w = \displaystyle\sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n$, $e_n(w) = (2/\pi)^{1/2}\sin(nu)$, where $w \in \mathcal{X}$ and $n = 1, 2, \cdots$, is the orthogonal set of eigenvectors of A. The analytic semigroup $\{T(t)\}_{t>0}$, $t \in J$, is compact and $\exists$ a constant $M \geq 1$ such that $\|\mathcal{T}(t)\|^2 \leq M$. Furthermore, for any $w \in \mathcal{X}$, we have

$$\mathcal{P}_{\frac{3}{4}}(t) = \frac{3}{4}\int_0^{\infty} \theta\psi_{\frac{3}{4}}(\theta)\mathcal{T}(t^{\frac{3}{4}}\theta)d\theta$$

$$\mathcal{P}_{\frac{3}{4}}(t)w = \frac{2}{5}\sum_{n=1}^{\infty}\int_0^t \theta\psi_{\frac{3}{4}}(\theta)\exp(-n^2 t^{frac34})d\theta\langle u, e_n\rangle e_n$$

We choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$ to define the operator $Q : \mathcal{K} \longrightarrow \mathcal{K}$, set $Qe_n = \lambda_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$

Define the process $B_Q(s)$ by

$$B_Q = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n,$$

where $\{\beta_n\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional mutually independent Brownian motions. Also, we define the following functions:

$$x(t)(\varsigma) = w(t,\varsigma), y(t)(\varsigma) = v(t,\varsigma) \ \ t \in J, \ \ \varsigma \in [0,\pi],$$

$$H_1(t,w(t,\varsigma),v(t,\varsigma)) = \frac{\alpha_1^2 w^2(t,\varsigma)}{2(1+\alpha_1^2)(1+w^2(t,\varsigma)+v^2(t,\varsigma))}, \quad \alpha_1 > 0, \ \ t \in J, \ \ \varsigma \in [0,\pi],$$

$$H_2(t,w(t,\varsigma),v(t,\varsigma)) = \frac{\alpha_2^2 v^2(t,\varsigma)}{2(1+\alpha_2^2)(1+w^2(t,\varsigma)+v^2(t,\varsigma))}, \quad \alpha_2 > 0, \ \ t \in J, \ \ \varsigma \in [0,\pi],$$

$$F(t,w(t,\varsigma),v(t,\varsigma)) = \frac{e^{-t}}{(1+e^{-t})}\sin(w(t,\varsigma)+v(t,\varsigma)))$$

$$G(t,w(t,\varsigma),v(t,\varsigma)) = \frac{e^{-2t}}{(1+e^{-2t})}\sin(w(t,\varsigma)+v(t,\varsigma)))$$

$$\sigma_1(t,w(t,\varsigma),v(t,\varsigma) = \frac{\sin(w(t,\varsigma)+v(t,\varsigma)))}{2t^{\frac{1}{4}}}$$

$$\sigma_2(t,w(t,\varsigma),v(t,\varsigma) = \frac{\sin(w(t,\varsigma)+v(t,\varsigma)))}{3t^{\frac{1}{4}}}$$

with the above choices, the system (5) can be revamped in the theoretical type of (1), since clearly functions H_i, F, G and σ_i are altogether uniformly bounded. Then again, it is simple to infer that all the conditions of Theorem 3.1 are fulfilled. In this manner, we conclude that the system (1) has a unique mild solution.

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