Filomat 37:2 (2023), 551–565
https://doi.org/10.2298/FIL2302551C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Dynamical analysis of an almost periodic multispecies mutualism system with impulsive effects and time delays

**Xuesong Chen**[a], **Demou Luo**[a,b]

[a]*School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510520, Guangdong, P. R. China*
[b]*School of Mathematics, Sun Yat-sen University, Guangzhou 510275, Guangdong, P. R. China*

**Abstract.** This article is concerned with a generalized almost periodic multispecies mutualism model with impulsive effects and time delays. By utilizing comparison theorem and constructing a feasible Lyapunov function. We establish some criteria to guarantee that the permanence, existence, uniqueness and global stability of almost periodic positive solution. A feasible numerical simulation will be provided to explain the suitability of our main criteria.

## 1. Introduction

In 1924-1926, Bohr [4–6] has established the theory about almost periodic function (APF) systematically. During the immediate decade, following Bohr's research, numerous significant works were finished to APF. We refer researchers to van Kampen [7], Bochner [8, 9] and von Neumann [10]. Almost periodic differential equations (APDEs) can be founded in various fields to characterize some phenomena such as celestial mechanics, mechanical vibration, electric or ecology system, engineering technology and so on. In view of its extensive applications from science to engineering, APDEs has been developed rapidly during the past three decades. Despite a lot of works devoted to the qualitative properties of periodic solutions (see [21]), but the study of almost periodic solutions can obtain a more general and extensive application in real world because of the different time-dependent coefficients in time period. As we know, the traditional tools of solving the qualitative problems of periodic model cannot be used to solve the same problems of almost periodic issues due to the compactness of operator. Furthermore, some results are obtained in recent decades, but there have still many unresolved problems, some of them were not even mentioned in literatures. Therefore, we claim that it will be significative to begin the investigation of almost periodic differential equations. In the field of biological dynamic, several useful researches on the APDEs have been published such as hematopoiesis system [22–27], cellular neural networks [28, 29] and so on.

Biological dynamic systems with time delays have been investigated by many researchers based on their real world applications. The mutualism model is one of the hot and interesting research subjects, which illustrates the mutualism between two or more species. What we mainly concern in this article is the qualitative properties of a multispecies mutualism system with Lotka-Volterra functional responses

2020 *Mathematics Subject Classification*. Primary 34C27, 34D20; Secondary 92D25.
*Keywords*. Multispecies mutualism system; Almost periodic solution; Impulsive effects; Time delays; Lotka-Volterra model.
Received: 29 July 2020; Accepted: 2 August 2020
Communicated by Maria Alessandra Ragusa
*Email addresses:* chenxs@gdut.edu.cn (Xuesong Chen), sysuldm@163.com (Demou Luo)

$$\begin{cases} \dot{x}_i(t) = x_i(t)\left(a_i(t) - b_i(t)x_i(t) + \displaystyle\sum_{j=1, i\neq j}^{n} c_{ij}(t)\frac{x_j(t-\tau_j(t))}{1+x_j(t-\tau_j(t))}\right), \ t \neq t_k, \\ x_i(t_k^+) = (1+h_{ik})x_i(t_k), \ k \in \mathbf{Z}^+, \ i = 1, 2, \cdots, n, \end{cases} \tag{1}$$

under the initial condition

$$x_i(\alpha) = \varphi_i(\alpha), \ \ \alpha \in [-\tau, 0], \ \ \varphi_i(\alpha) \in C([-\tau, 0], \mathbb{R}^+), \ \ i = 1, 2, \cdots, n. \tag{2}$$

The Lotka-Volterra models were introduced by Lotka and Volterra in the 1940s. They described many of the relationships between two or more species, such as competitive mechanism [18, 20], predator-prey mechanism [17, 19] and mutualism and so on. To study the mutual effects between two or more natural species of mutualism type, several biological models have been introduced and investigated by many ecologists, biologists and mathematicians [12–16, 32–34]. In 2014, Zhang et al [30] studied the dynamics of a Lotka-Volterra multispecies mutualism system with time delays

$$\dot{x}_i(t) = x_i(t)\left[a_i(t) - b_i(t)x_i(t-\tau_i(t)) + \sum_{j=1, j\neq i} \frac{c_{ij}(t)x_j(t)}{1+x_j(t)}\right], \tag{3}$$

where $x_i(t)$ stands for the population density of the species $i$ at time $t$. The time delay $\tau_i(t)$ appeared in system (3) illustrate the influence of the past history of the species $i$. However, this model did not describe the impacts of the past states of other species $j$. Recently, Lin et al [31] investigate the global asymptotic stability of a Holling type III multispecies competition-predator model with multi-time delays

$$\begin{cases} \dot{u}_i(t) = u_i(t)\left[a_i(t) - \displaystyle\sum_{k=1}^{n} b_{ik}(t)u_k(t-\tau_k(t)) - \sum_{k=1}^{m} \frac{c_{ik}u_i(t)v_k(t)}{u_i^2(t) + f_{ik}(t)}\right], \ i = 1, 2, \cdots, n, \\ \dot{v}_j(t) = v_i(t)\left[-d_j(t) + \displaystyle\sum_{k=1}^{n} \frac{e_{kj}u_k^2(t-\delta_k(t))}{u_k^2(t-\delta_k(t)) + f_{kj}(t)} - \sum_{k=1}^{m} g_{jk}(t)v_k(t-\sigma_k(t))\right], \ i, j = 1, 2, \cdots, m. \end{cases} \tag{4}$$

System (4) take into account the dependence on the past states of other species $j$. Therefore, motivated by the previous articles [30, 31], we are firmly convinced that this work can illustrate the effects of the time delays of other species. We also claim that it will be significant, interesting and beneficial to investigate the qualitative theory of system (1) as it extends previous theories and admits biological value.

Throughout the entire article, we assume that the following three conditions hold:

($H_1$) The biological coefficients $a_i(t)$, $b_i(t)$ and $c_{ij}(t)$ $(i, j = 1, 2, \cdots, n, \ i \neq j)$ are almost periodic continuous functions, where $a_i(t)$ stands for the growth rate of prey. $b_i(t)$ represents the prey population decays in the competition among the preys. $c_{ij}(t)$ is the prey is fed upon by the predators; all the parameters of the almost periodic model (1) satisfy the following two conditions:

$$\max_{i,j=1,2,\cdots,n, i\neq j} \left\{a_i(t), b_i(t), c_{ij}(t)\right\} < +\infty,$$
$$\min_{i,j=1,2,\cdots,n, i\neq j} \left\{a_i(t), b_i(t), c_{ij}(t)\right\} > 0.$$

($H_2$) $H_i(t) = \prod_{0<t_k<t}(1+h_{ik})$, $i = 1, 2, \cdots, n$, $k \in \mathbb{Z}^+$ denotes the almost periodic bounded function and there have two positive constants $H_i^l$ and $H_i^u$ satisfy $H_i^l \leq H_i(t) \leq H_i^u$.

($H_3$) Under the initial condition $\tau_i(0) = 0$, the time delay term $\tau_i(t)$ represent continuously differentiable and positive almost periodic functions on $\mathbb{R}^+$ with $\tau_i(t) < 1$. That is, $\phi_i(t) = t - \tau_i(t)$ possess the inverse function $\phi_i^{-1}(t)$. In addition, $\phi_i(t) < \phi_i^{-1}(t)$ for $t \geq 0$.

As far as I can survey, this is the first article to study the existence, uniqueness, permanence and global stability of almost periodic solutions (APS) of Lotka-Volterra multispecies mutualism model with impulsive effects and time delays on functional responses. The primary target of this article is to get several parameter conditions to guarantee the existence of a unique APS of (1) with global stability.

The organization of the rest part of this article is as follows: Section 2 contains some lemmas and the existence theorem of APS of (1). We offer the proof of the existence, permanence, global asymptotic stability and uniqueness of APS of model (1) in Section 3. In Section 4, we offer a numerical simulation to describe the applicability of our main results. In the last section, some conclusions and future orientations of investigation are performed.

## 2. Preliminaries

We restate some notations, lemmas and definitions which will be applied in the proofs of our main theorem.

**Lemma 2.1.** *Under the initial condition (2), all solutions to system (1) are positive, which means that*

$$\{(x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^{+n} | x_i(t_0) > 0 \text{ for } t_0 \geq 0,\ i = 1, 2, \cdots, n\}$$

*represents positive invariant for (1) and (2).*

*Proof.* Based on $x_i(t_0) > 0$ $(i = 1, 2, \cdots, n)$, we obtain

$$x_i(t) = x_i(t_0) \exp\left\{\int_{t_0}^{t} \left(a_i(s) - b_i(s)x_i(s) + \sum_{j=1, i\neq j} c_{ij}(s) \frac{x_j(s-\tau_j(s))}{1 + x_j(s-\tau_j(s))}\right) ds\right\}. \tag{5}$$

This ends the proof of Lemma 2.1. □

**Lemma 2.2 ([2], Lemma 2.2).** *If $a > 0$, $b > 0$, and $\dot{x} \geq (\leq) x(a - bx)$, when $t \geq 0$ and $x(0) > 0$, we deduce*

$$\liminf_{t\to\infty} x(t) \geq \frac{a}{b} \left(\limsup_{t\to\infty} x(t) \leq \frac{a}{b}\right).$$

**Lemma 2.3 (Brouwer fixed-point theorem).** *Assume that the continuous operator $A$ maps the closed and bounded convex set $Q \subset \mathbb{R}^n$ onto itself; then the operator $A$ has at least one fixed point in set $Q$.*

Here and subsequently, we focus on the system as follow

$$\dot{y}_i(t) = y_i(t) \left(a_i(t) - B_i(t)y_i(t) + \sum_{j=1, i\neq j} C_{ij}(t) \frac{x_j(t-\tau_j(t))}{1 + H_j(t)x_j(t-\tau_j(t))}\right),\ i = 1, 2, \cdots, n \tag{6}$$

under the initial condition

$$x_i(s) = \varphi_i(s),\ \ s \in [-\tau, 0],\ \ \varphi_i(s) \in C([-\tau, 0], \mathbb{R}^+),\ \ i \neq j$$

and

$$B_i(t) = \prod_{0<t_k<t} (1 + h_{ik})\, b_i(t),\ \ C_{ij}(t) = \prod_{0<t_k<t} (1 + h_{ik})\, c_{ij}(t),\ \ i \neq j.$$

**Lemma 2.4.** *All solutions to system (6) are positive, which means that* $\{(y_1(t), y_2(t), \cdots, y_n(t))^T \in \mathbb{R}^{+n} | y_i(t_0) > 0 \text{ for } t_0 \geq 0,\ i = 1, 2, \cdots, n\}$ *represents positive invariant for (6).*

**Lemma 2.5.** *The following two results hold:*

(i) *If $(y_1(t), y_2(t), \cdots, y_n(t))^T$ stands for a positive solutions of system (6), then*

$$\begin{aligned}&(x_1(t), x_2(t), \cdots, x_n(t))^T\\ =&\left(\prod_{0<t_k<t}(1+h_{1k})y_1(t), \prod_{0<t_k<t}(1+h_{2k})y_2(t), \cdots, \prod_{0<t_k<t}(1+h_{nk})y_n(t)\right)^T\end{aligned}$$

*represents a positive solution of system (1);*

(ii) *If $(x_1(t), x_2(t), \cdots, x_n(t))^T$ stands for a positive solutions of system (1), then*

$$\begin{aligned}&(y_1(t), y_2(t), \cdots, y_n(t))^T\\ =&\left(\prod_{0<t_k<t}(1+h_{1k})^{-1}x_1(t), \prod_{0<t_k<t}(1+h_{2k})^{-1}x_2(t), \cdots, \prod_{0<t_k<t}(1+h_{nk})^{-1}x_n(t)\right)^T\end{aligned}$$

*represents a positive solution of system (6);*

*for model (1) and (6).*

*Proof.* Assume that $(y_1(t), y_2(t), \cdots, y_n(t))^T$ stands for a positive solution of model (6). Set

$$x_i(t) = \prod_{0<t_k<t}(1+h_{ik})y_i(t),\ i = 1, 2, \cdots, n,$$

then for any $t \neq t_k$, by applying

$$y_i(t) = \prod_{0<t_k<t}(1+h_{ik})^{-1}x_i(t),\ i = 1, 2, \cdots, n$$

to model (6), we can effortlessly check that the first two equations of model (1) hold.

If $t = t_k$, $t \in \mathbb{Z}^+$, we obtain

$$\begin{aligned}x_i(t_k^+) =& \lim_{t\to t_k^+} x_i(t) = \lim_{t\to t_k^+}\prod_{0<t_k<t}(1+h_{ik})y_i(t) = \prod_{0<t_s\leq t_k}(1+h_{is})y_i(t_k)\\ =&(1+h_{ik})\prod_{0<t_s<t_k} y_i(t_k) = (1+h_{ik})x_i(t_k),\ i = 1, 2, \cdots, n\end{aligned} \tag{7}$$

Therefore, (7) can be applied to all the equations of system (1). Hence, $(x_1(t), x_2(t), \cdots, x_n(t))^T$ stands for a positive solution of system (1).

Now, we prove the second part of this theorem. We discover that $y_i(t)(i = 1, 2, \cdots, n)$ are continuous. Therefore, $y_i(t)(i = 1, 2, \cdots, n)$ are continuous on each interval $(t_k, t_{k+1}]$. It is effortless to verify the continuity of $y_i(t)$ at $t_k$ (impulse point), $t \in \mathbb{Z}^+$. Thanks to $y_i(t) = \prod_{0<t_k<t}(1+h_{ik})^{-1}x_i(t)$, we get

$$\begin{aligned}&y_i(t_k^+) = \prod_{0<t_s\leq t_k}(1+h_{is})^{-1}x_i(t_k^+) = \prod_{0<t_s<t_k}(1+h_{is})^{-1}x_i(t_k) = y_i(t_k),\\ &y_i(t_k^-) = \prod_{0<t_s\leq t_k}(1+h_{is})^{-1}x_i(t_k^-) = \prod_{0<t_s<t_k}(1+h_{is})^{-1}x_i(t_k) = y_i(t_k),\\ &i = 1, 2, \cdots, n.\end{aligned}$$

Hence, $y_i(t)(i = 1, 2, \cdots, n)$ are continuous on the positive axis $[0, +\infty)$ of $x$-axis. It is effortless to verify that $(y_1(t), y_2(t), \cdots, y_n(t))^T$ is a positive solution of system (6). The proof is complete now. □

**Lemma 2.6 (Barbalar Lemma).** *If $f : [0, +\infty) \to \mathbb{R}$ is uniformly continuous and $\lim_{t\to\infty}\int_0^t f(s)ds < +\infty$, then $\lim_{t\to\infty} f(t) = 0$.*

## 3. Main results

In this part, we prove the permanence, global asymptotic stability, existence and uniqueness of almost periodic solution of (1) and (6).

### 3.1. Permanence

**Theorem 3.1.** *Suppose that system (1) can satisfy ($H_1$)-($H_3$). Then system (1)-(2) possesses permanence. That is to say, any positive (almost periodic) solution $(x_1(t), x_2(t), \cdots, x_n(t))^T$ of system (1)-(2) fulfills*

$$m_i \leq \liminf_{t\to\infty} x_i(t) \leq \limsup_{t\to\infty} x_i(t) \leq M_i,\ i = 1, 2, \cdots, n.$$

*where*

$$M_i = \frac{a_i^u + \sum_{j=1, i\neq j}^n c_{ij}^u}{b_i^l},\ m_i = \frac{a_i^l}{b_i^u}.$$

*Proof.* Based on the first equation of model (1), we deduce that

$$\dot{x}_i(t) \leq \left(a_i^u - b_i^l x_i(t) + \sum_{j=1, i\neq j}^n c_{ij}^u\right),\ i = 1, 2, \cdots, n. \tag{8}$$

Employing Lemma 2.2 into (8), we obtain

$$\limsup_{t\to\infty} x_i(t) \leq \frac{a_i^u + \sum_{j=1, i\neq j}^n c_{ij}^u}{b_i^l} \equiv M_i,\ i = 1, 2, \cdots, n. \tag{9}$$

Based on the first equation of system (1), we get that

$$\dot{x}_i(t) \geq \left(a_i^l - b_i^u x_i(t)\right),\ i = 1, 2, \cdots, n. \tag{10}$$

Employing Lemma 2.2 into (10), we obtain

$$\liminf_{t\to\infty} x_i(t) \geq \frac{a_i^l}{b_i^u} \equiv m_i,\ i = 1, 2, \cdots, n. \tag{11}$$

This ends the proof of Theorem 3.1. □

**Theorem 3.2.** *Suppose that system (6) can satisfy ($H_1$)-($H_3$). Then system (6) possesses permanence. That is to say, any positive (almost periodic) solution $(y_1(t), y_2(t), \cdots, y_n(t))^T$ of system (6) fulfills*

$$\frac{m_i}{H_i^u} \leq \liminf_{t\to\infty} y_i(t) \leq \limsup_{t\to\infty} y_i(t) \leq \frac{M_i}{H_i^l},\ i = 1, 2, \cdots, n.$$

*Proof.* Suppose that $(y_1(t), y_2(t), \cdots, y_n)^T$ represents a positive (almost periodic) solution of system (6). Thanks to Lemma 2.5, we obtain that

$$(x_1(t), x_2(t), \cdots, x_n(t))^T = (H_1(t)y_1(t), H_2(t)y_2(t), \cdots, H_n(t)y_n(t))^T$$

represents a positive (almost periodic) solution of system (1)-(2). Together with Theorem 3.1, we have

$$m_i \leq \liminf_{t\to\infty} H_i(t)y_i(t) \leq \limsup_{t\to\infty} H_i(t)y_i(t) \leq M_i,\ i = 1, 2, \cdots, n,$$

which means that

$$\frac{m_i}{H_i^u} \leq \liminf_{t\to\infty} y_i(t) \leq \limsup_{t\to\infty} y_i(t) \leq \frac{M_i}{H_i^l},\ i = 1, 2, \cdots, n.$$

This finishes the proof of Theorem 3.2. □

**Theorem 3.3.** *Suppose that $\mathcal{S}$ represents the set of all solutions $(y_1(t), y_2(t), \cdots, y_n(t))^T$ of (6) satisfying*

$$\frac{m_i}{H_i^u} \leq y_i(t) \leq \frac{M_i}{H_i^l}$$

*for $t \in \mathbb{R}^+$. Then $\mathcal{S} \neq \emptyset$.*

*Proof.* According to the theory of almost periodic function, there possesses a sequence $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, such that

$$a_i(t + t_n) \to a_i(t),\ B_i(t + t_n) \to B_i(t),\ C_{ij}(t + t_n) \to C_{ij}(t),$$
$$H_i(t + t_n) \to H_i(t),\ \tau_i(t + t_n) \to \tau_i(t),\ i = 1, 2, \cdots, n,\ i \neq j,$$

uniformly on $\mathbb{R}^+$ as $n \to \infty$. Based on the Brouwer fixed-point theorem, we can propose a hypothesis that $(y_1(t), y_2(t), \cdots, y_n(t))^T$ represents a positive (almost periodic) solution of system (6) satisfying

$$\frac{m_i}{H_i^u} \leq y_i(t) \leq \frac{M_i}{H_i^l}$$

for $t > T \geq 0$, $i = 1, 2, \cdots, n$. It is obvious that the sequence $(y_1(t + t_n), y_2(t + t_n), \cdots, y_n(t + t_n))^T$ is equicontinuous and uniformly bounded on each bounded subset of $\mathbb{R}^+$. Applying Ascoli's theorem leads to

$$\lim_{k\to\infty} \mathbf{y}(t + t_k) = \lim_{k\to\infty} (y_1(t + t_k), y_2(t + t_k), \cdots, y_n(t + t_k))^T = \mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))^T.$$

where $\mathbf{y}(t + t_k)$ represents a subsequence of $\mathbf{y}(t + t_n)$ uniformly on each bounded subset, $\mathbf{z}(t)$ stands for a continuous function. For any given $T_1 \in \mathbb{R}^+$. For all positive integer $n$, we can suppose that $t_k + T_1 \geq T$. Therefore, if $t \geq 0$, we obtain

$$\begin{aligned} y_i(t + t_k + T_1) - y_i(t_k + T_1) = \int_{T_1}^{t+T_1} & y_i(s + t_k) \big(a_i(s + t_k) - B_i(s + t_k) y_i(s + t_k) \\ & + \sum_{j=1, i\neq j}^{n} C_{ij}(s + t_k) \frac{y_j(s + t_k - \tau_j(s + t_k))}{1 + H_j(s + t_k) y_j(s + t_k - \tau_j(s + t_k))}\Bigg) \end{aligned} \tag{12}$$

If we let $n \to \infty$ in (12), together with Lebesgue's dominated convergence theorem, we derive that:

$$y_i(t + T_1) - y_i(T_1) = \int_{T_1}^{t+T_1} y_i(s) \left(a_i(s) - B_i(s) y_i(s) + \frac{\sum_{j=1, i\neq j}^{n} C_{ij}(s) y_j(s - \tau_j(s))}{1 + H_j(s) y_j(s - \tau_j(s))}\right) \tag{13}$$

for all $t \geq 0$. $T_1 \in \mathbb{R}^+$ is selected randomly. For this reason, system (6) possesses a positive (almost periodic) solution $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))^T$ on $\mathbb{R}^+$. It is obvious that $\frac{m_i}{H_i^u} \leq y_i(t) \leq \frac{M_i}{H_i^l}$ for $t \in \mathbb{R}^+$. Therefore, $z(t) \in \mathcal{S}$. This ends the proof of Theorem 3.3. □

*3.2. Global asymptotic stability*

**Lemma 3.4.** *If $a, b \in \mathbb{R}$, then*

$$-\operatorname{sgn}(a) \cdot b \leq -|a| + |a - b|. \tag{14}$$

**Theorem 3.5.** *Suppose that the system (6) satisfy ($H_1$)-($H_3$) and the following condition:*

*($H_4$)* $\liminf P_i(t) > 0$, $i = 1, 2, \cdots, n$.

*where*

$$P_i(t) = \sum_{j=1, i\neq j}^{n} \frac{C_{ji}(t)}{(1+m_i)^2} - B_i(t) - \left[a_i(t) + \frac{2M_i}{H_i^l} B_i(t) + \sum_{j=1, i\neq j}^{n} \frac{M_j C_{ij}(t)}{H_j^l (1+m_j)^2}\right] \sum_{j=1, i\neq j}^{n} \int_t^{\phi_j^{-1}(t)} \frac{C_{ji}(u)}{(1+m_i)^2} du$$
$$- \frac{M_j C_i(\phi_i^{-1}(t))}{H_j^l \dot{\phi}_i\left(\phi_i^{-1}(t)\right)} \int_{\phi_i^{-1}(t)}^{\phi_i^{-1}(\phi_i^{-1}(t))} C_i(u) du$$

*in which $\phi_i^{-1}$ represents the inverse function of $\phi_i = t - \tau_i(t)$ ($i = 1, 2, \cdots, n$), respectively. Then the solution of system (6) is global asymptotically stable.*

*Proof.* We assume that system (6) possesses two positive solutions $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T$ and $\tilde{y}(t) = (\tilde{y}_1(t), \tilde{y}_2(t), \cdots, \tilde{y}_n(t))^T$. Thanks to Theorem 3.2, there has a positive constant $T$, such that

$$\frac{m_i}{H_i^u} \leq y_i(t) \leq \frac{M_i}{H_i^l}$$

for $t > T$, $i = 1, 2, \cdots, n$. We focus on the upper right derivatives of

$$V_{i1}(t) = \left|\ln \frac{\tilde{y}_i(t)}{y_i(t)}\right|, \ i = 1, 2, \cdots, n.$$

It is obvious by a direct computation that

$$\begin{aligned}
D^+ V_{i1}(t) &= \operatorname{sgn}\left(\tilde{y}_i(t) - y_i(t)\right)\left(\frac{\dot{\tilde{y}}_i(t)}{\tilde{y}_i(t)} - \frac{\dot{y}_i(t)}{y_i(t)}\right) \\
&= \operatorname{sgn}\left(\tilde{y}_i(t) - y_i(t)\right)\left[-B_i(t)\left(\tilde{y}_i(t) - y_i(t)\right) + \sum_{j=1, i\neq j}^{n} C_{ij}(t)\left(\frac{\tilde{y}_j(t-\tau_j(t))}{1 + H_j(t)\tilde{y}_j(t-\tau_j(t))} - \frac{y_j(t-\tau_j(t))}{1 + H_j(t) y_j(t-\tau_j(t))}\right)\right] \\
&= \operatorname{sgn}\left(\tilde{y}_i(t) - y_i(t)\right)\left[-B_i(t)\left(\tilde{y}_i(t) - y_i(t)\right) + \sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)\left[\tilde{y}_j(t-\tau_j(t)) - y_j(t-\tau_j(t))\right]}{\left[1 + H_j(t)\tilde{y}_j(t-\tau_j(t))\right]\left[1 + H_j(t) y_j(t-\tau_j(t))\right]}\right] \\
&\overset{(11)}{\leq} - \operatorname{sgn}\left(\tilde{y}_i(t) - y_i(t)\right)\left[-\sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left[\tilde{y}_j(t-\tau_j(t)) - y_j(t-\tau_j(t))\right] + B_i(t)\left(\tilde{y}_i(t) - y_i(t)\right)\right] \\
&\overset{(14)}{\leq} B_i(t)\left|\tilde{y}_i(t) - y_i(t)\right| - \sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\tilde{y}_j(t) - y_j(t)\right| + \sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\int_{t-\tau_j(t)}^{t}\left(\dot{\tilde{y}}_j(s) - \dot{y}_j(s)\right) ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\overset{(6)}{\le} B_i(t)\left|\tilde{y}_i(t)-y_i(t)\right| - \sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\tilde{y}_j(t)-y_j(t)\right| + \sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2}\\
&\quad\left|\int_{t-\tau_j(t)}^{t}\left\{\tilde{y}_j(s)\left[a_j(s)-B_j(s)\tilde{y}_j(s)+\sum_{i=1,j\neq i}^{n}C_{ji}(s)\frac{\tilde{y}_i(s-\tau_i(s))}{1+H_i(s)\tilde{y}_i(s-\tau_i(s))}\right]\right.\right.\\
&\quad\left.\left.-y_j(s)\left[a_j(s)-B_j(s)y_j(s)+\sum_{i=1,j\neq i}^{n}C_{ji}(s)\frac{y_i(s-\tau_i(s))}{1+H_i(s)y_i(s-\tau_i(s))}\right]\right\}ds\right|\\
&\le B_i(t)\left|\tilde{y}_i(t)-y_i(t)\right| - \sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\tilde{y}_j(t)-y_j(t)\right| + \sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2}\\
&\quad\left|\int_{t-\tau_j(t)}^{t}\left\{\left[a_j(s)-B_j(s)\tilde{y}_j(s)+\sum_{i=1,j\neq i}^{n}C_{ji}(s)\frac{\tilde{y}_i(s-\tau_i(s))}{1+H_i(s)\tilde{y}_i(s-\tau_i(s))}\right]\left(\tilde{y}_j(s)-y_j(s)\right)\right.\right.\\
&\quad -B_j(s)y_j(s)\left[\tilde{y}_j(s)-y_j(s)\right]\\
&\quad\left.\left.-y_j(s)\left[\sum_{i=1,j\neq i}^{n}C_{ji}(s)\frac{y_i(s-\tau_i(s))}{1+H_i(s)y_i(s-\tau_i(s))}-\sum_{i=1,j\neq i}^{n}C_{ji}(s)\frac{\tilde{y}_i(s-\tau_i(s))}{1+H_i(s)\tilde{y}_i(s-\tau_i(s))}\right]\right\}\right|\\
&\le B_i(t)\left|\tilde{y}_i(t)-y_i(t)\right| - \sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\tilde{y}_j(t)-y_j(t)\right| + \sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2}\\
&\quad\int_{t-\tau_j(t)}^{t}\left\{\left[a_j(s)+B_j(s)\tilde{y}_j(s)+\sum_{i=1,j\neq i}^{n}C_{ji}(s)\frac{\tilde{y}_i(s-\tau_i(s))}{1+H_i(s)\tilde{y}_i(s-\tau_i(s))}\right]\left|\tilde{y}_j(s)-y_j(s)\right|\right.\\
&\quad\left.+B_j(s)y_j(s)\left|\tilde{y}_j(s)-y_j(s)\right|+y_j(s)\sum_{i=1,j\neq i}^{n}\frac{C_{ji}(s)\left|\tilde{y}_i(s-\tau_i(s))-y_i(s-\tau_i(s))\right|}{\left[1+H_i(s)y_i(s-\tau_i(s))\right]\left[1+H_i(s)\tilde{y}_i(s-\tau_i(s))\right]}\right\}ds\\
&\le B_i(t)\left|\tilde{y}_i(t)-y_i(t)\right| - \sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\tilde{y}_j(t)-y_j(t)\right| + \sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2}\\
&\quad\int_{t-\tau_j(t)}^{t}\left\{\left[a_j(s)+\frac{M_j}{H_j^l}B_j(s)+\sum_{i=1,j\neq i}^{n}\frac{M_iC_{ji}(s)}{H_i^l(1+m_i)}\right]\left|\tilde{y}_j(s)-y_j(s)\right|+\frac{M_j}{H_j^l}B_j(s)\left|\tilde{y}_j(s)-y_j(s)\right|\right.\\
&\quad\left.+\frac{M_j}{H_j^l}\sum_{i=1,j\neq i}^{n}\frac{C_{ji}(s)}{(1+m_i)^2}\left|\tilde{y}_i(s-\tau_i(s))-y_i(s-\tau_i(s))\right|\right\}ds.
\end{aligned}
$$

Based on the appearance of integral term, we define

$$
\begin{aligned}
F_{ij}(s)=&\left[a_j(s)+\frac{M_j}{H_j^l}B_j(s)+\sum_{i=1,j\neq i}^{n}\frac{M_iC_{ji}(s)}{H_i^l(1+m_i)}\right]\left|\tilde{y}_j(s)-y_j(s)\right|+\frac{M_j}{H_j^l}B_j(s)\left|\tilde{y}_j(s)-y_j(s)\right|\\
&+\frac{M_j}{H_j^l}\sum_{i=1,j\neq i}^{n}\frac{C_{ji}(s)}{(1+m_i)^2}\left|\tilde{y}_i(s-\tau_i(s))-y_i(s-\tau_i(s))\right|
\end{aligned}
$$

and $G_{ij}(s)$ stands for a primitive function of $F_{ij}(s)$, $i = 1, 2, \cdots, n$. Thus,

$$\begin{aligned} D^+V_{i1}(t) \leq & B_i(t)\left|\tilde{y}_i(t) - y_i(t)\right| - \sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\tilde{y}_j(t) - y_j(t)\right| + \sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}\int_{t-\tau_j(t)}^{t} F_{ij}(s)ds \\ = & B_i(t)\left|\tilde{y}_i(t) - y_i(t)\right| - \sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\tilde{y}_j(t) - y_j(t)\right| + \sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left[G_{ij}(t) - G_{ij}(t-\tau_j(t))\right]. \end{aligned} \tag{15}$$

Set

$$C_i(t) = \sum_{j=1, i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}.$$

We denote that

$$V_{i2}(t) = \int_t^{\phi_i^{-1}(t)} \int_{\phi_i(u)}^{t} C_i(u) F_{ij}(s) ds du. \tag{16}$$

We can effortlessly check that

$$\begin{aligned} V_{i2}(t) = & \int_t^{\phi_i^{-1}(t)} C_i(u)\left[G_{ij}(t) - G_{ij}(\phi_j(u))\right] du \\ = & G_{ij}(t)\int_t^{\phi_i^{-1}(t)} C_i(u)du - \int_t^{\phi_i^{-1}(t)} C_i(u) G_{ij}(\phi_j(u))du. \end{aligned} \tag{17}$$

Therefore, if $t \geq T + \tau$, we get that

$$\begin{aligned} D^+V_{i2}(t) = & F_{ij}(t)\int_t^{\phi_i^{-1}(t)} C_i(u)du + G_{ij}(t)\left(\frac{C_i(\phi_i^{-1}(t))}{\dot{\phi}_i(t)} - C_i(t)\right) \\ & - \left(\frac{C_i(\phi_i^{-1}(t))}{\dot{\phi}_i(t)} G_{ij}(t) - C_i(t) G_{ij}(\phi_j(t))\right) \\ = & F_{ij}(t)\int_t^{\phi_i^{-1}(t)} C_i(u)du - C_i(t)\left(G_{ij}(t) - G_{ij}(\phi_j(t))\right). \end{aligned} \tag{18}$$

We denote that

$$V_{i3}(t) = \frac{M_j}{H_j^l}\int_{t-\tau_i(t)}^{t}\int_{\phi_i^{-1}(v)}^{\phi_i^{-1}(\phi_i^{-1}(v))} \frac{C_i(u) C_i(\phi_i^{-1}(v))}{\dot{\phi}_i\left(\phi_i^{-1}(v)\right)}\left|\tilde{y}_i(v) - y_i(v)\right| du dv. \tag{19}$$

It is obvious by a direct computation that

$$\begin{aligned} D^+V_{i3}(t) = & \frac{M_j C_i(\phi_i^{-1}(t))}{H_j^l \dot{\phi}_i\left(\phi_i^{-1}(t)\right)}\int_{\phi_i^{-1}(t)}^{\phi_i^{-1}(\phi_i^{-1}(t))} C_i(u)du \cdot \left|\tilde{y}_i(t) - y_i(t)\right| \\ & - \frac{M_j}{H_j^l} C_i(t)\left|\tilde{y}_i(t-\tau_i(t)) - y_i(t-\tau_i(t))\right|\int_t^{\phi_i^{-1}(t)} C_i(u)du. \end{aligned} \tag{20}$$

We construct a function of species $i$:

$$V_i(t) := V_{i1}(t) + V_{i2}(t) + V_{i3}(t), \ i = 1, 2, \cdots, n.$$

Thanks to (15), (18) and (20), it can easy to see that

$$\begin{aligned} D^+V_i(t) \leq & B_i(t)\left|\tilde{y}_i(t)-y_i(t)\right| - \sum_{j=1,i\neq j}^{n} \frac{C_{ij}(t)}{\left(1+m_j\right)^2}\left|\tilde{y}_j(t)-y_j(t)\right| + F_{ij}(t)\int_t^{\phi_i^{-1}(t)} \frac{C_{ij}(u)}{\left(1+m_j\right)^2}du \\ & + \frac{M_j C_i(\phi_i^{-1}(t))}{H_j^l \dot{\phi}_i\left(\phi_i^{-1}(t)\right)}\int_{\phi_i^{-1}(t)}^{\phi_i^{-1}(\phi_i^{-1}(t))} C_i(u)du \cdot \left|\tilde{y}_i(t)-y_i(t)\right| \\ & - \frac{M_j}{H_j^l}C_i(t)\left|\tilde{y}_i(t-\tau_i(t))-y_i(t-\tau_i(t))\right|\int_t^{\phi_i^{-1}(t)} C_i(u)du \end{aligned} \tag{21}$$

for $t \geq T+\tau$. We construct an appropriate Lyapunov function $V(t)$ as follows:

$$V(t) = \sum_{i=1}^{n} V_i(t).$$

Together with (21), we deduce that

$$D^+V(t) \leq -\sum_{i=1}^{n} P_i(t)\left|\tilde{y}_i(t)-y_i(t)\right|,$$

where the formula of $P_i(t)$ was proposed by Theorem 3.5.

Due to condition $(H_4)$, there possesses positive constants $T_0 \geq T+\tau$ and $\alpha_i (i=1,2,\cdots,n)$. If $t > T_0$, we get

$$P_i(t) \geq \alpha_i > 0. \tag{22}$$

We denote $\alpha^* = \min\{\alpha_1, \alpha_2, \cdots, \alpha_i\}$. Together with (21) and (22), we deduce that

$$D^+V(t) \leq -\alpha^* \sum_{i=1}^{n}\left|\tilde{y}_i(t)-y_i(t)\right|. \tag{23}$$

Integrating both sides of the last inequality from $T_0$ to $t$, we have

$$V(T_0) \geq V(t) + \alpha^* \int_{T_0}^{t}\sum_{i=1}^{n}\left|\tilde{y}_i(s)-y_i(s)\right|ds,$$

for $t \geq T_0$. Thus, it means that $V(t)$ is a bounded function on the interval $[T_0, +\infty)$. In addition,

$$\int_{T_0}^{+\infty}\sum_{i=1}^{n}\left|\tilde{y}_i(s)-y_i(s)\right|ds < +\infty.$$

Based on Theorem 3.2 and Eq.(6), we can deduce that $\tilde{y}_i(t)-y_i(t)(i=1,2,\cdots,n)$ and their derivatives are bounded functions on the interval $[T_0,+\infty)$. Hence, $\sum_{i=1}^{n}|\tilde{y}_i(t)-y_i(t)|$ remain uniformly continuous. Thanks to Lemma 2.6, we obtain that

$$\lim_{t\to+\infty}\sum_{i=1}^{n}\left|\tilde{y}_i(t)-y_i(t)\right| = 0.$$

Together with $\left|\tilde{y}_i(t)-y_i(t)\right| \geq 0$, we have , $i=1,2,\cdots,n$. Then the positive almost-periodic solution for system (6) is globally asymptotically stable. This finishes the proof of Theorem 3.5. □

*3.3. Existence and uniqueness of almost periodic solution*

**Theorem 3.6.** *Suppose that the conditions ($H_1$)-($H_4$) hold. Then the multi-species mutualism system (6) possesses a unique almost periodic positive solution which admits global asymptotic stability.*

*Proof.* Thanks to Theorem 3.3, the multi-species mutualism system (6) exists a positive bounded solution $(y_1(t), y_2(t), \cdots, y_n(t))^T$ fulfilling

$$\frac{m_i}{H_i^u} \leq y_i(t) \leq \frac{M_i}{H_i^l},\ t \in \mathbb{R}^+.$$

Hence, there possesses a sequence $\{\xi'_m\}$ with $\xi'_m \to \infty$ as $m \to \infty$ such that $(y_1(t+\xi'_m), y_2(t+\xi'_m), \cdots, y_n(t+\xi'_m))^T$ stands for a positive bounded solution of the following model

$$\dot{y}_i(t) = y_i(t)\left(a_i(t+\xi'_m) - B_i(t+\xi'_m)y_i(t) + \frac{\sum_{j=1,i\neq j} C_{ij}(t+\xi'_m)x_j(t-\tau_j(t))}{1 + H_j(t+\xi'_m)x_j(t-\tau_j(t))}\right). \tag{24}$$

Together with (24) and Theorem 3.1, we can come to a conclusion that not only $(y_1(t+\xi'_m), y_2(t+\xi'_m), \cdots, y_n(t+\xi'_m))^T$ but also $(\dot{y}_1(t+\xi'_m), \dot{y}_2(t+\xi'_m), \cdots, \dot{y}_n(t+\xi'_m))^T$ possesses uniformly bounded. Therefore, $(y_1(t+\xi'_m), y_2(t+\xi'_m), \cdots, y_n(t+\xi'_m))^T$ is equi-continuous. Thanks to Arzelà-Ascoli theorem, we can find a subsequence $\left\{(y_1(t+\xi_m), y_2(t+\xi_m), \cdots, y_n(t+\xi_m))^T\right\} \subseteq \left\{(y_1(t+\xi'_m), y_2(t+\xi'_m), \cdots, y_n(t+\xi'_m))^T\right\}$ with uniform convergence such that for any $\varepsilon > 0$, there admits a positive constant $\rho_0(\varepsilon) > 0$. In addition, if $m, l > \rho_0(\varepsilon)$, we get that

$$\left|y_i(t+\xi_m) - y_i(t+\xi_l)\right| < \varepsilon,\ i = 1, 2, \cdots, n,$$

which means that $(y_1(t+\xi_m), y_2(t+\xi_m), \cdots, y_n(t+\xi_m))^T$ represents a positive almost periodic asymptotic function, there admit two functions $P_i(t)$ and $Q_i(t)$ such that

$$y_i(t) = P_i(t) + Q_i(t),\ i = 1, 2, \cdots, n,\ t \in \mathbb{R}^+,$$

where

$$\lim_{m\to+\infty} P_i(t+\xi_m) = P_i(t),\ \lim_{m\to+\infty} Q_i(t+\xi_m) = 0,$$

$P_i(t)$ stand for almost periodic functions. It shows that

$$\lim_{m\to+\infty} y_i(t+\xi_m) = P_i(t),\ i = 1, 2, \cdots, n.$$

Meanwhile,

$$\begin{aligned}\lim_{m\to+\infty} \dot{y}_i(t+\xi_m) &= \lim_{m\to+\infty}\lim_{k\to 0} \frac{y_i(t+\xi_m+k) - y_i(t+\xi_m)}{k}\\ &= \lim_{k\to 0}\lim_{m\to+\infty} \frac{y_i(t+\xi_m+k) - y_i(t+\xi_m)}{k} = \lim_{k\to 0} \frac{P_i(t+k) - P_i(t)}{k},\end{aligned}$$

thus the almost periodic function $\dot{P}_i(t)$, $i = 1, 2, \cdots, n$ exist.

Here and subsequently, we need to verify that $P(t) = (P_1(t), P_2(t), \cdots, P_n(t))^T$ represents an almost periodic solution to model (6). According to the theory of almost periodic function [3], there admits a sequence $\left\{\xi_\rho\right\}$, $\xi_\rho \to \infty$ as $\rho \to +\infty$, such that

$$a_i(t+\xi_\rho) \to a_i(t),\ B_i(t+\xi_\rho) \to B_i(t),\ C_{ij}(t+\xi_\rho) \to C_{ij}(t),\ H_i(t+\xi_\rho) \to H_i(t),\ \tau_i(t+\xi_\rho) \to \tau_i(t),$$

as $\rho \to +\infty$ uniformly on $\mathbb{R}^+$, $i = 1, 2, \cdots, n$.

It is effortless to check that

$$\lim_{\rho\to+\infty} y_i(t+\xi_\rho) = P_i(t),\ i = 1,2,\cdots,n.$$

Therefore, we deduce that

$$\begin{aligned}\dot{P}_i(t) &= \lim_{\rho\to+\infty} \dot{y}_i(t+\xi_\rho)\\ &= \lim_{\rho\to+\infty} y_i(t+\xi_\rho)\left[a_i(t+\xi_\rho) - B_i(t+\xi_\rho)y_i(t+\xi_\rho) + \frac{\sum_{j=1,j\neq i} C_{ij}(t+\xi_\rho)y_j(t+\xi_\rho-\tau_j(t+\xi_\rho))}{1+H_j(t+\xi_\rho)y_j(t+\xi_\rho-\tau_j(t+\xi_\rho))}\right]\\ &= P_i(t)\left[a_i(t) - B_i(t)P_i(t) + \frac{\sum_{j=1,j\neq i} C_{ij}(t)P_j(t-\tau_j(t))}{1+H_j(t)P_j(t-\tau_j(t))}\right],\ i=1,2,\cdots,n.\end{aligned}$$

It means that $P(t) = (P_1(t), P_2(t), \cdots, P_n(t))$ can satisfy (6). Meanwhile, $P_i(t)$ represents an almost periodic solution of (6).

We proceed to show that system (6) has only one almost periodic positive solution. Suppose for the sake of contradiction that system (6) possesses two almost periodic positive solutions $P(t) = (P_1(t), P_2(t), \cdots, P_n(t))$ and $Q(t) = (Q_1(t), Q_2(t), \cdots, Q_n(t))$. We assert that $P_i(t) = Q_i(t)$ for all $t \in \mathbb{R}^+$. Conversely, there exists at least a positive number $\eta \in \mathbb{R}^+$ such that $P_i(\eta) \neq Q_i(\eta)$ for a certain integer $i > 0$. It means that

$$\Xi = \left|P_i(\eta) - Q_i(\eta)\right| = \left|\lim_{m\to+\infty} y_i(\eta+\xi_m) - \lim_{m\to+\infty} \tilde{y}_i(\eta+\xi_m)\right| = \lim_{m\to+\infty}\left|y_i(t) - \tilde{y}_i(t)\right| > 0,$$

which contradicts with $\lim_{t\to+\infty}\left|\tilde{y}_i(t) - y_i(t)\right| = 0$ proposed by Theorem 3.5. This finishes the proof of Theorem 3.6. □

**Theorem 3.7.** *Suppose that the conditions ($H_1$)-($H_4$) hold. Then the multi-species mutualism systems (1) and (6) possess a unique almost periodic positive solution which admits global asymptotic stability.*

*Proof.* Thanks to Lemma 2.5, we can discover that

$$(x_1(t), x_2(t), \cdots, x_n(t))^T = \left(\prod_{0<t_k<t}(1+h_{1k})y_1(t), \prod_{0<t_k<t}(1+h_{2k})y_2(t), \cdots, \prod_{0<t_k<t}(1+h_{nk})y_n(t)\right)^T$$

stands for an almost periodic solution of system (1). Because ($H_2$) holds, follow the proofs of Theorem 79 and Lemma 31 in [11], we can deduce that $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ stands for an almost periodic solution. Thus, based on the uniqueness and global stability of $y(t)$, $x(t)$ represents a unique globally asymptotically stable solution to system (1). This ends the proof. □

We can obtain the following corollary because the condition ($H_4$) has been simplified by setting $\tau_i(t) = \tau_i (i = 1,2,\cdots,n)$, where $\tau_i \geq 0$ are constants.

**Corollary 3.8.** *We denote $\tau_i(t) = \tau_i (i = 1,2,\cdots,n)$, where $\tau_i \geq 0$ are constants. Thanks to ($H_1$) and ($H_2$), we can suppose further that*

$$\begin{aligned}&\liminf_{t\to+\infty} P_i(t)\\ &= \liminf_{t\to+\infty}\Bigg\{\sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t)}{\left(1+m_j\right)^2} - B_i(t) - \left[a_j(t) + \frac{2M_i}{H_i^l}B_i(t) + \sum_{i=1,j\neq i}^{n}\frac{M_iC_{ji}(t)}{H_i^l(1+m_i)}\right]\int_t^{t+\tau_i}\sum_{j=1,i\neq j}^{n}\frac{C_{ij}(u)}{\left(1+m_j\right)^2}du\\ &\quad - \frac{M_j\sum_{j=1,i\neq j}^{n}\frac{C_{ij}(t+\tau_i)}{(1+m_j)^2}}{H_j^l}\int_{t+\tau_i}^{t+2\tau_i}\sum_{j=1,i\neq j}^{n}\frac{C_{ij}(u)}{\left(1+m_j\right)^2}du\Bigg\} > 0,\ i=1,2,\cdots,n.\end{aligned}$$

*Thus, (1) possesses a unique almost periodic positive solution which has global asymptotic stability.*

We can obtain the following corollary because the condition $(H_4)$ has been simplified by setting $h_{ik} \equiv 0$.

**Corollary 3.9.** *We denote $h_{ik} \equiv 0$. Thanks to $(H_1)$ and $(H_2)$, we can suppose further that*

$$\begin{aligned}
&\liminf_{t\to+\infty} P_i(t)\\
&=\liminf_{t\to+\infty}\Bigg\{\sum_{j=1,i\neq j}^{n}\frac{c_{ij}(t)}{\left(1+m_j\right)^2}-b_i(t)-\left[a_j(t)+2M_ib_i(t)+\sum_{i=1,j\neq i}^{n}\frac{M_ic_{ji}(t)}{(1+m_i)}\right]\int_t^{\phi_i^{-1}(t)}\sum_{j=1,i\neq j}^{n}\frac{c_{ij}(u)}{\left(1+m_j\right)^2}du\\
&-\frac{M_j\sum_{j=1,i\neq j}^{n}\frac{c_{ij}(\phi_i^{-1}(t))}{\left(1+m_j\right)^2}}{\dot{\phi}_i\left(\phi_i^{-1}(t)\right)}\int_{\phi_i^{-1}(t)}^{\phi_i^{-1}(\phi_i^{-1}(t))}\sum_{j=1,i\neq j}^{n}\frac{c_{ij}(u)}{\left(1+m_j\right)^2}du\Bigg\}>0,\ i=1,2,\cdots,n.
\end{aligned}$$

*Thus, (1) possesses a unique almost periodic positive solution which has global asymptotic stability.*

## 4. Example

In this section, we offer a multispecies mutualism system to verify the feasibility of our main theorem.

$$\begin{cases}
\dot{x}_1(t)=x_1(t)\left(0.3-0.05\sin(\sqrt{2}t)-(0.25-0.04\cos(\sqrt{3}t))x_1(t)+\dfrac{x_2(t-0.02)}{1+x_2(t-0.02)}+\dfrac{1.3x_3(t-0.015)}{1+x_3(t-0.015)}\right),\\
\dot{x}_2(t)=x_2(t)\left(0.2-0.02\cos(\sqrt{2}t)-(0.25-0.03\sin(\sqrt{3}t))x_2(t)+\dfrac{0.5x_1(t-0.01)}{1+x_1(t-0.01)}+\dfrac{0.8x_3(t-0.015)}{1+x_3(t-0.015)}\right),\\
\dot{x}_3(t)=x_3(t)\left(0.25-0.06\sin(\sqrt{3}t)-(0.3-0.05\sin(\sqrt{2}t))x_3(t)+\dfrac{0.7x_1(t-0.01)}{1+x_1(t-0.01)}+\dfrac{1.1x_2(t-0.020)}{1+x_2(t-0.020)}\right).
\end{cases}\tag{25}$$

By a direct computation, we can obtain the following table:

Table 1: The biological parameters of $x_1$, $x_2$ and $x_3$

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $a_i(t)$ | $0.3-0.05\sin(\sqrt{2}t)$ | $0.2-0.02\cos(\sqrt{2}t)$ | $0.25-0.06\sin(\sqrt{3}t)$ |
| $a_i^l$ | 0.25 | 0.18 | 0.19 |
| $a_i^u$ | 0.35 | 0.22 | 0.31 |
| $b_i(t)$ | $0.25-0.04\cos(\sqrt{3}t)$ | $0.25-0.03\sin(\sqrt{3}t)$ | $0.3-0.05\sin(\sqrt{2}t)$ |
| $b_i^l$ | 0.21 | 0.22 | 0.25 |
| $b_i^u$ | 0.29 | 0.28 | 0.35 |
| $c_{i1}(t)$ | $\varnothing$ | 0.5 | 0.7 |
| $c_{i2}(t)$ | 1 | $\varnothing$ | 1.1 |
| $c_{i3}(t)$ | 1.3 | 0.8 | $\varnothing$ |
| $m_i$ | 0.8621 | 0.6429 | 0.5429 |
| $M_i$ | 12.6190 | 6.9091 | 8.4400 |

Meanwhile, we obtain

$$\liminf_{t\to+\infty} P_1(t)>0.05>0,\ \liminf_{t\to+\infty} P_2(t)>0.04>0,\ \liminf_{t\to+\infty} P_3(t)>0.03>0.$$

It means that the conditions of $P_1(t)$, $P_2(t)$ and $P_3(t)$ proposed by $(H_4)$ are fulfilled. It is also shown that the almost periodic positive solution of system (26) admits existence, permanence, global asymptotic stability and uniqueness.

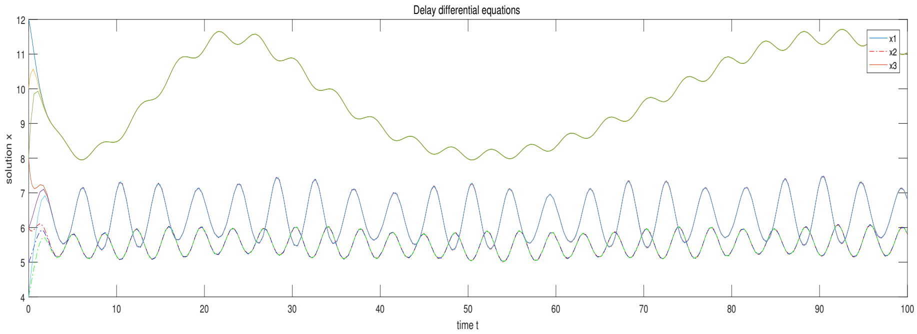


Figure 1: Numeric simulation of $x_1(t)$, $x_2(t)$ and $x_3(t)$ of (26) with the initial conditions $(x_1(0), x_2(0), x_3(0))^T = (1.5, 1, 0.5)^T$, $(x_1(0), x_2(0), x_3(0))^T = (2, 1.25, 1.5)^T$ and $(x_1(0), x_2(0), x_3(0))^T = (2.5, 1.5, 2.5)^T$.

## 5. Conclusion and future works

In this research, we investigated the qualitative behavior of a famous mutualism system, the denominated Lotka-Volterra model, assuming continuous and almost periodic parameters. It will be interesting to study the following two more general systems:

$$\begin{cases} \dot{x}_i(t) = x_i(t)\left(a_i(t) - b_i(t)x_i(t-\tau_i(t)) + \sum\limits_{j=1, i\neq j}^{n} c_{ij}(t)\dfrac{x_j(t-\tau_j(t))}{1+x_j(t-\tau_j(t))}\right), \ t \neq t_k, \\ x_i(t_k^+) = (1+h_{ik})x_i(t_k), \ k \in \mathbf{Z}^+, \ i = 1, 2, \cdots, n \end{cases} \tag{26}$$

and

$$\begin{cases} \dot{x}_i(t) = x_i(t)\left(a_i(t) - b_i(t)x_i(t) + \sum\limits_{j=1, i\neq j}^{K_1} c_{ij}(t)\dfrac{x_j^{m_j}(t-\tau_j(t))}{1+x_j(t-\tau_j(t))} + \sum\limits_{k=1, i\neq k}^{K_2} c_{ik}(t)\dfrac{x_k^{m_k}(t-\tau_k(t))}{1+x_k^{m}(t-\tau_k(t))}\right), \ t \neq t_k, \\ x_i(t_k^+) = (1+h_{ik})x_i(t_k), \ k \in \mathbf{Z}^+, \ i = 1, 2, \cdots, n, \end{cases} \tag{27}$$

where $0 < m_j \leq 1$, $0 < m_k \leq 1$, $m > 0$ and $K_1 + K_2 = n$. Clearly, system (26) is a more general model compared with system (1). Since we argue that the time-delay effect also exists in the species $i$. Biologically speaking, we describe (26) as *multispecies mutualism system with multiple delays*. However, we know that there might exist more than one kind of functional response in one model. Therefore, we claim that (27) also has biological significance. It stands for a *multispecies mutualism system with mixed monotone operator*.

It is a challenging work to investigate the qualitative properties such as existence, permanence, uniqueness and global asymptotic stability of system (26) or system (27). These tasks remain for the future.

## Acknowledgments

This study was funded by the Natural Science Foundation of Guangdong Province (No. 2018A030313505), China, and the Programme of Science and Technology of Guangzhou (No. 202002030448), China.

## Conflict of interest

The authors declare that they have no conflict of interest.

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