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Comment on "Analogies Between the Geodetic Number and the Steiner Number of Some Classes of Graphs"

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Abstract.

It is proved by counter example that one of the theorems presented in [FILOMAT 29:8 (2015), 1781-1788] does not always hold true. It is also proved by counter example that the necessary condition given in Theorem 3.7 [If $diam(H) = 2$, then $s(K_1 \odot H) = s(H)$] mentioned in the above cited paper does not hold true.

By a *graph* $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to [7]. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in G/uv \in E(G)\}$. The degree of a vertex $v \in V$ is $deg(v) = |N(v)|$. If $e = \{u, v\}$ is an edge of G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a pendant edge or end edge, u a leaf or end vertex and v a support. A vertex of degree $n - 1$ is called a *universal vertex*. The *minimum and maximum degrees* of G are denoted by $\delta(G)$ and $\triangle(G)$, respectively. The subgraph *induced by a set S* of vertices of G is denoted by $\langle S \rangle$ with $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. Let K and H be two graphs and let n be the order of H. The *corona product* $K \odot H$ is defined as the graph obtained from K and H by taking one copy of K and n copies of H and then joining by an edge, all the vertices from the i^{th}-copy of H to the i^{th}-vertex of K. The *closed interval* $I[u, v]$ consists of all vertices lying on some $u - v$ geodesic of G including the vertices u and v. For $S \subseteq V$, $I[S] = \cup_{u,v \in S} I[u, v]$. A set S of vertices is called a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number g(G)*. A geodetic set of cardinality *g(G)* is called a *g-set*. The geodetic number of a graph was studied in [5,6,9,10,12,18]. Recently tremendous work in geodetic number of a graph has been done by H. Abdollahzadeh Ahangar et al. in [1, 2, 3, 4]. It has applications in game theory, telephone switching centres, facility location, distributed computing, information retrieval, and communication networks. For $W \subseteq V$, the *Steiner distance d(W)* of W is the minimum size of a connected subgraph of G containing W. Necessarily, each subgraph is a tree and is called a *Steiner tree* with respect to W or a *Steiner W-tree*. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. If v is an end vertex of a Steiner W- tree, then $v \in W$. If v is a cut vertex of G, then v lies in every Steiner W - tree of G. Also if $\langle W \rangle$ is connected, then any Steiner W-tree contains the elements of W only. The Steiner distance was studied in [13]. For $W \subseteq V$, $S(W)$ denotes the set of all vertices that lie on Steiner W - trees. It is to be noted that $S(W) = W$ if and only if $\langle W \rangle$ is connected. A set $W \subseteq V$ is called a *Steiner set* of G if *S(W)=V*. A Steiner set of minimum cardinality is a *minimum Steiner set* or simply a *s-set* and this cardinality is the *Steiner number s(G)* of G. Let W be a Steiner set of G. Then $W \cup \{v\}$ is also a Steiner set

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of G if and only if v is a cut vetex of G. The Steiner number of a graph was introduced in [6]. The Steiner number of a graph was further studied in [8,9,10,11,12,14,15,16,17,18]. Steiner tree problem is used in combinatorial optimization and computer science especially in design of computer circuits. They have numerous applications in industries. Applying the Steiner tree concept improves the effectiveness in networks.

Geodetic and Steiner numbers of some class of graphs of diameter 2

Table 1

G	$K_{1,n-1}$	C_4	C_5	$K_{r,s}(2 \leq r \leq s)$	$W_{1,n-1}$	$K_n - \{e\}$
$g(G)$	$n-1$	2	3	$min\{4, r\}$	$\lceil \frac{n-1}{2} \rceil$	2
$s(G)$	$n-1$	2	3	r	$n-3$	2

1. Not every Steiner set of minimum cardinality in a graph of diameter 2 is a geodetic set of G

In [6], it is proved that every Steiner set in a connected graph is a geodetic set. Later in [14], I.M.Pelayo disproved that every Steiner set in a connected graph need not be a geodetic set. By Observing Table 1, all graphs has diameter 2 and the inequality $g(G) \leq s(G)$ holds. Hence, they deduced in [10] that *every minimum Steiner set is a geodetic set of a graph having diameter 2*. With regard to this derivation, the proof of Theorem 4.1 in [10] can be cited below.

Let W be a Steiner set of minimum cardinality in G and let n be the order of G. If $\langle W \rangle$ is connected, then $| W |= n$. Then $G \cong K_n$, which is a contradiction because G has diameter two. Thus, $\langle W \rangle$ is non-connected. Let $B_1, B_2, ...B_r$ be the connected components of $\langle W \rangle$. We assume that W is not a geodetic set. Then there exists a vertex x of G such that $x \notin I[W]$. Thus, $x \notin W$ and $x \notin I[u, v]$ for every $u, v \in W$. Hence, $N_W(x) \subseteq B_i$, for some $i \in \{1, 2, ..., r\}$. Since G has diameter two, any Steiner W-tree is formed by r Steiner B_i-trees connected by vertices $v_1, v_2, ..., v_t$, $t \geq 1$, not belonging to W such that $N_W(v_i) \nsubseteq B_j$, for every $i \in \{1, 2, ..., t\}$ and $j \in \{1, 2, ..., r\}$. Hence, $S(W) = (\cup_{i=1}^{r} S(B_i)) \cup (\cup_{i=1}^{t} \{v_i\}) = (\cup_{i=1}^{r} B_i) \cup (\cup_{i=1}^{t} \{v_i\})$, where the last equality comes from the connectivity $\langle B_i \rangle$. Therefore $x \notin S(W)$, which is a contradiction.

It can be proved in other-way that the statement is not necessarily true. For example consider the graph G given in Figure 1.1. The distance matrix of G is given in the Table 2

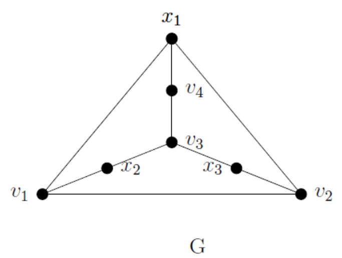


Figure 1.1

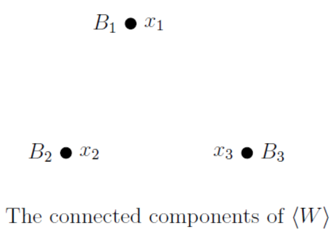
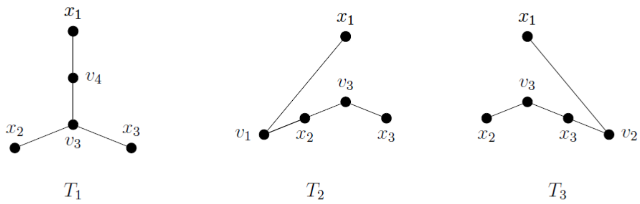


Figure 1.2

Table 2

$d(x,y)$	x_1	v_2	v_1	x_2	v_3	x_3	v_4
x_1	0	1	1	2	2	2	1
v_2	1	0	1	2	2	1	2
v_1	1	1	0	1	2	2	2
x_2	2	2	1	0	1	2	2
v_3	2	2	2	1	0	1	1
x_3	2	1	2	2	1	0	2
v_4	1	2	2	2	1	2	0



Steiner W - trees of G

Figure 1.1(a)

From the Table 2, we see that diameter of G is 2. It can be easily verified that no 2-element subsets of G is a Steiner set of G and so $s(G) \geq 3$. Let $W = \{x_1, x_2, x_3\}$. Then $S(W) = V$ and so W is a minimum Steiner set of G so that $s(G) = 3$. So by a careful analysis of proof by taking the minimum Steiner set $W = \{x_1, x_2, x_3\}$, we disprove Theorem 4.1 in [10]. We assume that W is not a geodetic set. Then there exists a vertex x of G such that $x \notin I[W]$. Thus, $x \notin W$ and $x \notin I[u, v]$ for every $u, v \in W$. Let us take the vertex x as v_4. Hence, $N_W(v_4) \subseteq B_1$.

The connected components of $\langle W \rangle$ are $B_1 = x_1$, $B_2 = x_2$ and $B_3 = x_3$ and they are shown in Figure 2.2. The element not belonging to W are v_1, v_2, v_3 and v_4. Now $N_W(v_1) = \{x_1, x_2\} \not\subseteq B_j$, $N_W(v_2) = \{x_1, x_3\} \not\subseteq B_j$ and $N_W(v_3) = \{x_2, x_3\} \not\subseteq B_j$ for some j $(1 \leq j \leq 3)$. But $N_W(v_4) = \{x_1\} = B_1$. Since $\langle B_1 \rangle = K_1$, the elements of Steiner B_1 - tree is x_1. Therefore $S(B_1) = x_1$. Similarly $S(B_2) = x_2$ and $S(B_3) = x_3$. Therefore $S(W) = S(B_1) \cup S(B_2) \cup S(B_3) = \{x_1, x_2, x_3\} = W$. Which implies $\langle W \rangle$ is connected. But, we have $\langle W \rangle$ is not connected, which is a contradiction. This contradiction arises because any Steiner B_i - trees, $(1 \leq i \leq 3)$ contain only one element. Therefore we cannot connect the vertices $v_1, v_2, ..., v_t (1 \leq t \leq 4)$, not belonging to W such that $N_W(v_i) \not\subseteq B_j$ for every $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$ to any Steiner B_i - tree, $(1 \leq i \leq 3)$ and also $x \notin I[u, v]$ is not used anywhere in the proof of Theorem 4.2 in [10]. Observe that the vertex $v_4 \notin I[W]$. Therefore W is not a geodetic set of G. Since $s(G) = 3$, $g(G) \geq 3$. It is easily verified that no 3-element subsets of G is not a geodetic set of G and so $g(G) \geq 4$. Let $S = \{x_1, v_1, v_2, v_3\}$. Then $I[S] = V$ and so S is a geodetic set of G so that $g(G) = 4$. Hence for the graph G given in the Figure 1.1, $s(G) > g(G)$ and diameter of G is 2. Therefore Corollary 4.2 in [10] is wrong.

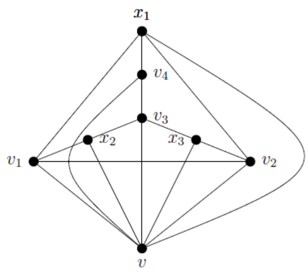
2. The Steiner number of corona product graphs

The necessary condition given in Theorem 3.7 in [10] is not true. i.e., If H is a graph having diameter 2, then $s(K_1 \odot H) = s(H)$ is not necessarily true. With regard to this derivation, the proof of first part of Theorem 3.7 in [10] is given below. Let B be a Steiner set of minimum cardinality in H and let v be the vertex of K_1. If $Diam(H) = 2$, then there exist three vertices of H such that $x, y \in B$ and $z \notin B$, $d_H(x, y) = 2$ and $x, y \in N_B(z)$. So, if we take a Steiner B-tree T in H containing the path xzy, then replacing the vertex z of T by the vertex v, and replacing every edge uz of T by a new edge uv, we obtain a Steiner B-tree T' in

$K_1 \odot H$. Hence, B is a Steiner set of $K_1 \odot H$. Therefore, $s(H) \geq s\ (K_1 \odot H)$ and, by Lemma 3.3, we conclude $s(H) = s(K_1 \odot H)$.

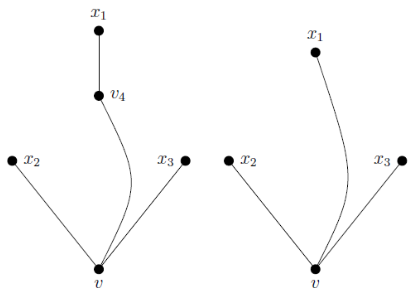
Again by observing Table 1 all graphs has diameter 2 and $s(K_1 \odot H) = s(H)$. Hence, they deduced that for a connected non-complete graph H, $s(K_1 \odot H) = s(H)$ if and only if diameter of H is 2.

The graph H given in Figure 1.1 has diameter 2. Take the minimum Steiner set $B = \{x_1, x_2, x_3\}$ and let v be the vertex of K_1. Since diameter of H is 2, there exists three vertices of H such that $x, y \in B$ and $z \notin B$, $d_H(x, y) = 2$ and $x, y \in N_B(z)$. Take $x = x_2$, $y = x_3$ and $z = v_3$ so that $x, y \in N_B(z)$. Consider the Steiner B-tree T of H containing the path xzy. It is the tree T_1 given in Figure 1.1(a), Let T' be a tree obtaining T_1 by deleting the edges x_2v_3, x_3v_3, v_4v_3 of T_1 and introducing the vertex v and adding the edges x_2v,x_3v and v_4v. Since $W \subseteq V(T')$ and $|V(T')| = |V(T_1)|$, T' is a Steiner B-tree of H. The tree T' is given in Figure 2.2(a). But, v is adjacent to x_1 in $K_1 \odot H$. Let T'' be a tree obtained from T' by introducing the edge vx_1 and deleting the edge x_1v_4 from T'. The tree T'' is given in Figure 2.2(b). Since $B \subset V(T'')$, the tree T'' is a Steiner B-tree of $K_1 \odot H$. Since $|V(T'')| = |V(T')| - 1 = 3$, T'' is the only Steiner B-tree of $K_1 \odot H$. Which implies $S(B) \neq V(K_1 \odot H)$. Hence it follows that B is not a Steiner set of $K_1 \odot H$. Therefore we cannot conclude that $s(K_1 \odot H) \leq s(H)$. It is easily verified that no 2-element or 3-element subsets of $K_1 \odot H$ is a Steiner set of $K_1 \odot H$ and so $s(K_1 \odot H) \geq 4$. Let $W' = \{v_1, v_3, x_1, x_3\}$. Then $S(W') = V(K_1 \odot H)$ and so W' is a Steiner set of $K_1 \odot H$ so that $s(K_1 \odot H) = 4$. Therefore for the graph $K_1 \odot H$ given in Figure 2.1, $s(H) \neq s(K_1 \odot H)$ with $Diam(H) = 2$. Hence Theorem 3.7 in [10] is wrong.



$K_1 \odot H$

Figure 2.1



T' (a) T'' (b)

Figure 2.2

3. Conclusion

In the presented article, it is validated that $g(G) \leq s(G)$ is disproved for a connected graph having diameter two. Alternatively it can be further investigated that under what condition $g(G) \leq s(G)$ holds for a connected graph having diameter two.

References

[1] H. Abdollahzadeh Ahangar, Graphs with large total geodetic number, *Filomat* 31:13 (2017), 4297-4304.
[2] H. Abdollahzadeh Ahangar and V. Samodivkin, The total geodetic number of a graph, *Utilitas Mathematica*, 100 (2016), 253-268.
[3] H. Abdollahzadeh Ahangar, V. Samodivkin, S. M. Sheikholeslami and A. Khodkar, The restrained geodetic number of a graph, *Bulletin of the Malaysian Mathematical Sciences Society*, 38 (3) (2015) 1143-1155
[4] H. Abdollahzadeh Ahangar, S. Kosari, S. M. Sheikholeslami and L. Volkmann, Graphs with large geodetic number, *Filomat* 29:6 (2015), 1361-1368.
[5] G. Chartrand, F. Harary and P. Zhang, On the geodetic number of a graph, *Networks* 39 (2002), 1-6.
[6] G. Chartrand, P. Zhang, The Steiner number of a graph, *Discrete Math.*, 242 (2002),41 - 54.
[7] Douglas B. West, Introduction to graph theory, *Second Edition Pearson Education, New Delhi, India*.
[8] M. B. Frondoza and S. R. Canoy Jr., The edge Steiner number of a graph, *Bulletin of the Malaysian Mathematical Sciences Society*,35 (2012),53 - 69.

[9] C. Hernando, T. Jiang, M. Mora, I. M. Pelayo and C. Seara, On the Steiner, geodetic and hull number of graphs, *Discrete Math.*, 293 (2005), 139 - 154.

[10] Ismael G. Yero and Juan A. Rodriguez-Velazquez, Analogies between the geodetic number and the Steiner number of some classes of graphs, *FILOMAT*, 29:8 (2015), 1781-1788.

[11] J. John and M. S. Malchijah Raj, The upper restrained Steiner number of a graph, *Discrete Mathematics, Algorithms and Applications*, 12(1) (2019),2050004.

[12] Li-Da Tong, Geodetic sets and Stenier sets in graphs, *Discrete Math.*, 309 (12) (2009), 4205-4207.

[13] Y. Mao, Steiner distance in graphs — A Survey, *Research Gate* (2017).

[14] I. M. Pelayo, Comment on "The Steiner number of a graph" by G. Chartrand and P. Zhang, *Discrete Math.*, 280 (2004), 259 - 263.

[15] A. P. Santhakumaran and J. John, The edge Steiner number of a graph, *J. Discrete math. Sci. Cryptogr.*,10 (2007),677 - 696.

[16] A. P. Santhakumaran and J. John, The upper Steiner number of a graph, *Graph Theory Notes of New York LIX*, (2010), 9 - 14.

[17] A. P. Santhakumaran and J. John, The forcing Steiner number of a graph, *Discussiones Mathematicae Graph Theory*, 31 (2011), 171 - 181.

[18] A. P. Santhakumaran and J. John, On the forcing geodetic and the forcing Steiner numbers of a graph, *Discussiones Mathematicae Graph Theory*, 31 (2011), 611- 624.