Filomat 37:2 (2023), 627–642
https://doi.org/10.2298/FIL2302627K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The study of error bounds for generalized vector inverse mixed quasi-variational inequalities

Jong Kyu Kim[a], Salahuddin[b], A.A.H. Ahmadini[b]

[a]*Department of Mathematics Education, Kyungnam University, Cgangwon, Gyeongnam, 51767, Korea*
[b]*Department of Mathematics, Jazan University, Jazan-45142, Saudi Arabia*

Abstract. In this study, generalized multivalued vector inverse quasi-variational inequality problems are developed, and error bounds are obtained in terms of the residual gap function, the regularized gap function, and the $\mathcal{D}$-gap function. With the help of these constraints, one can effectively estimate the distances between any feasible point and the solution set of problems involving generalized multivalued vector inverse quasi-variational inequality.

1. Introduction

The theory of variational (Quasi-variational) inequality is quite application oriented and thus developed much in recent years in many different disciplines. This theory provides us with a framework to understand and solve many problems arising in the field of economics, optimization, transportation, elasticity and applied sciences. The fundamental goal in the theory of variational (Quasi-variational) inequality is to develop a streamline algorithm for solving a variational inequality and its various forms. These methods include the projection method and its novel forms, approximation methods, Newton's methods and the methods derived from auxiliary principle techniques.

In 1980, vector variational inequalities were initiated in setting of the finite dimensional Euclidean space, see [5]. This is a generalization of scalar variational inequalities to the vector case by virtue of multi-criteria consideration. The inverse variational inequalities were introduced by He et al. [7] and have many applications in various fields such as market equilibrium problems in economics and telecommunication networks, see [8–12, 17, 18, 20, 25].

In 2014, Li et al. [14] introduced a new class of inverse mixed variational inequality in the setting of Hilbert spaces, which has simple traffic network equilibrium control problem as an application. For the analysis of optimization problems, the idea of gap function was first introduced and plays an important

2020 *Mathematics Subject Classification*. Primary 49J40, 49H09 ; Secondary 47J20, 54H25.
Keywords. Generalized multivalued vector inverse quasi-variational inequality; Residual gap function; Regularized gap function; $\mathcal{D}$-gap function; Error bounds; Generalized projection operator.
Received: 10 November 2020; Accepted: 21 April 2021
Communicated by Ivana Djolović
Correspondng author: Jong Kyu Kim
This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).
Email addresses: jongkyuk@kyungnam.ac.kr (Jong Kyu Kim), smohammad@jazanu.edu.sa (Salahuddin), aahmadini@jazanu.edu.sa (A.A.H. Ahmadini)

role in developing iterative algorithms, but more importantly in evaluating their convergence properties and obtaining useful stopping rules for iterative algorithms, see [1, 3, 13, 16, 21]. Error bounds are very important and useful because they provide a measure of the distance between a solution set and a feasible arbitrary point.

Solodov [22] developed some merit functions associated with a generalized mixed variational inequality and used those functions to obtain mixed variational inequality error limits. Recently, Aussel et al. [2] introduced a new inverse quasi-variational inequality (IQVI), obtained local (global) error bounds for IQVI in terms of some gap functions to demonstrate IQVI's applicability, and gave an example of problems with road pricing. Sun and Chai [23] introduced regularized gap functions for generalized vector variational inequalities (GVVI) and obtained GVVI error bounds for regularized gap functions.

Our goal in this paper is to initiate a study of a problem of generalized mixed vector inverse quasi-variational inequality with point to set-valued mappings. We propose three gap functions, the residual gap function, the regularized gap function, and the D-gap function, and obtain error bounds for generalized mixed vector inverse quasi-variational inequality using these gap functions and generalized ζ-projection operator under the strong monotonicity, relaxed monotonicity and Lipschitz continuity of underlying mappings.

2. Preliminaries

Throughout this paper, we assume that the set of non-negative real numbers is denoted by $\mathbf{R}_+$, the origin of all finite dimensional spaces is denoted by $\mathbf{0}$, and the norms and the inner products of all finite dimensional spaces are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $\Omega, \mathcal{F}, \mathcal{T} : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be the mappings with nonempty closed convex values and $\mathbb{P}_i, \mathbb{Q}_i : \mathbf{R}^n \to \mathbf{R}^n$ $(i = 1, 2, \cdots, m)$ be the point to point mappings. Let $p : \mathbf{R}^n \to \mathbf{R}^n$ be a point to point mapping and $\zeta_i : \mathbf{R}^n \to \mathcal{R}$ $(i = 1, 2, \cdots, m)$ be real-valued convex functions.

We put

$$\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_m), \quad \mathbb{P} = (\mathbb{P}_1, \mathbb{P}_2, \cdots, \mathbb{P}_m)$$

and for any $x, w \in \mathbf{R}^n$,

$$\langle \mathbb{P}(x), w\rangle = (\langle \mathbb{P}_1(x), w\rangle, \langle \mathbb{P}_2(x), w\rangle, \cdots, \langle \mathbb{P}_n(x), w\rangle).$$

We consider the following generalized vector inverse mixed quasi-variational inequality (GVIMQVI) for finding $\bar{u} \in \mathcal{F}(\bar{x}), \bar{v} \in \mathcal{T}(\bar{x})$ and $\bar{x} \in \Omega(\bar{x})$ such that

$$\langle \mathbb{P}(\bar{u}) + \mathbb{Q}(\bar{v}), y - p(\bar{x})\rangle + \zeta(y) - \zeta(p(\bar{x})) \notin -int\mathbf{R}^m_+, \ \forall y \in \Omega(\bar{x}), . \tag{1}$$

where $int\mathbf{A}$ denotes the interior of the set $\mathbf{A}$. The solution set of (1) is denoted by sol(GVIMQVI).

Special Cases:

(i) If we note that $\mathcal{T}$ is a point to point mapping and $\mathbb{Q}$ is a zero mapping, then problem GVIMQVI reduces to the following problem for finding $\bar{u} \in \mathcal{F}(\bar{x})$ and $\bar{x} \in \Omega(\bar{x})$ such that

$$\langle \mathbb{P}(\bar{u}), y - p(\bar{x})\rangle + \zeta(y) - \zeta(p(\bar{x})) \notin -int\mathbf{R}^m_+, \ \forall y \in \Omega(\bar{x}). \tag{2}$$

(ii) If $\mathcal{F}$ is the identity, then (2) reduces to the following vector inverse mixed quasi-variational inequality (VIMQVI) problem for finding $\bar{x} \in \Omega(\bar{x})$ such that

$$\langle \mathbb{P}(\bar{x}), y - p(\bar{x})\rangle + \zeta(y) - \zeta(p(\bar{x})) \notin -int\mathbf{R}^m_+, \ \forall y \in \Omega(\bar{x}), \tag{3}$$

which was studied in [24].

(iii) If $C \subset \mathbf{R}^n$ is a nonempty, closed and convex subset, $p(x) = x$ and $\Omega(x) = C$ for all $x \in \mathbf{R}^n$, then (3) collapses to the following generalized vector variational inequality (GVVI) for finding $\bar{x} \in C$ such that

$$\langle \mathbb{P}(\bar{x}), y - \bar{x}\rangle + \zeta(y) - \zeta(\bar{x}) \notin -int\mathbf{R}^m_+, \ \ \forall y \in C, \tag{4}$$

which was considered in [23].

(iv) If $\zeta(x) = 0$ for all $x \in \mathbf{R}^n$, then (4) reduces to vector variational inequality (VVI) problem introduced and studied by [4, 6, 19]. Obviously, for $m = 1$, (3) collapses to the following inverse mixed quasi-variational inequality (IMQVI) problem for finding $\bar{x} \in \Omega(\bar{x})$ such that

$$\langle \mathbb{P}_1(\bar{x}), y - p(\bar{x})\rangle + \zeta_1(y) - \zeta_1(p(\bar{x})) \geq 0, \ \forall y \in \Omega(\bar{x}), \tag{5}$$

which was studied in [15].

(v) If $\zeta_1(x) = 0$ for all $x \in \bar{\mathbf{R}}^n$, then inverse mixed quasi-variational inequality problem collapses to the following inverse quasi-variational inequality ((IQVI) problem for finding $\bar{x} \in \Omega(\bar{x})$ such that

$$\langle \mathbb{P}_1(\bar{x}), y - p(\bar{x})\rangle \geq 0, \ \forall y \in \Omega(\bar{x}). \tag{6}$$

(vi) If $C \subset \mathbf{R}^n$ is a nonempty closed and convex subset and $\Omega(x) = C$ for all $x \in \mathbf{R}^n$, then inverse mixed quasi-variational inequality problem collapses to the following mixed variational inequality (MVI) problem for finding $\bar{x} \in C$ such that

$$\langle \mathbb{P}_1(\bar{x}), y - p(\bar{x})\rangle + \zeta_1(y) - \zeta_1(p(\bar{x})) \geq 0, \ \ \forall y \in C, \tag{7}$$

which was studied in [2]. When $C = \mathbf{R}^n$, mixed variational inequality was introduced by Solodov [22].

(vii) When $\mathbb{P}_1(x) = x$, for all $x \in \mathbf{R}^n$, mixed variational inequality becomes inverse mixed variational inequality (IMVI) which was studied by [14].

(viii) For $i = 1, 2, \cdots, m$, we denote the inverse mixed quasi-variational inequality (IMQVI) associated with $\mathbb{P}_i$, p, Ω, and ζ_i as $(\text{IMQVI})^i$. The solution sets of $(\text{IMQVI})^i$ are denoted by $sol(\text{IMQVI})^i$.

In this paper, we intend to study several scalar-valued gap functions and error bounds for generalized vector inverse mixed quasi-variational inequality problem with point to set-valued mappings. In order to do this, we shall revoke some notations and definitions, which will be used in the sequel.

Definition 2.1. [2] Let $\mathbb{P} : \mathbf{R}^n \to \mathbf{R}^n$ and $p : \mathbf{R}^n \to \mathbf{R}^n$ be two maps.

(i) $(\mathbb{P}, p)$ is said to be a strongly monotone couple with modulus μ if there exists a constant $\mu > 0$ such that

$$\langle \mathbb{P}(y) - \mathbb{P}(x), p(y) - p(x)\rangle \geq \mu\|y - x\|^2, \ \forall x, y \in \mathbf{R}^n;$$

(ii) $(\mathbb{P}, p)$ is said to be a relaxed monotone couple with modulus μ if there exists a constant $\mu > 0$ such that

$$\langle \mathbb{P}(y) - \mathbb{P}(x), p(y) - p(x)\rangle \geq -\mu\|y - x\|^2, \ \forall x, y \in \mathbf{R}^n;$$

(iii) p is said to be ℓ-Lipschitz continuous on $\mathbf{R}^n$ if there exists a constant $\ell > 0$ such that

$$\|p(x) - p(y)\| \leq \ell\|x - y\|, \forall x, y \in \mathbf{R}^n.$$

For any fixed $\rho > 0$, let $G : \mathbf{R}^n \times \tilde{\Omega} \to (-\infty, +\infty]$ be a function defined as follows:

$$G(\varphi, x) = \|x\|^2 - 2\langle \varphi, x\rangle + \|\varphi\|^2 + 2\rho\zeta(x), \ \forall \varphi \in \mathbf{R}^n, \ x \in \tilde{\Omega}, \tag{8}$$

where $\tilde{\Omega} \subset \mathbf{R}^n$ is a nonempty closed convex subset and $\zeta : \mathbf{R}^n \to \mathcal{R}$ is convex.

Definition 2.2. [25] We say that $\urcorner^{\zeta}_{\tilde{\Omega}} : \mathbf{R}^n \to 2^{\tilde{\Omega}}$ is a generalized ζ-projection operator if

$$\urcorner^{\zeta}_{\tilde{\Omega}}\varphi = \left\{w \in \tilde{\Omega} : G(\varphi, w) = \inf_{y \in \tilde{\Omega}} G(\varphi, y)\right\}, \ \forall \varphi \in \mathbf{R}^n.$$

Remark 2.3. If $\zeta(x) = 0$ for all $x \in \tilde{\Omega}$, then the generalized ζ-projection operator $\urcorner^{\zeta}_{\tilde{\Omega}}$ is equivalent to the following metric projection operator:

$$P_{\tilde{\Omega}}(\varphi) = \Big\{ w \in \tilde{\Omega} : \|w - \varphi\| = \inf_{y \in \tilde{\Omega}} \|y - \varphi\| \Big\}, \ \forall \varphi \in \mathbf{R}^n.$$

Lemma 2.4. [14, 25] *The following statements hold:*

(i) For any given $\varphi \in \mathbf{R}^n$, $\urcorner^{\zeta}_{\tilde{\Omega}}\varphi$ *is nonempty and* $\urcorner^{\zeta}_{\tilde{\Omega}}$ *is a single-valued mapping;*

(ii) For any given $\varphi \in \mathbf{R}^n, x = \urcorner^{\zeta}_{\tilde{\Omega}}\varphi$ *if and only if*

$$\langle x - \varphi, y - x \rangle + \rho\zeta(y) - \rho\zeta(x) \geq 0, \ \forall y \in \tilde{\Omega};$$

(iii) $\urcorner^{\zeta}_{\tilde{\Omega}} : \mathbf{R}^n \to \Omega$ *is nonexpansive, that is,*

$$\|\urcorner^{\zeta}_{\tilde{\Omega}}x - \urcorner^{\zeta}_{\tilde{\Omega}}y\| \leq \|x - y\|, \ \forall x, y \in \mathbf{R}^n.$$

Lemma 2.5. [15] *Let m be a positive number,* $\mathcal{B} \subset \mathbf{R}^n$ *be a nonempty subset such that*

$$\|w\| \leq m, \ \forall \ w \in \mathcal{B}.$$

Let $\Omega : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ *be a mapping such that, for each* $x \in \mathbf{R}^n$, $\Omega(x)$ *is a closed convex set, and let* $\zeta : \mathbf{R}^n \to \mathcal{R}$ *be a convex function on* $\mathbf{R}^n$. *Assume that*

(i) there exists a constant $\alpha > 0$ *such that*

$$\mathcal{H}(\Omega(x), \Omega(y)) \leq \alpha\|x - y\|, x, y \in \mathbf{R}^n;$$

where $\mathcal{H}(\Omega(x), \Omega(y))$ *is the Hausdorff distance between* $\Omega(x)$ *and* $\Omega(y)$, *that is,*

$$\mathcal{H}(\Omega(x), \Omega(y)) := \max\Big\{ \sup_{u \in \Omega(x)} \inf_{v \in \Omega(y)} \|u - v\|, \sup_{v \in \Omega(y)} \inf_{u \in \Omega(x)} \|u - v\| \Big\}.$$

(ii) $0 \in \bigcap_{w \in \mathbf{R}^n} \Omega(w)$,

(iii) ζ *is* ℓ*-Lipschitz continuous on* $\mathbf{R}^n$.

Then there exists a constant $\kappa = \sqrt{6\alpha(m + \rho\ell)}$ *such that*

$$\|\urcorner^{\zeta}_{\Omega(x)}z - \urcorner^{\zeta}_{\Omega(y)}z\| \leq \kappa\|x - y\|, \forall x, y \in \mathbf{R}^n, z \in \mathcal{B}.$$

Definition 2.6. A function $r : \mathbf{R}^n \to R$ is said to be a gap function for a generalized vector inverse mixed quasi-variational inequality on a set $\tilde{\mathcal{S}} \subset \mathbf{R}^n$ if it satisfies the following properties:

(i) $r(x) \geq 0$ for any $x \in \tilde{\mathcal{S}}$;

(ii) $r(\bar{x}) = 0, \ \bar{x} \in \tilde{\mathcal{S}}$ if and only if $\bar{x}$ is a solution of GVIMQVI.

3. The residual gap functions

In this section, we will give the residual gap function for generalized vector inverse mixed quasi-variational inequality (GVIMQVI) problem and we prove the error bounds related to the residual gap function. Now, we define the residual gap function for GVIMQVI as follows:

$$r_\rho(x) = \min_{1\leq i\leq m}\Big\{\|p(x) - \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}(\bar{u}) + \mathbb{Q}(\bar{v}))]\|\Big\}, x \in \mathbf{R}^n, u \in \mathcal{F}(x), v \in \mathcal{T}(x), \rho > 0. \tag{9}$$

Theorem 3.1. *Let $\mathcal{F}, \mathcal{T} : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be mappings and $\mathbb{P}_i, \mathbb{Q}_i : \mathbf{R}^n \to \mathbf{R}^n (i = 1, 2, \cdots, m)$ be point to point mappings. Assume that $p : \mathbf{R}^n \to \mathbf{R}^n$ is a point to point mapping. Then for any $\rho > 0$, $r_\rho(x)$ is a gap function for GVIMQVI on $\mathbf{R}^n$.*

Proof. It is clear that,

$$r_\rho(x) \geq 0 \text{ for any } x \in \mathbf{R}^n.$$

Next, for $\bar{x} \in \mathbf{R}^n$. if

$$r_\rho(\bar{x}) = 0,$$

then there exists $0 \leq i_0 \leq m$ such that

$$p(\bar{x}) = \daleth^{\zeta_{i_0}}_{\Omega(\bar{x})}[p(\bar{x}) - \rho(\mathbb{P}_{i_0}(\bar{u}) + \mathbb{Q}_{i_0}(\bar{v}))], \ \forall \bar{u} \in \mathcal{F}(\bar{x}), \bar{v} \in \mathcal{T}(\bar{x}).$$

Lemma 2.4 implies that

$$\langle p(\bar{x}) - [p(\bar{x}) - \rho(\mathbb{P}_{i_0}(\bar{u}) + \mathbb{Q}_{i_0}(\bar{v}))], y - p(\bar{x})\rangle + \rho\zeta(y) - \rho\zeta(p(\bar{x})) \leq 0, \ \forall y \in \Omega(\bar{x}),$$

and so

$$\langle \mathbb{P}_{i_0}(\bar{u}) + \mathbb{Q}_{i_0}(\bar{v}), y - p(\bar{x})\rangle + \zeta(y) - \zeta(p(\bar{x})) \leq 0, \ \forall y \in \Omega(\bar{x}), \ \bar{u} \in \mathcal{F}(\bar{x}), \bar{v} \in \mathcal{T}(\bar{x}).$$

It means that

$$\langle \mathbb{P}(\bar{u}) + \mathbb{Q}(\bar{v}), y - p(\bar{x})\rangle + \zeta(y) - \zeta(p(\bar{x})) \notin -intR^m_+, \ \forall y \in \Omega(\bar{x}), \ \bar{u} \in \mathcal{F}(\bar{x}), \bar{v} \in \mathcal{T}(\bar{x}).$$

Thus, $\bar{x}$ is a solution of GVIMQVI.

Conversely, if $\bar{x}$ is a solution of GVIMQVI, there exists $1 \leq i_0 \leq m$ such that

$$\langle \mathbb{P}_{i_0}(\bar{u}) + \mathbb{Q}_{i_0}(\bar{v}), y - p(\bar{x})\rangle + \zeta_{i_0}(y) - \zeta_{i_0}(p(\bar{x})) \geq 0, \ \forall y \in \Omega(\bar{x}), \bar{u} \in \mathcal{F}(\bar{x}), \bar{v} \in \mathcal{T}(\bar{x}).$$

By Lemma 2.4, we have

$$p(\bar{x}) = \daleth^{\zeta_{i_0}}_{\Omega(\bar{x})}[p(\bar{x}) - \rho(\mathbb{P}_{i_0}(\bar{u}) + \mathbb{Q}_{i_0}(\bar{v}))], \forall \ \bar{u} \in \mathcal{F}(\bar{x}), \bar{v} \in \mathcal{T}(\bar{x}).$$

This means that

$$r_\rho(\bar{x}) = \min_{1\leq i\leq m}\{\|p(\bar{x}) - \daleth^{\zeta_i}_{\Omega(\bar{x})}[p(\bar{x}) - \rho(\mathbb{P}_i(\bar{u}) + \mathbb{Q}_i(\bar{v}))]\|\} = 0.$$

This completes the proof. □

Next, we will give the error bounds for GVIMQVI in term of the residual gap function r_ρ.

Theorem 3.2. *Let $\mathcal{F}, \mathcal{T} : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be $\mathcal{H}$-δ-Lipschitz continuous and $\mathcal{H}$-η-Lipschitz continuous, respectively, and $\mathbb{P}_i, \mathbb{Q}_i : \mathbf{R}^n \to \mathbf{R}^n (i = 1, 2, \cdots, m)$ be σ_i-Lipschitz continuous and ϱ_i-Lipschitz continuous, respectively. Let $p : \mathbf{R}^n \to \mathbf{R}^n$ be ℓ-Lipschitz continuous, and for $i = 1, 2, \cdots, m$, $(\mathbb{P}_i, p)$ be a strongly monotone couple with modulus μ_i and $(\mathbb{Q}_i, p)$ be a relaxed monotone couple with modulus π_i. Let*

$$\bigcap_{i=1}^{m} sol(GVIMQVI)^i \neq \emptyset.$$

Assume that there exists $\kappa_i \in \Big(0, \dfrac{\mu_i - \pi_i}{\delta\sigma_i + \eta\varrho_i}\Big)$ *such that*

$$\|\urcorner^{\zeta_i}_{\Omega(x)} z - \urcorner^{\zeta_i}_{\Omega(y)} z\| \leq \kappa_i \|x - y\|, \ \forall x, y \in \mathbf{R}^n, \tag{10}$$

where $z \in \{w \mid w = p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))\}$*, for all* $u \in \mathcal{F}(x),\ v \in \mathcal{T}(x)$*. Then, for any* $x \in \mathbf{R}^n$*,* $\mu_i > \pi_i + \kappa_i(\delta\sigma_i + \eta\varrho_i)$ *and*

$$\rho > \frac{\kappa_i \ell}{\mu_i - \pi_i - \kappa_i(\delta\sigma_i + \eta\varrho_i)},$$

we have

$$d(x, sol(GVIMQVI)) \leq \frac{\rho(\delta\sigma_i + \eta\varrho_i) + \ell}{\rho\mu_i - \rho\pi_i - \rho\kappa_i(\delta\sigma_i + \eta\varrho_i) - \kappa_i\ell} r_\rho(x),$$

where

$$d(x, sol(GVIMQVI)) := \inf_{\bar{x} \in sol(GVIMQVI)} \|x - \bar{x}\|$$

denotes the distance between the point x *and the set sol(GVIMQVI).*

Proof. Since

$$\bigcap_{i=1}^{m} sol(GVIMQVI)^i \neq \emptyset,$$

we assume that $\bar{x} \in \Omega(\bar{x})$ is a common solution of $(\text{GVIMQVI})^i$, $i = 1, \cdots, m$, and thus for any $i \in \{1, \cdots, m\}$, we have

$$\langle \mathbb{P}_i(\bar{u}) + \mathbb{Q}_i(\bar{v}), y - p(\bar{x})\rangle + \zeta_i(y) - \zeta_i(p(\bar{x})) \geq 0, \ \forall y \in \Omega(\bar{x}), \bar{u} \in \mathcal{F}(\bar{x}), \bar{v} \in \mathcal{T}(\bar{x}). \tag{11}$$

By definition of $\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]$ and Lemma 2.4 implies that

$$\begin{aligned}&\langle \urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - (p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))),\ y - \urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) \\ &\quad - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]\rangle + \rho\zeta_i(y) - \rho\zeta_i(\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]) \geq 0,\end{aligned} \tag{12}$$

for all $y \in \Omega(\bar{x}), u \in \mathcal{F}(x), v \in \mathcal{T}(x)$.

Since $\bar{x} \in \bigcap_{i=1}^{m} sol(GVIMQVI)^i$ and $p(\bar{x}) \in \Omega(\bar{x})$, replacing y by $p(\bar{x})$ in (12), we get

$$\begin{aligned}&\langle \urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - (p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))),\ p(\bar{x}) - \urcorner^{\zeta}_{\Omega(\bar{x})}[p(x) \\ &\quad - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]\rangle + \rho\zeta_i(p(\bar{x})) - \rho(\zeta_i(\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))])) \geq 0.\end{aligned} \tag{13}$$

Since $\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] \in \Omega(\bar{x})$, it follows from (11) that

$$\begin{aligned}&\langle \rho(\mathbb{P}_i(\bar{u}) + \mathbb{Q}_i(\bar{v})),\ \urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - p(\bar{x})\rangle \\ &\quad + \rho\zeta_i(\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]) - \rho\zeta_i(p(\bar{x})) \geq 0.\end{aligned} \tag{14}$$

From (13) and (14), we have

$$\begin{aligned}&\langle \rho(\mathbb{P}_i(\bar{u}) + \mathbb{Q}_i(\bar{v})) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v)) - \urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] + p(x), \\ &\qquad\qquad \urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - p(\bar{x})\rangle \geq 0,\end{aligned}$$

which also implies

$$\begin{aligned}&\langle\rho(\mathbb{P}_i(\bar{u})+\mathbb{Q}_i(\bar{v}))-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v)),\ \urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]-p(x)\rangle\\&\quad-\langle\rho(\mathbb{P}_i(\bar{u})+\mathbb{Q}_i(\bar{v}))-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v)),p(\bar{x})-p(x)\rangle\\&\quad+\langle p(x)-\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))],\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]-p(x)\rangle\\&\quad+\langle p(x)-\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))],p(x)-p(\bar{x})\rangle\geq 0.\end{aligned}$$

Since, for $i=1,2,\cdots,m,(\mathbb{P}_i,p)$ are strongly monotone couples with modulus μ_i, and $(\mathbb{Q}_i,p)$ are relaxed monotone couples with modulus π_i, we have

$$\begin{aligned}&\langle\rho(\mathbb{P}_i(\bar{u})+\mathbb{Q}_i(\bar{v}))-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v)),\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]-p(x)\rangle\\&\qquad-\|p(x)-\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]\|^2\\&\qquad+\langle p(x)-\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))],p(x)-p(\bar{x})\rangle\\&\quad\geq\rho\mu_i\|x-\bar{x}\|^2-\rho\pi_i\|x-\bar{x}\|^2.\end{aligned}$$

By inserting

$$\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]$$

and using the Cauchy—Schwarz inequality along with the triangular inequality, we have

$$\begin{aligned}&\|\rho(\mathbb{P}_i(\bar{u})+\mathbb{Q}_i(\bar{v}))-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))\|\times\Big\{\|\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]\\&\qquad-\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]\|+\|\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]-p(x)\|\Big\}\\&\qquad+\|p(x)-p(\bar{x})\|\times\Big\{\|p(x)-\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]\|\\&\qquad+\|\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]-\urcorner^{\zeta_i}_{\Omega(\bar{x})}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]\|\Big\}\\&\quad\geq\rho(\mu_i-\pi_i)\|x-\bar{x}\|^2.\end{aligned}$$

Using the Lipschitz continuity of $\mathbb{P}_i,\mathbb{Q}_i,p$, $\mathcal{H}$-δ-Lipschitz continuity of $\mathcal{F}$, $\mathcal{H}$-η-Lipschitz continuity of $\mathcal{T}$ and condition (10), we have

$$\begin{aligned}&\rho(\delta\sigma_i+\eta\varrho_i)\|\bar{x}-x\|\times\Big\{\kappa_i\|\bar{x}-x\|+\|\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]-p(x)\|\Big\}\\&\qquad+\ell\|x-\bar{x}\|\times\Big\{\|p(x)-\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]\|+\kappa_i\|x-\bar{x}\|\Big\}\\&\quad\geq\rho(\mu_i-\pi_i)\|x-\bar{x}\|^2.\end{aligned}$$

Hence, for any $x\in\mathbf{R}^n$, $i\in\{1,2,\cdots,m\}$, $\mu_i>\pi_i+\kappa_i(\delta\sigma_i+\eta\varrho_i)$ and

$$\rho>\frac{\kappa_i\ell}{\mu_i-\pi_i-\kappa_i(\delta\sigma_i+\eta\varrho_i)},$$

we have

$$\|x-\bar{x}\|\leq\frac{\rho(\delta\sigma_i+\eta\varrho_i)+\ell}{\rho\mu_i-\rho\pi_i-\rho\kappa_i(\delta\sigma_i+\eta\varrho_i)-\kappa_i\ell}\|p(x)-\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]\|.$$

This implies

$$\|x-\bar{x}\|\leq\frac{\rho(\delta\sigma_i+\eta\varrho_i)+\ell}{\rho\mu_i-\rho\pi_i-\rho\kappa_i(\delta\sigma_i+\eta\varrho_i)-\kappa_i\ell}\min_{1\leq i\leq m}\Big\{\|p(x)-\urcorner^{\zeta_i}_{\Omega(x)}[p(x)-\rho(\mathbb{P}_i(u)+\mathbb{Q}_i(v))]\|\Big\}.$$

which means that

$$d(x, sol(GVIMQVI)) \leq \|x - \bar{x}\| \leq \frac{\rho(\delta\sigma_i + \eta\varrho_i) + \ell}{\rho\mu_i - \rho\pi_i - \rho\kappa_i(\delta\sigma_i + \eta\varrho_i) - \kappa_i\ell}\, r_\rho(x).$$

This completes the proof. □

Remark 3.3. Lemma 2.5 implies that condition (10) holds under certain appropriate assumptions.

4. The regularized gap function

In general, the residual gap function fails to be smooth but for the algorithmic purpose, it is desirable to deal with a smooth optimization problems. Sun and Chai [23] studied the regularize gap function for generalized vector variational inequalities. Taking motivation from these works, we design a regularize gap function for GVIMQVI and develop corresponding error bounds for GVIMQVI.

The regularized gap function for GVIMQVI is defined for all $x \in \mathbf{R}^n$ as follows:

$$\phi_\rho(x) = \min_{1\leq i\leq m} \sup_{y\in\Omega(x)} \Big\{ \langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), p(x) - y\rangle + \zeta_i(p(x)) - \zeta_i(y) - \frac{1}{2\rho}\|p(x) - y\|^2 \Big\},$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x),\ \rho > 0$.

Lemma 4.1. *We have*

$$\phi_\rho(x) = \min_{1\leq i\leq m} \Big\{ \langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\rho(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\rho(x)) - \frac{1}{2\rho}\|\mathbf{R}^i_\rho(x)\|^2 \Big\}, \tag{15}$$

where

$$\mathbf{R}^i_\rho(x) = p(x) - \urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))], \forall x \in \mathbf{R}^n,\ u \in \mathcal{F}(x), v \in \mathcal{T}(x).$$

And if

$$x \in p^{-1}(\Omega) := \Big\{ \xi \in \mathbf{R}^n \Big| p(\xi) \in \Omega(\xi) \Big\},$$

then

$$\phi_\rho(x) \geq \frac{1}{2\rho} r_\rho(x)^2. \tag{16}$$

Proof. For given $x \in \mathbf{R}^n$ and $i \in \{1, 2, \cdots, m\}$, set

$$\psi_i(x, y) = \langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), p(x) - y\rangle + \zeta_i(p(x)) - \zeta_i(y) - \frac{1}{2\rho}\|p(x) - y\|^2,$$

for all $y \in \mathbf{R}^n, u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. Consider the following optimization problem:

$$g_i(x) = \max_{y\in\Omega(x)} \psi_i(x, y).$$

Since $\psi_i(x, \cdot)$ is a strongly concave function and $\Omega(x)$ is nonempty, closed and convex, the above optimization problem has a unique solution, say $z \in \Omega(x)$. Evoking the condition of optimality at z, we get

$$0 \in \mathbb{P}_i(u) + \mathbb{Q}_i(v) + \partial\zeta_i(z) + \frac{1}{\rho}(z - p(x)) + N_{\Omega(x)}(z),$$

where $N_{\Omega(x)}(z)$ is the normal cone at z to $\Omega(x)$ and $\partial\zeta_i(z)$ denotes the subdifferential of ζ_i at z. Therefore,

$$\langle z - (p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))), y - z\rangle + \rho\zeta_i(y) - \rho\zeta_i(z) \geq 0,\ \forall y \in \Omega(x),$$

and so

$$z = \urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))],\ \forall u \in \mathcal{F}(x), v \in \mathcal{T}(x).$$

Hence $g_i(x)$ can be rewritten as

$$\begin{aligned} g_i(x) = &\langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), p(x) - \urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]\rangle \\ &+ \zeta_i(p(x)) - \zeta_i(\urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]) \\ &- \frac{1}{2\rho}\|p(x) - \urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]\|^2, \end{aligned}$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$.

Letting

$$\mathbf{R}^i_\rho(x) = p(x) - \urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))],\ \forall u \in \mathcal{F}(x), v \in \mathcal{T}(x),$$

then we get

$$g_i(x) = \langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\rho(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\rho(x)) - \frac{1}{2\rho}\|\mathbf{R}^i_\rho(x)\|^2, \tag{17}$$

and so

$$\phi_\rho(x) = \min_{1\le i\le m}\Big\{\langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\rho(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\rho(x)) - \frac{1}{2\rho}\|\mathbf{R}^i_\rho(x)\|^2\Big\}.$$

From the definition of projection (Lemma 2.4).

$$\urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))],$$

we have

$$\begin{aligned} &\langle \urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - p(x) + \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v)), y - \urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]\rangle \\ &\quad + \rho\zeta_i(y) - \rho\zeta_i(\urcorner^{\zeta_i}_{\Omega(x)}[p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]) \ge 0, \end{aligned} \tag{18}$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. For any $x \in p^{-1}(\Omega)$, we have

$$p(x) \in \Omega(x),$$

and therefore, by taking $y = p(x)$ in (18), we get

$$\langle \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v)) - \mathbf{R}^i_\rho(x),\ \mathbf{R}^i_\rho(x)\rangle + \rho\zeta_i(p(x)) - \rho\zeta_i(p(x) - \mathbf{R}^i_\rho(x)) \ge 0,$$

that is,

$$\begin{aligned} \langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\rho(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\rho(x)) &\ge \frac{1}{\rho}\langle \mathbf{R}^i_\rho(x), \mathbf{R}^i_\rho(x)\rangle \\ &= \frac{1}{\rho}\|\mathbf{R}^i_\rho(x)\|^2, \end{aligned}$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. From the definition of $r_\rho(x)$ and (15), we get

$$\phi_\rho(x) \ge \frac{1}{2\rho} r_\rho(x)^2.$$

This completes the proof. □

Theorem 4.2. *For $\rho > 0$, ϕ_ρ is a gap function for GVIMQVI on the set*

$$p^{-1}(\Omega) = \{\xi \in \mathbf{R}^n | p(\xi) \in \Omega(\xi)\}.$$

Proof. From the definition of ϕ_ρ, we have

$$\phi_\rho(x) \geq \min_{1 \leq i \leq m} \Big\{ \langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), p(x) - y \rangle + \zeta_i(p(x)) - \zeta_i(y) - \frac{1}{2\rho} \|p(x) - y\|^2 \Big\},$$

for all $y \in \Omega(x)$, $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. Therefore, for any $x \in p^{-1}(\Omega)$, by setting $y = p(x)$, we have

$$\phi_\rho(x) \geq 0.$$

Next, suppose that $\bar{x} \in p^{-1}(\Omega)$ with $\phi_\rho(\bar{x}) = 0$. From (16), it follows that

$$r_\rho(\bar{x}) = 0,$$

which implies that $\bar{x}$ is a solution of GVIMQVI.

Conversely, if $\bar{x}$ is a solution of GVIMQVI, there exists $1 \leq i_0 \leq m$ such that

$$\langle \mathbb{P}_{i_0}(\bar{u}) + \mathbb{Q}_{i_0}(\bar{v}), p(\bar{x}) - y \rangle + \zeta_{i_0}(p(\bar{x})) - \zeta_{i_0}(y) \leq 0,$$

for all $y \in \Omega(\bar{x}), \bar{u} \in \mathcal{F}(\bar{x}), \bar{v} \in \mathcal{T}(\bar{x})$ which means that

$$\min_{1 \leq i \leq m} \Big\{ \sup_{y \in \Omega(\bar{x})} \Big\{ \langle \mathbb{P}_i(\bar{u}) + \mathbb{Q}_i(\bar{v}), p(\bar{x}) - y \rangle + \zeta_i(p(\bar{x})) - \zeta_i(y) - \frac{1}{2\rho} \|p(\bar{x}) - y\|^2 \Big\} \Big\} \leq 0.$$

Thus,

$$\phi_\rho(\bar{x}) \leq 0.$$

The preceding claim leads to

$$\phi_\rho(\bar{x}) \geq 0$$

and it implies that

$$\phi_\rho(\bar{x}) = 0.$$

This completes the proof. □

Since ϕ_ρ can act as a gap function for GVIMQVI , according to Theorem 4.2, investigating the error-bound properties that can be obtained with ϕ_ρ is interesting. The following corollary is obtained directly from Theorem 3.2 and (13).

Corollary 4.3. *Let $\mathcal{F}, \mathcal{T} : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be $\mathcal{H}$-δ-Lipschitz continuous; $\mathcal{H}$-η-Lipschitz continuous, respectively, $\mathbb{P}_i, \mathbb{Q}_i : \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1, 2, \cdots, m)$ be σ_i-Lipschitz continuous and ϱ_i-Lipschitz continuous, respectively. Let $p : \mathbb{R}^n \to \mathbb{R}^n$ be ℓ-Lipschitz continuous, and for $i = 1, 2, \cdots, m$, $(\mathbb{P}_i, p)$ be a strongly monotone couple with modulus μ_i and $(\mathbb{Q}_i, p)$ be a relaxed monotone couple with modulus π_i. Let*

$$\bigcap_{i=1}^{m} sol(GVIMQVI)^i \neq \emptyset.$$

Assume that there exists $\kappa_i \in \Big(0, \dfrac{\mu_i - \pi_i}{\delta\sigma_i + \eta\varrho_i}\Big)$ such that

$$\|\urcorner^{\zeta_i}_{\Omega(x)} z - \urcorner^{\zeta_i}_{\Omega(y)} z\| \leq \kappa_i \|x - y\|,$$

for all $x, y \in \mathbf{R}^n$ and $z \in \Big\{ w \mid w = p(x) - \rho(\mathbb{P}_i(u) + \mathbb{Q}_i(v)) \Big\}$ for $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. Then, for any $x \in p^{-1}(\Omega)$ and $\rho > \dfrac{\kappa_i \ell}{\mu_i - \pi_i - \kappa_i(\delta\sigma_i + \eta\varrho_i)}$, we have

$$d(x, sol(GVIMQVI)) \leq \frac{\rho(\delta\sigma_i + \eta\varrho_i) + \ell}{\rho\mu_i - \rho\pi_i - \rho\kappa_i(\delta\sigma_i + \eta\varrho_i) - \kappa_i \ell} \sqrt{2\rho\phi_\rho(x)}.$$

If $\mathcal{T}$ is an identity mapping and $\mathbb{Q}$ is zero mapping, then Corollary 4.3 will be as follows:

Corollary 4.4. *Let $\mathcal{F} : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be $\mathcal{H}$-δ-Lipschitz continuous and $\mathbb{P}_i : \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1, 2, \cdots, m)$ be σ_i-Lipschitz continuous. Let $p : \mathbb{R}^n \to \mathbb{R}^n$ be ℓ-Lipschitz continuous, and for $i = 1, 2, \cdots, m$, $(\mathbb{P}_i, p)$ be a strongly monotone couple with modulus μ_i. Let*

$$\bigcap_{i=1}^{m} sol(GVIMQVI)^i \neq \emptyset.$$

Assume that there exists $\kappa_i \in \left(0, \frac{\mu_i}{\delta\sigma_i}\right)$ such that

$$\|\urcorner^{\zeta_i}_{\Omega(x)} z - \urcorner^{\zeta_i}_{\Omega(y)} z\| \leq \kappa_i \|x - y\|,$$

for all $x, y \in \mathbf{R}^n$ and $z \in \left\{w \mid w = p(x) - \rho\mathbb{P}_i(u)\right\}$ for $u \in \mathcal{F}(x)$. Then, for any $x \in p^{-1}(\Omega)$ and $\rho > \frac{\kappa_i \ell}{\mu_i - \kappa_i\delta\sigma_i}$, we have

$$d(x, sol(2)) \leq \frac{\rho\delta\sigma_i + \ell}{\rho\mu_i - \rho\kappa_i\delta\sigma_i - \kappa_i\ell} \sqrt{2\rho\phi_\rho(x)},$$

where $sol(2)$ is the set of all solutions of the variational inequality problem (2).

If $\mathcal{F}$ is a point to point mapping, then Corollary 4.4 will be as follows:

Corollary 4.5. *Let $\mathbb{P}_i : \mathbf{R}^n \to \mathbf{R}^n$ $(i = 1, 2, \cdots, m)$ be σ_i-Lipschitz continuous, $p : \mathbf{R}^n \to \mathbf{R}^n$ be ℓ-Lipschitz continuous, and for $i = 1, 2, \cdots, m$, $(\mathbb{P}_i, p)$ be a strongly monotone couple with modulus μ_i. Let*

$$\bigcap_{i=1}^{m} sol(2)^i \neq \emptyset.$$

Assume that there exists $\kappa_i \in \left(0, \frac{\mu_i}{\sigma_i}\right)$ such that

$$\|\urcorner^{\zeta_i}_{\Omega(x)} z - \urcorner^{\zeta_i}_{\Omega(y)} z\| \leq \kappa_i \|x - y\|,$$

for all $x, y \in \mathbf{R}^n$ and $z \in \left\{w \mid w = p(x) - \rho\mathbb{P}_i(x)\right\}$. Then, for any $x \in p^{-1}(\Omega)$ and $\rho > \frac{\kappa_i \ell}{\mu_i - \kappa_i\sigma_i}$, we have

$$d(x, sol(VIMQVI)) \leq \frac{\rho\sigma_i + \ell}{\rho\mu_i - \rho\kappa_i\sigma_i - \kappa_i\ell} \sqrt{2\rho\phi_\rho(x)}.$$

5. The *D*-Gap functions

It is surprising that the regularized gap function ϕ_ρ does not provide global error bounds for GVIMQVI on $\mathbf{R}^n$. Solodov [22] proposed the *D*-gap function for mixed variational inequality and obtained error bounds for mixed variational inequality related to the *D*-gap function. With this motivation, we develop the *D*-gap function for GVIMQVI, which gives $\mathbf{R}^n$ the global error bounds for GVIMQVI.

For GVIMQVI with $\gamma > \lambda > 0$, the *D*-gap function for GVIMQVI is defined as follows:

$$G_{\gamma\lambda}(x) = \min_{1\leq i\leq m} \Big\{ \sup_{y\in\Omega(x)} \Big\{\langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), p(x) - y\rangle + \zeta_i(p(x)) - \zeta_i(y) - \frac{1}{2\gamma}\|p(x) - y\|^2\Big\}$$
$$- \sup_{y\in\Omega(x)} \Big\{\langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), p(x) - y\rangle + \zeta_i(p(x)) - \zeta_i(y) - \frac{1}{2\lambda}\|p(x) - y\|^2\Big\}\Big\},$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$.

By (15) in Lemma 4.1, we know that $G_{\gamma\lambda}$ can be rewritten as

$$G_{\gamma\lambda}(x) = \min_{1\le i\le m} \Big\{\langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\gamma(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\gamma(x)) - \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x)\|^2 - \Big(\langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\lambda(x)\rangle + \zeta_i(p(x)) - \zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) - \frac{1}{2\lambda}\|\mathbf{R}^i_\lambda(x)\|^2\Big)\Big\},$$

where

$$\mathbf{R}^i_\gamma(x) = p(x) - \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]$$

and

$$\mathbf{R}^i_\lambda(x) = p(x) - \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v))],\ \forall x \in \mathbf{R}^n,\ u \in \mathcal{F}(x), v \in \mathcal{T}(x).$$

Theorem 5.1. *For any $x \in \mathbf{R}^n$, $\gamma > \lambda > 0$, we have*

$$\frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)r^2_\lambda(x) \le G_{\gamma\lambda}(x) \le \frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)r^2_\gamma(x). \tag{19}$$

Proof. From the definition of $G_{\gamma\lambda}(x)$, it follows that

$$G_{\gamma\lambda}(x) = \min_{1\le i\le m} \Big\{\langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle - \zeta_i(p(x) - \mathbf{R}^i_\gamma(x)) - \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x)\|^2 + \zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) + \frac{1}{2\lambda}\|\mathbf{R}^i_\lambda(x)\|^2\Big\},$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. For any given $i \in \{1, 2, \cdots, m\}$, we set

$$g^i_{\gamma\lambda}(x) = \langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle - \zeta_i(p(x) - \mathbf{R}^i_\gamma(x)) - \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x)\|^2 + \zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) + \frac{1}{2\lambda}\|\mathbf{R}^i_\lambda(x)\|^2, \tag{20}$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. Since $\daleth^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] \in \Omega(x)$, by Lemma 2.4, we know

$$\langle \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - (p(x) - \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v))), \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]\rangle + \gamma\zeta_i(\daleth^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]) - \gamma\zeta_i(\daleth^{\zeta_i}_{\Omega(x)}[p(x) - \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]) \ge 0,$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. Hence we have

$$\langle \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v)) - \mathbf{R}^i_\gamma(x), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle + \gamma\zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) - \gamma\zeta_i(p(x) - \mathbf{R}^i_\gamma(x)) \ge 0. \tag{21}$$

Combining (20) and (21), we get

$$g^i_{\gamma\lambda}(x) \ge \frac{1}{\gamma}\langle \mathbf{R}^i_\gamma(x), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle - \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x)\|^2 + \frac{1}{2\lambda}\|\mathbf{R}^i_\lambda(x)\|^2 = \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\|^2 + \frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)\|\mathbf{R}^i_\lambda(x)\|^2. \tag{22}$$

And also, since $\daleth^{\zeta_i}_{\Omega(x)}[p(x) - \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] \in \Omega(x)$, by Lemma 2.4, we have

$$\langle \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - (p(x) - \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v))), \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v))] - \daleth^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]\rangle + \lambda\zeta_i(\daleth^{\zeta_i}_{\Omega(x)}[p(x) - \gamma(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]) - \lambda\zeta_i(\daleth^{\zeta_i}_{\Omega(x)}[p(x) - \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v))]) \ge 0,$$

for all $u \in \mathcal{F}(x), v \in \mathcal{T}(x)$. Hence, we have

$$\langle \lambda(\mathbb{P}_i(u) + \mathbb{Q}_i(v)) - \mathbf{R}^i_\lambda(x), \mathbf{R}^i_\lambda(x) - \mathbf{R}^i_\gamma(x)\rangle + \lambda\zeta_i(p(x) - \mathbf{R}^i_\gamma(x)) - \lambda\zeta_i(p(x) - \mathbf{R}^i_\lambda(x)) \geq 0,$$

and so

$$\begin{aligned}\frac{1}{\lambda}\langle \mathbf{R}^i_\lambda(x), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle &\geq \langle \mathbb{P}_i(u) + \mathbb{Q}_i(v), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle \\ &\quad - \zeta_i(p(x) - \mathbf{R}^i_\gamma(x)) + \zeta_i(p(x) - \mathbf{R}^i_\zeta(x)).\end{aligned}$$

This and (21) imply that

$$\begin{aligned}g^i_{\gamma\lambda}(x) &\leq \frac{1}{\lambda}\langle \mathbf{R}^i_\lambda(x), \mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\rangle - \frac{1}{2\gamma}\|\mathbf{R}^i_\gamma(x)\|^2 + \frac{1}{2\lambda}\|\mathbf{R}^i_\lambda(x)\|^2 \\ &= -\frac{1}{2\lambda}\|\mathbf{R}^i_\gamma(x) - \mathbf{R}^i_\lambda(x)\|^2 + \frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)\|\mathbf{R}^i_\gamma(x)\|^2. \qquad (23)\end{aligned}$$

From (22) and (23), for any $i \in \{1, 2, \cdots, m\}$, we obtain

$$\frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)\|\mathbf{R}^i_\lambda(x)\|^2 \leq g^i_{\gamma\lambda}(x) \leq \frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)\|\mathbf{R}^i_\gamma(x)\|^2.$$

Hence

$$\frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)\min_{1\leq i\leq m}\Big\{\|\mathbf{R}^i_\lambda(x)\|^2\Big\} \leq \min_{1\leq i\leq m}\Big\{g^i_{\gamma\lambda}(x)\Big\} \leq \frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)\min_{1\leq i\leq m}\Big\{\|\mathbf{R}^i_\gamma(x)\|^2\Big\},$$

and so

$$\frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)r^2_\lambda(x) \leq G_{\gamma\lambda}(x) \leq \frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\gamma}\Big)r^2_\gamma(x).$$

This completes the proof. □

Now we prove that $G_{\gamma\lambda}$ in the set $\mathbf{R}^n$ is a global gap function for GVIMQVI.

Theorem 5.2. *For $0 < \lambda < \gamma$, $G_{\gamma\lambda}$ is a gap function for GVIMQVI on $\mathbf{R}^n$.*

Proof. From (20), we have

$$G_{\gamma\lambda}(x) \geq 0, \quad \forall x \in \mathbf{R}^n.$$

Suppose that $\bar{x} \in \mathbf{R}^n$ with $G_{\gamma\lambda}(\bar{x}) = 0$. Then (20) implies that

$$r_\lambda(\bar{x}) = 0.$$

From Theorem 3.1, we know that $\bar{x}$ is a solution of GVIMQVI.

Conversely, if $\bar{x}$ is a solution of GVIMQVI, than from Theorem 3.1, it follows that

$$r_\gamma(\bar{x}) = 0.$$

Obviously, (20) shows that

$$G_{\gamma\lambda}(\bar{x}) = 0.$$

The proof is completed. □

From Theorem 3.2 and (20), we immediately get a global error bound in the set $\mathbf{R}^n$ for GVIMQVI.

Corollary 5.3. *Let $\mathcal{F},\mathcal{T}:\mathbf{R}^n\to 2^{\mathbf{R}^n}$ be $\mathcal{H}$-δ-Lipschitz continuous, $\mathcal{H}$-η-Lipschitz continuous, respectively, and $\mathbb{P}_i,\mathbb{Q}_i:\mathbf{R}^n\to\mathbf{R}^n$ $(i=1,2,\cdots,m)$ be σ_i-Lipschitz continuous, ϱ_i-Lipschitz continuous, respectively. Let $p:\mathbf{R}^n\to\mathbf{R}^n$ be ℓ-Lipschitz continuous, and for $i=1,2,\cdots,m$, $(\mathbb{P}_i,p)$ be a strongly monotone couple with modulus μ_i and $(\mathbb{Q}_i,p)$ be a ralaxed monotone couple with modulus with π_i. Let*

$$\bigcap_{i=1}^{m} sol(GVIMQVI)^i\neq\emptyset.$$

Assume that there exists $\kappa_i\in\Big(0,\dfrac{\mu_i-\pi_i}{\delta\sigma_i+\eta\varrho_i}\Big)$ such that

$$\|\urcorner^{\zeta_i}_{\Omega(x)}z-\urcorner^{\zeta_i}_{\Omega(y)}z\|\leq\kappa_i\|x-y\|,$$

for all $x,y\in\mathbf{R}^n$ and $z\in\Big\{w\mid w=p(x)-\lambda(\mathbb{P}_i(u)+\mathbb{Q}_i(v))\Big\}$, for $u\in\mathcal{F}(x),v\in\mathcal{T}(x)$. Then, for any $x\in\mathbf{R}^n$ and $\lambda>\dfrac{\kappa_i\ell}{\mu_i-\pi_i-\kappa_i(\delta\sigma_i+\eta\varrho_i)}$, we have

$$d(x,sol(GVIMQVI))\leq\frac{\lambda(\delta\sigma_i+\eta\varrho_i)+\ell}{\lambda\mu_i-\lambda\pi_i-\lambda\kappa_i(\delta\sigma_i+\eta\varrho_i)-\kappa_i\ell}\sqrt{\frac{2\gamma\lambda}{\gamma-\lambda}G_{\gamma\lambda}(x)}.$$

Note that if $\mathcal{T}$ is a point to point mappings and $\mathbb{Q}$ is zero mapping, then Corollary 5.3 reduces to the following:

Corollary 5.4. *Let $\mathcal{F}:\mathbf{R}^n\to\mathbf{R}^n$ be $\mathcal{H}$-δ-Lipschitz continuous and $\mathbb{P}_i:\mathbf{R}^n\to\mathbf{R}^n$ $(i=1,2,\cdots,m)$ be σ_i-Lipschitz continuous. Let $p:\mathbf{R}^n\to\mathbf{R}^n$ be ℓ-Lipschitz continuous, and for $i=1,2,\cdots,m$, $(\mathbb{P}_i,p)$ be a strongly monotone couple with modulus μ_i. Let*

$$\bigcap_{i=1}^{m} sol(2)^i\neq\emptyset.$$

Assume that there exists $\kappa_i\in\Big(0,\dfrac{\mu_i}{\delta\sigma_i}\Big)$ such that

$$\|\urcorner^{\zeta_i}_{\Omega(x)}z-\urcorner^{\zeta_i}_{\Omega(y)}z\|\leq\kappa_i\|x-y\|,$$

for all $x,y\in\mathbf{R}^n$ and $z\in\Big\{w\mid w=p(x)-\lambda\mathbb{P}_i(u)\Big\}$, for $u\in\mathcal{F}(x)$. Then, for any $x\in\mathbf{R}^n$ and any $\lambda>\dfrac{\kappa_i\ell}{\mu_i-\kappa_i\delta\sigma_i}$, we have

$$d(x,sol(2))\leq\frac{\lambda\delta\sigma_i+\ell}{\lambda\mu_i-\lambda\kappa_i\delta\sigma_i-\kappa_i\ell}\sqrt{\frac{2\gamma\lambda}{\gamma-\lambda}G_{\gamma\lambda}(x)},$$

where $sol(2)$ is the set of all solutions of the variational inequality problem (2).

Note that if $\mathcal{F}$ is an identity mappings and $\mathbb{P}$ is a point to point mapping, then Corollary 5.4 reduces to the following:

Corollary 5.5. *Let $\mathbb{P}_i:\mathbf{R}^n\to\mathbf{R}^n(i=1,2,\cdots,m)$ be σ_i-Lipschitz continuous, $p:\mathbf{R}^n\to\mathbf{R}^n$ be ℓ-Lipschitz continuous, and for $i=1,2,\cdots,m$, $(\mathbb{P}_i,p)$ be a strongly monotone couple with modulus μ_i. Let*

$$\bigcap_{i=1}^{m} sol(VIMQVI)^i\neq\emptyset.$$

Assume that there exists $\kappa_i\in\Big(0,\dfrac{\mu_i}{\sigma_i}\Big)$ such that

$$\|\urcorner^{\zeta_i}_{\Omega(x)} z - \urcorner^{\zeta_i}_{\Omega(y)} z\| \leq \kappa_i \|x - y\|,$$

for all $x, y \in \mathbf{R}^n$ *and* $z \in \left\{w \mid w = p(x) - \lambda \mathbb{P}_i(x)\right\}$. *Then, for any* $x \in \mathbf{R}^n$ *and* $\lambda > \dfrac{\kappa_i \ell}{\mu_i - \kappa_i \sigma_i}$, *we have*

$$d(x, sol(VIMQVI)) \leq \frac{\lambda \sigma_i + \ell}{\lambda \mu_i - \lambda \kappa_i \sigma_i - \kappa_i \ell} \sqrt{\frac{2\gamma\lambda}{\gamma - \lambda} G_{\gamma\lambda}(x)}.$$

Remark 5.6. We note that if $i = 1$ and $\zeta_1(x) = 0$ for all $x \in \mathbf{R}^n$, then the results obtained in this paper collapse to the corresponding ones in [2] and [15].

6. Conclusions

One of the classical approaches in the interpretation of a variational inequality and its variants is to transformed it into an equivalent optimization problem through the notion of gap functions. In contrast, gap functions play a pivotal role in deriving the so-called error bounds that provide a measure of the distances between the solution set and an arbitrary feasible point. These motivate us for GVIMQVI to research and evaluate various gap functions and error bounds.

References

[1] D. Aussel, R. Correa, H. Marechal, *Gap functions for quasi-variational inequalities and generalized Nash equilibrium problems*, J. Optim. Theory Appl. **151** (2011), 474-–488..

[2] D. Aussel, R. Gupta, A. Mehra, *Gap functions and error bounds for inverse quasi-variational inequality problems*, J. Math. Anal. Appl. **407** (2013), 270—280.

[3] G.Y. Chen, C.J. Goh, X.Q. Yang, *On gap functions for vector variational inequalities*, In: Giannessi, F. (ed.) Vector Variational Inequality and Vector Equilibria: Mathematical Theories, Kluwer Academic, Boston, 2000.

[4] G.Y. Chen, X.X. Huang, X.Q. Yang, *Vector Optimization: Set-Valued and Variational Analysis.* Lecture Notes in Economics and Mathematical Systems. Springer, Berlin, 2005.

[5] F. Giannessi, *Theorems of the alternative, quadratic programs and complementarity problems*, In: Cottle, R.W., Giannessi, F., Lions, J.L. Eds., Variational Inequalities and Complementarity Problems, John Wiley and Sons, Chichester, 1980.

[6] F. Giannessi, *Vector Variational Inequalities and Vector Equilibria: Mathematical Theories*. Kluwer Academic, Dordrecht, 2000.

[7] B.S He, H.X. Liu, *Inverse variational inequalities in the economic field: applications and algorithms*, Sciencepaper Online, 2006 http://www.paper.edu.cn/releasepaper/content/200609-260.

[8] J.K. Kim, Salahuddin, *Local sharp vector variational type inequality and optimization problems*, Mathematics **8** (10) (2020), 1-10. https://doi.org/10.3390/math8101844

[9] J.K. Kim, Slahuddin, *System of hierarchical nonlinear mixed variational inequalities*, Nonlinear Funct. Anal. and Appl. **24**(2) (2019), 207-220.

[10] B.S. Lee, Salahuddin, *Minty lemma for inverted vector variational inequalities*, Optimization, **66** (3) (2017), 351–359.

[11] S.J. Li, K.L. Teo, X.Q. Yang, S.Y. Wu, *Gap functions and existence of solutions to generalized vector quasi-equilibrium problems*, J. Glob. Optim. **34** (2006), 427-–440.

[12] C.Q.Li, J. Li, *Merit functions and error bounds for constrained mixed set-valued variational inequalities via generalized f-projection operators*, Optimization **65**(8) (2016), 1569-–1584.

[13] S.J. Li, H. Yan, G.Y. Chen, *Differential and sensitivity properties of gap functions for vector variational inequalities*, Math. Methods Oper. Res. **57** (2003), 377-–391.

[14] X. Li, X.S. Li, N.J. Huang, *A generalized f-projection algorithm for inverse mixed variational inequalities*, Optim. Lett. **8** (2014), 1063-–1076.

[15] X. Li, Y.Z. Zou, *Existence result and error bounds for a new class of inverse mixed quasi-variational inequalities*, J. Inequal. Appl. **2016** (2016), Article ID 42.

[16] D.V. Luu, T.T. Mai, *On optimality conditions for Henig efficiency and superefficiency in vector equilibrium problems*, Nonlinear Funct. Anal. Optim. **39** (16) (2018), 1833-1854. doi.org/10.1080/01630563.2018.1501580

[17] K.W. Meng, S.J. Li, *Differential and sensitivity properties of gap functions for Minty vector variational inequalities*, J. Math. Anal. Appl. **337** (2008). 386-398.

[18] S.N. Mishra, P.K. Das, S.K. Mishra, *On generalized harmonic vector variational inequalities using* HC_**- condition*, Nonlinear Funct Anal. Appl. **24** (3) (2019), 639-649.

[19] M. Pappalardo, G. Mastroeni, M. Passacantando, *Merit functions: a bridge between optimization and equilibria*, Ann. Oper. Res. **240** (2016), 271—299.

[20] T. Ram, A. A. Khanna, *On perturbed quasi-equilibrium problems with operator solutions,* Nonlinear Funct Anal. Appl. **22** (2) (2017), 385-394.
[21] Salahuddin *Regularization techniques for Inverse variational inequalities involving relaxed cocoercive mapping in Hilbert spaces,* Nonlinear Anal.s Forum **19** (2014), 65–76.
[22] M.V. Solodov, *Merit functions and error bounds for generalized variational inequalities,* J. Math. Anal. Appl. **287** (2003), 405-–414.
[23] X.K. Sun, Y. Chai, *Gap functions and error bounds for generalized vector variational inequalities,* Optim. Lett. **8** (2014), 1663—1673.
[24] Z.B. Wang, Z.Y. Chen, Z. Chen, *Gap functions and error bounds for vector inverse mixed quasi-variational inequalities problems,* Fixed Point Theo. Appl. **14** (2019), 1-–14.
[25] K.Q. Wu, N.J. Huang, *The generalized f-projection operator with an application*, Bull. Aust. Math. Soc. **73**, 307-–317 (2006).