Filomat 37:2 (2023), 387–391
https://doi.org/10.2298/FIL2302387B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cosine families of operators have the SVEP

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Abstract. Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators on a Banach space X with infinitesimal generator A. In this paper, we prouve that A has the SVEP if and only if $C(t)$ has the SVEP for all $t \in \mathbb{R}$ if and only if $C(t_0)$ has the SVEP for some $t_0 \in \mathbb{R}$.

1. Introduction

Throughout, X denotes a complex Banach space, let A be a closed linear operator on X with domain $\mathcal{D}(A)$, we denote by A^* and $\sigma(A)$, respectively the adjoint and the spectrum of A. The operator A is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open disc $D_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0, the only analytic function $f : D_{\lambda_0} \longrightarrow D(A)$ which satisfies the equation $(A - zI)f(z) = 0$ for all $z \in D_{\lambda_0}$ is the function $f \equiv 0$. The operator A is said to have the SVEP if A has the SVEP for every $\lambda \in \mathbb{C}$. Denote by

$$\mathcal{S}(A) = \{\lambda \in \mathbb{C} : A \text{ has not the SVEP at } \lambda\}.$$

Note that $\mu \in S(A)$ if and only if there exists a sequence $(x_i)_{i\geq 0} \subseteq D(A)$ not all of them equal to zero such that $(A - \mu)x_{i+1} = x_i$, with $x_0 = 0$ and $\sup_i \|x_i\|^{\frac{1}{i}} < \infty$. $\mathcal{S}(A)$ is open and contained in the interior of the spectrum $\sigma(A)$. For further information, see [1, 2].

Consider in X the well-posed Cauchy problem

$$(*)\begin{cases} u''(t) = Au(t), & t \in \mathbb{R} \\ u(0) = u_0 \\ u'(0) = u_1 & . \end{cases}$$

Where $A : X \longrightarrow X$ is a densely defined closed operator with nonempty resolvent set $\rho(A)$. The problem $(*)$ is (see [3] and [6]) well-posed if and only if A generates a strongly continuous cosine operator function $(C(t))_{t\in\mathbb{R}}$, i.e., a family of operators satisfying the following conditions:

1. $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.
2. $C(0) = I$ (the identity operator).
3. $t \to C(t)$ is continuous on $\mathbb{R}$ with respect to the operator norm topology on $\mathcal{B}(X)$.

2020 *Mathematics Subject Classification*. Primary 47A10, Secondary 47D62.
Keywords. Cosine; Generator; SVEP.
Received: 12 October 2020; Accepted: 30 December 2020
Communicated by Dragan S. Djordjević
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There exist some $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

If $(C(t))_{t\in\mathbb{R}}$ is a strongly continuous cosine operator function, then the infinitesimal generating operator A is defined by

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{s\to 0} \frac{2(C(s)x - x)}{s^2} \text{ exists} \right\}$$

and

$$Ax = \lim_{s\to 0} \frac{2(C(s)x - x)}{s^2} = C''(0).$$

A solution of problem (*) is given with the help of a strongly continuous cosine operator function by the formula $u(t) = C(t)u_0 + S(t)u_1$ for all $t \in \mathbb{R}$, where $S(t)$ is the sine operator function associated with the $(C(t))_{t\in\mathbb{R}}$ and is defined as $S(t)x := \int_0^t C(s)xds,\ t \in \mathbb{R},\ x \in X$. In this work we will use the theory of integration in the sense of Bochner. For $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, $S_\lambda(t)x := \int_0^t \sinh \lambda(t-s)C(s)xds,\ \ x \in X$ defines a bounded linear operator commutes with A, and $(A - \lambda^2)S_\lambda(t)x = \lambda(C(t) - \cosh \lambda t)x$, for all $x \in X$, see ([5, Lemma.4]).

If $(C(t))_{t\in\mathbb{R}}$ is a uniformly continuous operator cosine function then there is an $A \in \mathcal{B}(X)$ with $C(t) = \cosh t\sqrt{A}$, $t \in \mathbb{R}$. We have $A = \lim_{s\to 0} \frac{2(C(s) - I)}{s^2}$ in the uniform operator topology, see [4, Theorem.2.18]. For $t \in \mathbb{R}$, the function $f : z \in \mathbb{C} \mapsto \cosh(t\sqrt{z})$ defines an entire function. Thus, according to the spectral mapping theorem, we have $\cosh t\sqrt{\mathcal{S}(A)} = \mathcal{S}(C(t))$, for all $t \in \mathbb{R}$. Which implies that A has the SVEP if and only if $C(t)$ has the SVEP for all $t \in \mathbb{R}$ if and only if $C(t_0)$ has the SVEP for some $t_0 \in \mathbb{R}$. It's normal to ask the following question: Does this property remain true when replacing a family uniformly continuous cosine function of operators with a family strongly continuous cosine function of operators? In this article, we have given a positive answer to this question. More precisely, we show that if $(C(t))_{t\in\mathbb{R}}$ is a strongly continuous operator cosine function, then A has the SVEP if and only if $C(t)$ has the SVEP for all $t \in \mathbb{R}$ if and only if $C(t_0)$ has the SVEP for some $t_0 \in \mathbb{R}$.

2. Main results

Theorem 2.1. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators with infinitesimal generator A. Then for all $t \in \mathbb{R}$, we have the following equality :*

$$\mathcal{S}(C(t)) \cup \{-1, 1\} = \cosh t\sqrt{\mathcal{S}(A)} \cup \{-1, 1\}.$$

Proof. Suppose that $\cosh(\lambda t) - C(t)$ has not SVEP at 0, then there exists $x_i \in X$ such that $x_0 = 0,\ x_1 \neq 0$, $\forall i \geq 1,\ (\cosh(\lambda t) - C(t))x_i = x_{i-1}$ and $\sup_i \|x_i\|^{\frac{1}{i}} < \infty$. Since $(\cosh(\lambda t) - C(t))x_1 = 0$ and $x_1 \neq 0$. Choose now $x_1^* \in X^*$ satisfying $< x_1, x_1^* > \neq 0$ and consider the t-periodic function f defined by:

$$f(s) = \begin{cases} < \sinh \lambda(t-s)C(s)x_1, x_1^* > & \text{if } s \in [0, t[\\ \sinh \lambda t < x_1, x_1^* > & \text{if } s = t \end{cases}$$

For $m \in \mathbb{Z}$, we have,

$$\begin{aligned} < S_{\lambda_m}(t)x_1, x_1^* > &= \int_0^t \sinh \lambda_m(t-s) < C(s)x_1, x_1^* > ds \\ &= \frac{1}{2}e^{\lambda_m t}\int_0^t e^{-\lambda_m s} < C(s)x_1, x_1^* > ds - \frac{1}{2}e^{-\lambda_m t}\int_0^t e^{\lambda_m s} < C(s)x_1, x_1^* > ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}e^{\lambda t}\int_0^t e^{-\lambda_m s} < C(s)x_1, x_1^* > ds - \frac{1}{2}e^{-\lambda t}\int_0^t e^{\lambda_m s} < C(s)x_1, x_1^* > ds \\
&= \frac{1}{2}\int_0^t e^{\lambda(t-s)}e^{-2i\pi ms/t} < C(s)x_1, x_1^* > ds - \frac{1}{2}\int_0^t e^{-\lambda(t-s)}e^{2i\pi ms/t} < C(s)x_1, x_1^* > ds.
\end{aligned}
$$

For $n \in \mathbb{N}^*$ we have,

$$
\begin{aligned}
\sum_{m=-n}^{n} < S_{\lambda_m}(t)x_1, x_1^* > &= \frac{1}{2}\int_0^t e^{\lambda(t-s)}D_n(s/t) < C(s)x_1, x_1^* > ds - \frac{1}{2}\int_0^t e^{-\lambda(t-s)}D_n(s/t) < C(s)x_1, x_1^* > ds \\
&= \int_0^t D_n(s/t)f(s)ds \\
&= t\int_0^1 D_n(u)f(tu)ds,
\end{aligned}
$$

where $D_n(u) = \sum_{m=-n}^{n} e^{2in\pi u}$. Then for $q \in \mathbb{N}^*$, we have,

$\frac{1}{q+1}\sum_{n=0}^{q}\sum_{m=-n}^{n} < S_{\lambda_m}(t)x_1, x_1^* >= t\int_0^t F_q(u)f(tu)du$, where $F_q(s) = \frac{1}{q+1}\left(\sum_{n=0}^{q} D_n(s)\right)$. We show that $\lim_{q\to\infty}\int_0^t F_q(u)f(tu)du = \frac{1}{2}e^{\lambda t} < x_1, x_1^* >$. Indeed let $\epsilon > 0$. Since $\lim_{s\to 0^+} f(s) = \sinh \lambda t < x_1, x_1^* >$ and $\lim_{s\to 0^-} f(s) = \cosh \lambda t < x_1, x_1^* >$, then there exists $\eta > 0$, such that $\left|f(tu)du - \sinh \lambda t < x_1, x_1^* >\right| \leq \epsilon/4$ for all $s \in [0, \eta]$ and $\left|f(tu)du - \cosh \lambda t < x_1, x_1^* >\right| \leq \epsilon/4$ for all $s \in [-\eta, 0]$. As we can decrease t without changing the above implications, we can assume $\eta < 1/2$. Note that:

$$
\begin{aligned}
\left|\int_0^{1/2} F_q(u)f(tu)du - 1/2\sinh(\lambda t) < x_1, x_1^* >\right| &= \left|\int_0^{1/2} F_q(u)\left[f(tu) - \sinh(\lambda t) < x_1, x_1^* >\right]du\right| \\
&\leq \int_0^{1/2} F_q(u)\left|f(tu) - \sinh(\lambda t) < x_1, x_1^* >\right|du \\
&\leq \epsilon/2 + \int_\eta^{1/2} F_q(u)\left|f(tu) - \sinh(\lambda t) < x_1, x_1^* >\right|du \\
&\leq \epsilon/2 + 2\|f\|_\infty \int_\eta^{1/2} F_q(u)du \leq \epsilon \text{ for } q \geq q_0.
\end{aligned}
$$

Similarly, by increasing q_0 if necessary, we have, for $q \geq q_0$:

$$
\begin{aligned}
\left|\int_{-1/2}^{0} F_q(u)f(tu)du - 1/2\cosh(\lambda t) < x_1, x_1^* >\right| &= \left|\int_{-1/2}^{0} F_q(u)\left[f(tu) - \cosh(\lambda t) < x_1, x_1^* >\right]du\right| \\
&\leq \int_{-1/2}^{0} F_q(u)\left|f(tu) - \cosh(\lambda t) < x_1, x_1^* >\right|du \\
&\leq \epsilon/2 + \int_\eta^{1/2} F_q(u)\left|f(tu) - \cosh(\lambda t) < x_1, x_1^* >\right|du \\
&\leq \epsilon/2 + 2\|f\|_\infty \int_{-1/2}^{-\eta} F_q(u)du \leq \epsilon.
\end{aligned}
$$

Therefore $\lim_{q\to\infty}\frac{1}{q+1}\sum_{n=0}^{q}\sum_{m=-n}^{n} < S_{\lambda_m}(t)x_1, x_1^* >= \frac{1}{2}te^{\lambda t} < x_1, x_1^* >\neq 0$. So necessarily, there exists $p \in \mathbb{Z}$ such that $S_{\lambda_p}(t)x_1 \neq 0$. Let $y_i = S^i_{\lambda_p}(t)x_i$, then $(y_i)_{i\geq 0} \subseteq D(A)$, $y_0 = x_0 = 0$, $y_1 = S_{\lambda_p}(t)x_1 \neq 0$, and we have, for all $i \geq 1$:

$$\begin{aligned}
(\lambda_p^2 - A)y_i &= (\lambda_p^2 - A)S_{\lambda_p}(t)S^{i-1}_{\lambda_p}(t)x_i\\
&= (\cosh(\lambda_p t) - C(t))S^{i-1}_{\lambda_p}(t)x_i\\
&= S^{i-1}_{\lambda_p}(t)(\cosh(\lambda_p t) - C(t))x_i\\
&= S^{i-1}_{\lambda_p}(t)x_{i-1}\\
&= y_{i-1}.
\end{aligned}$$

Therefore, $(\lambda_p^2 - A)y_i = y_{i-1}$. On the other hand $\|y_i\| = \|S^i_{\lambda_p}(t)x_i\| \leq \|S^i_{\lambda_p}(t)\|\|x_i\| \leq M^i\|x_i\|$, where $M = \|S_{\lambda_p}(t)\| > 0$. Then $\sup_i \|y_i\|^{\frac{1}{i}} < \infty$. Hence $\lambda_p \in S(A)$ and $\cosh(\lambda t) = \cosh(\lambda_p t) \in \cosh(t\sqrt{S(A)})$. Finally $S(C(t)) \subseteq \cosh(t\sqrt{S(A)})$.

Conversely, let $\cosh(\lambda_0 t) \notin S(C(t)) \cup \{-1, 1\}$, then $C(t)$ has SVEP at $\cosh(\lambda_0 t)$. Let us show that $\lambda_0^2 \notin S(A)$. Let $D_{\lambda_0^2}$ the open disc centered at λ_0^2, $f : D_{\lambda_0^2} \longrightarrow D(A)$ an analytic function such that for all $\mu \in D_{\lambda_0^2}$, $(\mu - A)f(\mu) = 0$. Show that f is identically zero on $D_{\lambda_0^2}$. Consider the analytic function $\varphi_t : z \in D_{\lambda_0^2} \longmapsto \cosh(t\sqrt{z})$. Since $\cosh(\lambda_0 t) \neq \pm 1$, then $\varphi_t'(\lambda_0^2) \neq 0$. By the inverse function theorem, there exists a neighborhood V of λ_0^2 such that $V \subseteq D_{\lambda_0^2}$, $\varphi_t(V)$ is open and the function $\varphi_t : V \longrightarrow \varphi_t(V)$ is bijective. The function $\varphi_t^{-1} : \varphi_t(V) \longrightarrow V$ is analytic. Then $g : z \in \varphi_t(V) \longrightarrow f(\varphi_t^{-1}(z))$ is also analytic. Now, let $z \in \varphi_t(V)$, there exists $\mu \in V$ such that $z = \cosh(t\sqrt{\mu})$. Therefore we have

$$\begin{aligned}
(z - C(t))g(z) &= (\mu - A)S_{\sqrt{\mu}}(t)f(\varphi_t^{-1}(z))\\
&= (\mu - A)S_{\sqrt{\mu}}(t)f(\mu)\\
&= S_{\sqrt{\mu}}(t)(\mu - A)f(\mu) = 0.
\end{aligned}$$

Since g has the SVEP at $\cosh(\lambda_0 t)$, Then g is identically zero, so f is identically zero on V. As f is analytic function, then f is identically zero on $D_{\lambda_0^2}$. which implies A has the SVEP at λ_0^2. So the proof is complete. □

Corollary 2.2. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators with infinitesimal generator A. The following assertions are equivalent:*

1. *A has the SVEP.*
2. *For all $t \geq 0$, $C(t)$ has the SVEP.*
3. *There exists $t_0 \geq 0$, $C(t_0)$ has the SVEP.*

Proof. (1) ⇒ (2): If A has the SVEP, then $\mathcal{S}(A) = \emptyset$. By theorem 2.1, we have $\mathcal{S}(C(t)) \subseteq \{-1, 1\}$ for all $t \in \mathbb{R}$. Since $\mathcal{S}(C(t))$ is open, then $\mathcal{S}(C(t)) = \emptyset$. Which implies that $C(t)$ has the SVEP for all $t \in \mathbb{R}$.
(2) ⇒ (3): Obvious.
(3) ⇒ (1): If $C(t_0)$ has the SVEP for some $t_0 \in \mathbb{R}$, then $\mathcal{S}(C(t_0)) = \emptyset$. By theorem 2.1, we have $\cosh(t_0\sqrt{\mathcal{S}(A)}) \subseteq \{-1, 1\}$. Since $\mathcal{S}(A)$ is open and the function $z \mapsto \cosh(t\sqrt{z})$ is open, then $\cosh(t_0\sqrt{\mathcal{S}(A)})$ is open. Which implies that $\mathcal{S}(A) = \emptyset$. Therefore A has the SVEP. □

Example 2.3. *Let X be the complex ℓ^2 space, and for $(z_n)_{n\in\mathbb{N}} \in \ell^2, s \in \mathbb{R}$, we put $C(s)(z_n)_n = (\cos(ns)z_n)_n$. Then $A(z_n)_n = (-n^2 z_n)_n$ with $\mathcal{D}(A) = \{(z_n)_n \in \ell^2 : \sum_{n=1}^{\infty} n^4|z_n|^2 < \infty\}$ and $\sigma(A) = \{-n^2 : n \in \mathbb{N}^*\}$. So A has the SVEP. According to corollary 2.2, $C(t)$ has the SVEP for all $t \geq 0$.*

Example 2.4. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators with infinitesimal generator A. If $C(.)$ is is 2π-periodic, then $C(2\pi)$ has the SVEP. From corollary 2.2, A and $C(t)$ are the SVEP for all $t \geq 0$.*

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