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# Fractional differential equations with maxima on time scale via Picard operators

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**Abstract.** In this paper, we prove a result of existence and uniqueness of solutions for the following class of problem of initial value for differential equations with maxima and Caputo's fractional order on the time scales:

$$ {}^{c}\Delta_a^{\omega} u(\vartheta) = \zeta(\vartheta, u(\vartheta), \max_{\varsigma\in[a,\vartheta]} u(\varsigma)), \quad \vartheta \in \mathcal{J} := [a,b]_{\mathbb{T}},\ 0 < \omega \leq 1, $$
$$ u(a) = \phi, $$

We used the techniques of the Picard and weakly Picard operators to obtain some data dependency on the parameters results.

## 1. Introduction and Preliminaries

During the last decades, the theory of differential equations on time scales has developed very intensively (see for example [2, 3, 7–10, 17–20, 23] and the references therein). Indeed, it was in 1988 that Hilger [13, 24] introduced the concept of "calculation of chains of measures" in order to unify the discrete and continuous analysis. On the other hand, the differential equations of fractional order has become very important in recent years due to their applications in various fields, in : physiology, rheology, control, viscoelasticity, electrochemistry, electromagnetism, etc. For moor details, see [4–6, 11, 15, 16, 37] and the references therein. Many authors have considered fractional differential equations (FDE) with maxima (see [1, 14, 21, 25–29]). In [27], Otrocol discussed the following system:

$$ \begin{cases} u'(t) = f(t, u(t)) + g(t, \max_{\varsigma\in[0,t]} u(\varsigma)), \\ u(0) = \phi, \end{cases} $$

where $t \in [0, b]$, $b \in \mathbb{R}^p$, $\phi \in \mathbb{R}^p$, and $f, g \in [0, b] \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$.

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In this article, a result of existence and uniqueness has been established through Banach's principle of contraction. In addition, we used the Picard and weakly Picard operator [27, 29–36]) techniques to obtain data dependency results on the parameters. We considered the following FDE on time scale with maxima

$$ {}^{c}\Delta_{a}^{\omega}u(\vartheta) = \zeta(\vartheta, u(\vartheta), U(\vartheta)), \ \ \vartheta \in \mathcal{J} := [a,b]_{\mathbf{T}} = [a,b] \cap \mathbf{T}, \ 0 < \omega \le 1, \tag{1} $$

$$ u(a) = \phi, \tag{2} $$

where $U(\vartheta) = \max\limits_{\varsigma\in[a,\vartheta]} u(\varsigma)$, $b > a$, ${}^{c}\Delta_{a}^{\omega}$ is the Caputo fractional derivative operator or order $\omega$ defined on $\mathbf{T}$, $\zeta : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\phi$ is a real constant.

The expression $C(\mathcal{J}, \mathbb{R})$ denotes Banach space of continuous functions $y$ with the norm $\|y\|_\infty = \sup\{|y(\vartheta)| : \vartheta \in \mathcal{J}\}$, where $\mathcal{J}$ is a bounded interval. A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$ (see [19, 20]).

**Definition 1.1.** *A function $h : \mathbf{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbf{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbf{T}$. $C_{rd}$ denote the set of rd-continuous functions $h : \mathbf{T} \to \mathbb{R}$.*

**Definition 1.2.** *A function $H$ from a closed bounded interval of $\mathbf{T}$ to $\mathbb{R}$ is called a delta antiderivative of $h : [\kappa, \mu) \to \mathbb{R}$ provided $F$ is continuous on $[\kappa, \mu]$, delta differentiable on $[\kappa, \mu)$, and $H^{\Delta}(\vartheta) = f(\vartheta)$ for all $\vartheta \in [\kappa, \mu)$. Then, we define the $\Delta$-integral of $h$ from $a$ to $b$ by*

$$ \int_{\kappa}^{\mu} h(\vartheta)\Delta\vartheta := H(\mu) - H(\kappa). $$

**Lemma 1.3.** *[12] Suppose $\mathbf{T}$ is a time scale and $h$ is an increasing continuous function on the time-scale interval $[\kappa, \mu]$. If $H$ is the extension of $h$ to the real interval $[\kappa, \mu]$ given by*

$$ H(\vartheta) := \begin{cases} h(\vartheta) & \text{if } \vartheta \in \mathbf{T}, \\ h(s) & \text{if } \vartheta \in (s, \sigma(s)) \notin \mathbf{T}, \end{cases} $$

*then*

$$ \int_{\kappa}^{\mu} h(\vartheta)\Delta\vartheta \le \int_{\kappa}^{\mu} H(\vartheta)d\vartheta. $$

**Definition 1.4 (Fractional integral on time scales).** *Suppose $\mathbf{T}$ is a time scale, $[a, b] \subset \mathbf{T}$, and $\zeta$ is an integrable function on $[a, b]$. Let $0 < \omega < 1$. Then the fractional integral of order $\omega$ of $\zeta$ is defined by*

$$ {}^{\mathbf{T}}I_{a}^{\omega}\zeta(\vartheta) := \frac{1}{\Gamma(\omega)} \int_{a}^{\vartheta} (\vartheta - s)^{\omega-1}\zeta(s)\Delta s. $$

**Definition 1.5 (Caputo fractional derivative on time scales).** *Let $\mathbf{T}$ be a time scale, $\vartheta \in \mathbf{T}$, $0 < \omega < 1$, and $\zeta : \mathbf{T} \to \mathbb{R}$. The Caputo $\Delta-$fractional derivative of order $\omega$ of $\zeta$ is defined by*

$$ {}^{c}\Delta_{a^+}^{\omega}\zeta(\vartheta) := \frac{1}{\Gamma(n-\omega)} \int_{a}^{\vartheta} (\vartheta - s)^{n-\omega-1}\zeta^{\Delta^n}(s)\Delta s, \tag{3} $$

*where $n = [\omega] + 1$ and $[\omega]$ denotes the integer part of $\omega$.*

**Theorem 1.6.** *(semigroup property) Let $\omega, \varpi > 0$, and $\zeta$ is an integrable function on $[a, b]$. Then*

$$ {}^{\mathbf{T}}I_{a}^{\omega}\, {}^{\mathbf{T}}I_{a}^{\varpi}\zeta(\vartheta) = {}^{\mathbf{T}}I_{a}^{\omega+\varpi}\zeta(\vartheta). $$

Let $(\mathbb{X}, d)$ be a metric space and $\Re : \mathbb{X} \to \mathbb{X}$ be an operator. Let $Fix(\Re)$ the set of the fixed points of $\Re$, i.e.

$$Fix(\Re) := \{x \in \mathbb{X} : \ x = \Re(x)\}.$$

We denote by $\mathcal{P}(\Re)$ the family of all nonempty subsets of $\mathbb{X}$, i.e.

$$\mathcal{P}(\Re) := \{\Upsilon \subseteq \mathbb{X}, \Upsilon \neq \emptyset\},$$

and by $\mathcal{I}(\Re)$ the family of the nonempty invariant subsets of $\Re$, i.e.

$$\mathcal{I}(\Re) := \{\Upsilon \subset \mathbb{X}, \Re(y) \subset \Upsilon, \Upsilon \neq \emptyset\}.$$

We also denote $\Re^0 := 1_{\mathbb{X}}, \ \Re^1 := \Re, \dots, \Re^{n+1} := \Re \circ \Re^n; \ n \in \mathbb{N}$ the iterations of the operator $\Re$.

**Definition 1.7.** *(Picard operator: ([31, 33, 35])) The operator $\Re : \mathbb{X} \to \mathbb{X}$ is a Picard operator (**P.o.**) if there exists $p \in \mathbb{X}$ such that:*

*(i) $Fix(\Re) = \{p\}$;*
*(ii) The sequence $(\Re^n(x_0))_{n \in \Re}$ converges to $p$ for all $x_0 \in \mathbb{X}$.*

**Example 1.8.** *[30] Let $(\mathbb{X}, d)$ be a metric space and $\Phi, \Psi : \mathbb{X} \to \mathbb{X}$ such that*

$$d(\Phi(u), \Psi(v)) \leq \kappa[d(u, \Phi(u)) + d(v, \Psi(v))]$$

*for all $u, v \in \mathbb{X}$ and for some $\kappa \in (0, 1)$. Then $\Phi$ and $\Psi$ are Picard operators.*

**Definition 1.9.** *(Weakly Picard operator: Rus 1993) The operator $\Re : \mathbb{X} \to \mathbb{X}$ is a weakly Picard operator (**w.P.o.**) if the sequence $(\Re^n(x))_{n \in \mathbb{N}}$ converges for all $x \in \mathbb{X}$, and its limit (which may depend on $x$) is a fixed point of $\Re$.*

**Example 1.10.** *[30] Let $\mathbb{X} = C[0, 1]$, $d(u, v) = \|u - v\|_\infty$,*

$$\Upsilon(u)(t) = u(0) + \int_0^t K(t, s)u(s)ds, \ \ t \in [0, 1],$$

*where $K \in C([0, 1] \times [0, 1])$. Then $\Upsilon$ is **w.P.o.**.*

If $\Re$ is weakly Picard operator then we consider the operator $\Re^\infty$ defined by

$$\Re^\infty : \mathbb{X} \to \mathbb{X}; \ \Re^\infty(x) = \lim_{n \to \infty} \Re^n(x).$$

**Remark 1.11.** *It is clear that $\Re^\infty(\mathbb{X}) = Fix(\Re)$.*

**Definition 1.12.** *(c-weakly Picard operator) Let $\Re$ be a weakly Picard operator and $c > 0$. The operator $\Re$ is c-weakly Picard operator (**c-w.P.o.**) if*

$$d(x, \Re^\infty(x)) \leq c \ d(x, \Re(x)); \ x \in \mathbb{X}.$$

**Example 1.13.** *[33] Let $(\mathbb{X}, d)$ be a complete metric space. If $\Re : \mathbb{X} \to \mathbb{X}$ is an $\kappa$-contraction then the operator $\Re$ is (**c-w.P.o.**) with $c = (1 - \kappa)^{-1}$.*

**Lemma 1.14.** *[33, 34] Let $(\mathbb{X}, d, \leq)$ be an ordered metric space and $\Re : \mathbb{X} \to \mathbb{X}$ be an operator. If*

*(i) $\Re$ is monotone increasing,*
*(ii) $\Re$ is **w.P.o.**,*

*then $\Re^\infty$ is monotone increasing.*

**Theorem 1.15.** *(Rus 1993 [32]) Let $(\mathbb{X}, d, \leq)$ be an ordered metric space and $\mathfrak{N} : \mathbb{X} \to \mathbb{X}$ be an operator. The operator $\mathfrak{N}$ is **w.P.o.** (**c-w.P.o.**) if and only if there exists a partition of $\mathbb{X}$,*

$$\mathbb{X} = \bigcup_{\phi \in \Lambda} \mathbb{X}_\phi,$$

*where $\Lambda$ is the indices'set of partition, such that*

1) $\mathbb{X}_\phi \in \mathcal{I}(\mathfrak{N})$;
2) $\mathfrak{N} \mid \mathbb{X}_\phi : \mathbb{X}_\phi \to \mathbb{X}_\phi$ *is a Picard (c-Picard) operator, for all* $\phi \in \Lambda$.

**Lemma 1.16.** *(Abstract comparison principle [33, 34]). Let $(\mathbb{X}, d, \leq)$ be an ordered metric space and $\mathfrak{N}, \mathfrak{M}, \mathfrak{P} : \mathbb{X} \to \mathbb{X}$ operators. If*

(i) $\mathfrak{N} \leq \mathfrak{M} \leq \mathfrak{P}$,
(ii) $\mathfrak{N}, \mathfrak{M}, \mathfrak{P}$ *are **w.P.o.**s,*
(iii) $\mathfrak{M}$ *is monotone increasing,*

*then*

$$u \leq v \leq v \Rightarrow \mathfrak{N}^\infty(u) \leq \mathfrak{M}^\infty(v) \leq \mathfrak{P}^\infty(w).$$

**Theorem 1.17.** *(General data dependence theorem: Rus 2001 [36]) Let $(\mathbb{X}, d)$ be a complete metric space and $\mathfrak{M}, \mathfrak{N} : \mathbb{X} \to \mathbb{X}$ two operators. We suppose that such that*

(i) $\mathfrak{M}$ *is a* $\kappa$*-contraction;*
(ii) $Fix(\mathfrak{N}) \neq \emptyset$;
(iii) *there exists* $\nu > 0$ *such that* $d(\mathfrak{M}(x), \mathfrak{N}(x)) \leq \nu$, *for all* $x \in \mathbb{X}$.

*Then, if $Fix(\mathfrak{M}) = \{p_{\mathfrak{M}}\}$ and $p_{\mathfrak{N}} \in Fix(\mathfrak{N})$, we have*

$$d(p_{\mathfrak{M}}, p_{\mathfrak{N}}) \leq \frac{\nu}{1-\kappa}.$$

## 2. Existence of Solutions

**Definition 2.1.** *A function $u \in C^1_{rd}(\mathcal{J}, \mathbb{R})$ is said a solution of (1)-(2) if $u$ satisfies the condition $u(a) = \phi$, and the equations ${}^c\Delta^\omega_a u(\vartheta) = \zeta(\vartheta, u(\vartheta), \max_{\varsigma \in [a,\vartheta]} u(\varsigma))$ on $\mathcal{J}$.*

**Lemma 2.2.** *Let $0 < \omega < 1$, and $\zeta : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be rd-continuous. Function $u \in C^1_{rd}(\mathcal{J}, \mathbb{R})$ is a solution of problem (1)-(2) if and only if it is a solution of the following integral equation:*

$$u(\vartheta) = \phi + \frac{1}{\Gamma(\omega)} \int_a^\vartheta (\vartheta - s)^{\omega-1} \zeta(s, u(s), U(s)) \Delta s,$$

*where $U(\vartheta) = \max_{\varsigma \in [a,\vartheta]} u(\varsigma)$.*

**Proof.** We have ${}^{\mathbf{T}}I^\omega_a \circ ({}^c\Delta^\omega_a(u(\vartheta))) = u(\vartheta) - u(a)$. Then, from (3) we have

$$u(\vartheta) = \phi + \frac{1}{\Gamma(\omega)} \int_a^\vartheta (\vartheta - s)^{\omega-1} \zeta(s, u(s), U(s)) \Delta s,$$

where $U(\vartheta) = \max_{\varsigma \in [a,\vartheta]} u(\varsigma)$.

Set $C_{rd} = C_{rd}(\mathcal{J}, \mathbb{R})$.

**Theorem 2.3.** *Assume*

*($Ax_I$) The function $\zeta : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous.*

*($Ax_{II}$) There exists a constant $k_\zeta > 0$ such that*

$$|\zeta(\vartheta, \overline{u}, \overline{v}) - \zeta(\vartheta, \underline{u}, \underline{v})| \leq k_\zeta \; max(|\overline{u} - \underline{u}|, |\overline{v} - \underline{v}|)$$

*for all $\overline{u}$, $\underline{u}$, $\overline{v}$, $\underline{v} \in \mathbb{R}$, and $\vartheta \in \mathcal{J}$.*

*($Ax_{III}$)*

$$\pi := \frac{(b-a)^\omega}{\Gamma(1+\omega)} k_\zeta < 1. \tag{4}$$

*Then the problem* (1) – (2) *has unique a unique solution $\widehat{u}$ in $C_{rd}(\mathcal{J}, \mathbb{R})$.*

**Proof.** Let the Banach space $\Omega = (C_{rd}, \|\cdot\|_\infty)$ and consider the operator $\Phi : \Omega \to \Omega$, defined by

$$\Phi u(\vartheta) = \phi + \frac{1}{\Gamma(\omega)} \int_a^\vartheta (\vartheta - s)^{\omega-1} \zeta(s, u(s), U(s)) \Delta s, \tag{5}$$

where $U(\vartheta) = \max\limits_{\xi \in [a,\vartheta]} u(\xi)$.

We show that the operator $\Phi$ defined in (5) has a unique fixed point $\widehat{u}$ in $C_{rd}$. Let $u, w \in C_{rd}$, and $\vartheta \in \mathcal{J}$. Then

$$\Phi u(\vartheta) = \phi + \frac{1}{\Gamma(\omega)} \int_a^\vartheta (\vartheta - s)^{\omega-1} \zeta(s, u(s), U(s)) \Delta s,$$

and

$$\Phi w(\vartheta) = \phi + \frac{1}{\Gamma(\omega)} \int_a^\vartheta (\vartheta - s)^{\omega-1} \zeta(s, w(s), W(s)) \Delta s,$$

where $U(t) = \max\limits_{\xi \in [a,t]} u(\xi)$, and $W(t) = \max\limits_{\xi \in [a,t]} w(\xi)$.
By ($Ax_{\mathrm{II}}$) we get

$$\begin{aligned} & |\Phi u(\vartheta) - \Phi w(\vartheta)| \\ \leq \; & \frac{1}{\Gamma(\omega)} \int_a^\vartheta (\vartheta - s)^{\omega-1} \mid \zeta(s, u(s), U(s)) - \zeta(s, w(s), W(s)) \mid \Delta s \\ \leq \; & \frac{k_\zeta}{\Gamma(\omega)} \int_a^\vartheta (\vartheta - s)^{\omega-1} \max_{s \in [a,b]} (\mid u(s) - w(s) \mid, |U(s) - W(s)|) \, \Delta s. \end{aligned} \tag{6}$$

According to "max" property, (see [29])

$$\max_{s \in [a,b]} |\max_{\tau \in [a,s]} u(\tau) - \max_{\tau \in [a,s]} v(\tau)| \leq \max_{s \in [a,b]} |u(s) - v(s)|,$$

we obtain

$$\begin{aligned} & \max_{s \in [a,b]} (\mid u(s) - w(s) \mid, |U(s) - W(s)|) \\ = \; & \max_{s \in [a,b]} \left( \mid u(s) - w(s) \mid, |\max_{\tau \in [a,s]} u(\tau) - \max_{\tau \in [a,s]} w(\tau)| \right) \\ \leq \; & \max_{s \in [a,b]} |u(s) - w(s)|. \end{aligned}$$

So, by (6) and Lemma 1.3, we get

$$\begin{aligned}|\Phi u(\vartheta)-\Phi w(\vartheta)| &\leq \frac{k_\zeta}{\Gamma(\omega)}\max_{s\in[a,b]}|u(s)-w(s)|\int_a^\vartheta(\vartheta-s)^{\omega-1}\Delta s\\ &\leq \frac{k_\zeta\max\limits_{s\in[a,b]}|u(s)-w(s)|}{\Gamma(\omega)}\int_a^\vartheta(\vartheta-s)^{\omega-1}ds\\ &\leq \frac{(\vartheta-a)^\omega k_\zeta}{\Gamma(1+\omega)}\max_{s\in[a,b]}|u(s)-w(s)|\\ &\leq \frac{(b-a)^\omega k_\zeta}{\Gamma(1+\omega)}\max_{s\in[a,b]}|u(s)-w(s)|.\end{aligned}$$

Then

$$\|\Phi u-\Phi w\| \quad\leq\quad \frac{(b-a)^\omega k_\zeta}{\Gamma(1+\omega)}\|u-w\|.$$

By $(A_{\mathbf{III}})$, the operator $\Phi$ is a $\pi-$contraction. Hence, by Banach's contraction principle, $\Phi$ has a unique fixed point $\widehat{u}$ which is a unique solution of the problem (1)–(2). □

**Remark 2.4.** *It is clear that equation (1) is equivalent to the integral equation*

$$u(\vartheta)=u(a)+\frac{1}{\Gamma(\omega)}\int_a^\vartheta(\vartheta-s)^{\omega-1}\zeta(s,u(s),U(s))\Delta s,$$

*and problem (1)-(2) is equivalent to the integral equation*

$$u(\vartheta)=\phi+\frac{1}{\Gamma(\omega)}\int_a^\vartheta(\vartheta-s)^{\omega-1}\zeta(s,u(s),U(s))\Delta s,$$

*where* $U(t)=\max\limits_{\xi\in[a,t]}u(\xi)$, *and* $u\in C^1_{rd}$.

Define the operator $\Psi:\Omega\to\Omega$ by

$$\Psi(u)(\vartheta)=u(a)+\frac{1}{\Gamma(\omega)}\int_a^\vartheta(\vartheta-s)^{\omega-1}\zeta(s,u(s),\max_{\xi\in[a,s]}u(\xi))\Delta s.$$

Put

$$\Omega_\phi:=\{u\in C_{rd},u(a)=\phi\}.$$

Note that

$$\Omega=\bigcup_{\phi\in\mathbb{R}}\Omega_\phi,$$

is a partition of $\Omega$. We deduce the following auxiliary lemma

**Lemma 2.5.** *[29] If $(A_I)$ is satisfied, then for each $\phi\in\mathbb{R}$ :*

*(i)* $\Phi(\Omega)\subset\Omega_\phi$ *and* $\Phi(\Omega_\phi)\subset\Omega_\phi$;
*(ii)* $\Phi\mid(\Omega_\phi)=\Psi\mid(\Omega_\phi)$.

**Remark 2.6.** $\Phi$ *is (**P.o.**), and for all* $\phi\in\mathbb{R}$,

*(i)* $\Phi\mid(\Omega_\phi)=\Psi\mid(\Omega_\phi)$, *and*
*(ii)* $\Omega=C_{rd}=\bigcup\limits_{\phi\in\mathbb{R}}\Omega_\phi$, *and*
*(ii)* $\Omega_\phi\in\mathcal{I}(\Psi)$,

*we deduce that* $\Psi$ *is (**w.P.o.**). Moreover*

$$Fix(\Psi)\cap\Omega_\phi=\{\widehat{u}\},$$

*for all* $\phi\in\mathbb{R}$, *where* $\widehat{u}$ *is the unique solution of the problem (1)–(2).*

*2.1. Data dependance*

Let the operators $\Phi$ and $\Psi$ on the ordered Banach space $(C_{rd}, \|\cdot\|, \leq)$, and consider the following differential inequality

$$ {}^{c}\Delta_a^{\omega} v(\vartheta) \leq \zeta(\vartheta, v(\vartheta), V(\vartheta)), \quad \vartheta \in \mathcal{J}, \; 0 < \omega \leq 1, \tag{7}$$

where $V(\vartheta) = \max\limits_{\xi \in [a,\vartheta]} v(\xi)$.

**Lemma 2.7.** *Let $\zeta \in C_{rd}(\mathcal{J} \times \mathbb{R} \times \mathbb{R})$, and assume*

*(i) $(Ax_{\boldsymbol{I}}) - (Ax_{\boldsymbol{III}})$ hold;*

*(ii) $\zeta(\vartheta, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is increasing, for all $\vartheta \in \mathcal{J}$.*

*If $\widehat{u}$ is the unique solution of equation (1) and $\widehat{v}$ a solution of inequality (7), then*

$$\widehat{v}(a) \leq \widehat{u}(a) \Rightarrow v \leq u.$$

**Proof.** We have, in the terms of the operator $\Psi$ : $u = \Psi(u)$ and $v \leq \Psi(v)$, and $u(a) \leq v(a)$. By Remark 2.6 the operator $\Psi$ is (**w.P.o.**). Moreover, from Lemma 1.14 and condition (ii) $\Psi^{\infty}$ is increasing.
For $\phi \in \mathbb{R}$ we define the constant function

$$\widetilde{\phi} : \mathcal{J} \to \mathbb{R}, \;\; \widetilde{\phi}(\vartheta) = \phi, \;\; \text{for all } \; \vartheta \in \mathcal{J}.$$

So, we have

$$\widehat{v} \leq \Psi(\widehat{v}) \leq \Psi^2(\widehat{v}) \leq \cdots \leq \Psi^{\infty}(\widehat{v}).$$

But

$$\Psi^{\infty}(\widehat{v}) = \Psi^{\infty}(\widetilde{\widehat{v}(a)}) \leq \Psi^{\infty}(\widetilde{\widehat{u}(a)}) = \widehat{u}.$$

Thus

$$\widehat{v} \leq \widehat{u}.$$

□

We present the following monotonicity result.

**Theorem 2.8.** *Let $\zeta_k \in C_{rd}(\mathcal{J} \times \mathbb{R} \times \mathbb{R})$, where $k \in \{1, 2, 3\}$, and assume $(Ax_{\boldsymbol{I}}) - (Ax_{\boldsymbol{III}})$ hold, and*

*(i) $\zeta_1 \leq \zeta_2 \leq \zeta_3$;*

*(ii) $\zeta_2(\vartheta, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is increasing, for all $\vartheta \in \mathcal{J}$.*

*Let $\widehat{u_k} \in C^1_{rd}(\mathcal{J})$ be a solution of equation*

$${}^{c}\Delta_a^{\omega} u_k(\vartheta) = \zeta_k(\vartheta, u_k(\vartheta), \max_{\varsigma \in [a,\vartheta]} u_k(\varsigma)), \; k \in \{1, 2, 3\}, \; \vartheta \in \mathcal{J}, \; 0 < \omega \leq 1.$$

*Then we have*

$$\widehat{u_1}(a) \leq \widehat{u_2}(a) \leq \widehat{u_3}(a) \Rightarrow \widehat{u_1} \leq \widehat{u_2} \leq \widehat{u_3},$$

*i.e. the unique solution of problem (1)–(2) is increasing with respect to $\phi$, and $\zeta$.*

**Proof.** $\Psi_k$, $k \in \{1, 2, 3\}$ is **w.P.o.** (see proof. of Lemma 2.7). Moreover, from (ii) $\Psi_2$ is a monotone increasing operator. By (i) we deduce $\Psi_1 \leq \Psi_2 \leq \Psi_3$.
Let $\widetilde{u_k}(a) \in C_{rd}$ defined by $\widetilde{u_k}(a) = u_k(a)$ for $\vartheta \in \mathcal{J}$, and $k \in \{1, 2, 3\}$. Note that

$$\widetilde{u_1}(a) \leq \widetilde{u_2}(a) \leq \widetilde{u_3}(a), \;\; \text{for all } \; \vartheta \in \mathcal{J}.$$

By abstract comparison principle (Lemma 1.16), we deduce

$$\Psi_1^{\infty}(\widetilde{\widehat{u_1}}(a)) \leq \Psi_2^{\infty}(\widetilde{\widehat{u_2}}(a)) \leq \Psi_3^{\infty}(\widetilde{\widehat{u_3}}(a)).$$

And as $\Psi_k^{\infty}(\widetilde{\widehat{u_k}}(a)) = \widehat{u_k}$, where $k \in \{1, 2, 3\}$, we deduce

$$\widehat{u_1} \leq \widehat{u_2} \leq \widehat{u_3}.$$

□

In the following result, we prove the continuous dependence of the solution for problem (1)–(2).

**Theorem 2.9.** *Let $\phi_k \in \mathbb{R}$, and $\zeta_k \in C_{rd}(\mathcal{J} \times \mathbb{R} \times \mathbb{R})$, where $k \in \{1,2\}$, and assume $(Ax_I) - (Ax_{III})$ hold, and*

*(i)* $|\phi_1(\vartheta) - \phi_2(\vartheta)| \leq \aleph_1$, *for all* $\vartheta \in \mathcal{J}$;

*(ii)* $|\zeta_1(\vartheta, u_1, u_2) - \zeta_2(\vartheta, u_1, u_2)| \leq \aleph_2$ , *for all* $\vartheta \in \mathcal{J}$, $u_k \in \mathbb{R}$, $k \in \{1,2\}$.

*Then*

$$\|\widehat{u}(\vartheta;\phi_1,\zeta_1) - \widehat{u}(\vartheta;\phi_2,\zeta_2)\| \leq \frac{\aleph_1\Gamma(1+\omega) + \aleph_2(b-a)^{\omega}}{\Gamma(1+\omega) - K_{\zeta}(b-a)^{\omega}},$$

*where $\widehat{u}(t;\phi_k,\zeta_k)$, $k \in \{1,2\}$, are the solutions of problem (1)–(2) with respect to $\phi_k, \zeta_k$, $k \in \{1,2\}$, and $K_\zeta = max(k_{\zeta_1}, k_{\zeta_2})$.*

**Proof.** Let the operators $\Phi_{\phi_k,\zeta_k}$ where $k \in \{1,2\}$. By Theorem 2.3, $\Phi_{\phi_k,\zeta_k}$ are a $\pi-$contraction. Then, for all $u, v \in \mathbb{R}$ we deduce

$$\begin{aligned} \|\Phi_{\phi_1,\zeta_1}(u) - \Phi_{\phi_1,\zeta_1}(v)\| &\leq \pi\|u-v\| \\ &= \frac{(b-a)^{\omega}k_{\zeta}}{\Gamma(1+\omega)}\|u-v\|. \end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned} &|\Phi_{\phi_1,\zeta_1}(u) - \Phi_{\phi_2,\zeta_2}(u)| \\ &\quad\leq |\phi_1 - \phi_2| \\ &\quad+ \int_a^{\vartheta} \frac{(\vartheta-s)^{\omega-1}}{\Gamma(\omega)} |\zeta_1(s,u(s),U(s)) - \zeta_2(s,u(s),U(s))|\,\Delta s, \end{aligned}$$

where $U(t) = \max\limits_{\xi\in[a,t]} u(\xi)$.

Then by Lemma 1.3 we get

$$\begin{aligned} &|\Phi_{\phi_1,\zeta_1}(u) - \Phi_{\phi_2,f_2}(u)| \\ &\quad\leq |\phi_1 - \phi_2| \\ &\quad+ \int_a^{\vartheta} \frac{(\vartheta-s)^{\omega-1}}{\Gamma(\omega)} |\zeta_1(s,u(s),U(s)) - \zeta_2(s,u(s),U(s))|\,ds \\ &\quad\leq \aleph_1 + \frac{\aleph_2(b-a)^{\omega}}{\Gamma(1+\omega)}, \end{aligned} \tag{9}$$

where $U(t) = \max\limits_{\xi\in[a,t]} u(\xi)$. Then

$$\begin{aligned} \|\widehat{u}_{(\vartheta,\phi_1,f_1)} - \widehat{u}_{(\vartheta,\phi_2,f_2)}\| &= \|\Phi_{\phi_1,\zeta_1}(\widehat{u}_{(\vartheta,\phi_1,\zeta_1)}) - \Phi_{\phi_2,\zeta_2}(\widehat{u}_{(\vartheta,\phi_2,\zeta_2)})\| \\ &\leq \|\Phi_{\phi_1,\zeta_1}(\widehat{u}_{(\vartheta,\phi_1,\zeta_1)}) - \Phi_{\phi_1,\zeta_1}(\widehat{u}_{(\vartheta,\phi_2,\zeta_2)})\| \\ &+ \|\Phi_{\phi_1,\zeta_1}(\widehat{u}_{(\vartheta,\phi_2,\zeta_2)}) - \Phi_{\phi_2,\zeta_2}(\widehat{u}_{(\vartheta,\phi_2,\zeta_2)})\|. \end{aligned}$$

Thus, by (8) and (9) we get

$$\|\widehat{u}_{(\vartheta,\phi_1,\zeta_1)} - \widehat{u}_{(\vartheta,\phi_2,\zeta_2)}\| \leq \pi\|\widehat{u}_{(\vartheta,\phi_1,\zeta_1)}) - \widehat{u}_{(\vartheta,\phi_2,\zeta_2)})\| + \aleph_1 + \frac{\aleph_2(b-a)^{\omega}}{\Gamma(1+\omega)}.$$

Put, in Theorem 1.17

$$\mathfrak{M} := \Phi_{\phi_1,\zeta_1},\ \mathfrak{N} := \Phi_{\phi_2,\zeta_2},$$

and

$$\nu := \aleph_1 + \frac{\aleph_2(b-a)^{\omega}}{\Gamma(1+\omega)},\ \ \kappa := \frac{(b-a)^{\omega}}{\Gamma(1+\omega)}K_{\zeta},$$

where $K_\zeta = max(k_{\zeta_1}, k_{\zeta_2})$, we get

$$\begin{aligned}\|\widehat{u}(t;\phi_1,\zeta_1) - \widehat{u}(t;\phi_2,\zeta_2)\| &\leq \frac{\nu}{1-\kappa} \\ &= \frac{\aleph_1\Gamma(1+\omega) + \aleph_2(b-a)^\omega}{\Gamma(1+\omega) - K_\zeta(b-a)^\omega}.\end{aligned}$$

□

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