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Landau-Bloch type theorems of certain subclasses of biharmonic mappings

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Abstract. In this paper, we first establish a Landau-Bloch type theorem for certain bounded and normalized biharmonic mappings $F(z) = |z|^2 g(z) + h(z)$, where $g(z)$ and $h(z)$ are harmonic in the unit disk with $|g(z)| \leq M_1, |h(z)| \leq M_2$. In particular, our result is sharp when $M_1 = M_2 = 1$. Then, we establish several new versions of Landau-Bloch type theorems for certain normalized biharmonic mappings with the coefficients condition in place of $|h(z)| \leq M_2$ or $|g(z)| \leq M_1$, and obtain several sharp results.

1. Introduction

Suppose D is a domain in the complex plane $\mathbb{C}$. For $z = x + iy \in D$, the formal derivatives of a complex-valued function $F(z) = u(z) + iv(z)$ are defined respectively by

$$F_z = \frac{1}{2}(F_x - iF_y), \quad F_{\overline{z}} = \frac{1}{2}(F_x + iF_y).$$

Define the Laplacian of F as follow:

$$\Delta F = 4F_{z\overline{z}} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}.$$

Then a two times continuously differentiable complex-valued function $F(z)$ is said to be a *harmonic function* in a domain $D \subseteq \mathbb{C}$ if $\Delta F(z) = 0$ for all $z \in D$.

Lewy's theorem [13] from 1936 states that a harmonic mapping $F(z)$ is locally univalent if and only if its Jacobian $J_F(z) = |F_z|^2 - |F_{\overline{z}}|^2 \neq 0$ for $z \in D$. If D is simply connected, $F(z)$ can be written as $F = h + \overline{g}$ with $F(0) = h(0)$, where g and h are analytic on D (for details see [10]). Thus,

$$J_F(z) = |h'(z)|^2 - |g'(z)|^2.$$

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Suppose that $F(z)$ is a four continuously differentiable complex-valued function in a domain $D \subseteq \mathbb{C}$. Then $F(z)$ is said to be a *biharmonic mapping* in a domain D if and only if F satisfies the biharmonic equation $\Delta(\Delta F)(z) = 0$ for all $z \in D$. In other words, $F(z)$ is biharmonic in a domain D if and only if ΔF is harmonic in the domain D.

It is well known [1] that a mapping $F(z)$ is biharmonic in a simply connected domain D if and only if $F(z)$ has the following representation:

$$F(z) = |z|^2 g(z) + h(z), \tag{1}$$

where $g(z)$ and $h(z)$ are complex-valued harmonic mappings in D.

For a continuously differentiable complex-valued function F, we define

$$\begin{aligned} \Lambda_F(z) &= \max_{0\le\theta\le 2\pi} |F_z(z) + e^{-2i\theta} F_{\bar{z}}(z)| = |F_z(z)| + |F_{\bar{z}}(z)|, \\ \lambda_F(z) &= \min_{0\le\theta\le 2\pi} |F_z(z) + e^{-2i\theta} F_{\bar{z}}(z)| = ||F_z(z)| - |F_{\bar{z}}(z)||, \end{aligned}$$

which are the maximum and the minimum dilations of the mapping F respectively.

The harmonic mappings are regarded as the generalization of analytic functions, and the biharmonic mappings are regarded as the generalization of harmonic mappings.

The classical Landau's theorem states that if f is an analytic function on the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{U}$, then f is univalent in the disk $\mathbb{U}_{\rho_0} = \{z \in \mathbb{C} : |z| < \rho_0\}$ with

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}}, \tag{2}$$

and $f(\mathbb{U}_{\rho_0})$ contains a disk $|w| < \sigma_0$ with $\sigma_0 = M\rho_0^2$. This result is sharp, with the extremal function $f_0(z) = Mz\frac{1-Mz}{M-z}$ (see [16]).

For bounded harmonic mappings in $\mathbb{U}$, Landau-Bloch type theorems had been obtained by Chen et al. [4, 5]. Liu improved the results of Landau-Bloch type theorems for bounded harmonic mappings, and obtained the sharp result when $M = 1$ (see [16]). Recently, Khalfallah, Mateljević and Mhamdi studied some properties of mappings admitting general Poisson representations, they proved a Landau-type theorem for T_α-harmonic functions in [14]. Liu *et al.* also proved the sharp result of Landau-Bloch type theorem for strongly-bounded harmonic mappings when $M > 1$ in [19], and obtained several new versions of Landau-Bloch type theorems of harmonic mappings. One of their results is the following result.

Theorem A ([19, Theorem 3.5]) Suppose that $M > 1$. Let $f(z)$ be a harmonic mapping in the unit disk $\mathbb{U}$ with $f(0) = \lambda_f(0) - 1 = 0$, and

$$f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z^n}$$

satisfying the following inequality

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) r^{n-1} \le \frac{(M^2-1)(2Mr - r^2)}{(M-r)^2}, \quad 0 \le r \le \rho_0 = M - \sqrt{M^2 - 1}. \tag{3}$$

Then $f(z)$ is univalent in the disk $\mathbb{U}_{\rho_0}$ and $f(\mathbb{U}_{\rho_0})$ contains a schlicht disk $\mathbb{U}_{\sigma_0}$, where $\rho_0 = \frac{1}{M+\sqrt{M^2-1}}$, $\sigma_0 = M\rho_0^2$. This result is sharp, with $f_0(z) = Mz\frac{1-Mz}{M-z}$ being an extremal mapping.

In 2008, Abdulhadi and Muhanna first obtained two versions of Landau-Bloch type theorems for biharmonic mappings (see [2]). From that on, many authors also considered the Landau-Bloch type theorems for certain biharmonic mappings (see [2, 3, 7, 15, 17, 18, 20]). In 2008, Liu established the following result by establishing the better coefficients estimates of bounded and normalized harmonic mappings (see [15]).

Theorem B ([15, Theorem 2.10]) Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping in the unit disk $\mathbb{U}$, with $F(0) = h(0) = \lambda_F(0) - 1 = 0$ and $|g(z)| \le M_1, |h(z)| \le M_2$ for $z \in \mathbb{U}$. Then, F is univalent in the disk $\mathbb{U}_{\rho_1}$, and $F(\mathbb{U}_{\rho_1})$ contains a schlicht disk $\mathbb{U}_{\sigma_1}$, where ρ_1 is the minimum positive root of the following equation :

$$1 - 2rM_1 - 2M_1 \cdot \frac{r^2}{(1-r)^2} - \sqrt{2M_2^2 - 2} \cdot \frac{2r - r^2}{(1-r)^2} = 0, \tag{4}$$

and

$$\sigma_1 = \rho_1 - \frac{2M_1\rho_1^3}{1-\rho_1} - \sqrt{2M_2^2 - 2}\frac{\rho_1^2}{1-\rho_1}. \tag{5}$$

Later, Zhu and Liu improved Theorem B by applying Schwarz's inequality as follows.

Theorem C ([25, Theorem 3.2]) Suppose that $F(z) = |z|^2 g(z) + h(z)$ is a biharmonic mapping of the unit disk $\mathbb{U}$ such that $|g(z)| \le M_1$ and $|h(z)| \le M_2$ for $z \in \mathbb{U}$ with $\lambda_F(0) = 1$.
(1) If $M_2 \ge 1$ and $M_1 > 0$, the F is univalent in the disk $\mathbb{U}_{\rho_2}$, and $F(\mathbb{U}_{\rho_2})$ contains a schlicht disk $\mathbb{U}_{\sigma_2}(F(0))$, where $\rho_2 = \rho_2(M_1, M_2)$ is the minimum positive root of the following equation:

$$1 - 2M_1 r - \frac{4M_1 r^2}{\pi(1-r^2)} - \sqrt{2(M_2^2 - 1)} \cdot \frac{r\sqrt{4 - 3r^2 + r^4}}{(1-r^2)^{\frac{3}{2}}} = 0, \tag{6}$$

and

$$\sigma_2 = \rho_2 - M_1\rho_2^2 - \sqrt{2(M_2^2 - 1)} \cdot \frac{\rho_2^2}{(1-\rho_2^2)^{\frac{1}{2}}}. \tag{7}$$

(2) If $M_2 = 1$ and $M_1 = 0$, then F is univalent in the $\mathbb{U}$ and $F(\mathbb{U}) = \mathbb{U}$.

In [17], Liu *et al.* established a Landau-Bloch type theorem of biharmonic mappings of the form $F(z) = |z|^2 g(z)$ as follows, which improved a corresponding result of Abdulhadi and Muhanna in [2].

Theorem D ([17, Theorem 2.10]) Let $g(z)$ be harmonic in the unit disk $\mathbb{U}$, with $g(0) = \lambda_g(0) - 1 = 0$ and $|g(z)| \le M$ for $z \in \mathbb{U}$. Then, $F(z) = |z|^2 g(z)$ is univalent in the disk $\mathbb{U}_{\rho_3}$, and $F(\mathbb{U}_{\rho_3})$ contains a schlicht disk $\mathbb{U}_{\sigma_3}$, where

$$\rho_3 = \frac{1}{1 + 2K(M) + \sqrt{K(M) + 4K(M)^2}}, \quad K(M) = \min\{\sqrt{2M^2 - 2}, \frac{4M}{\pi}\},$$

and

$$\sigma_3 = \begin{cases} \rho_3^3 - K(M)\frac{\rho_3^4}{1-\rho_3}, & \text{if } M > 1, \\ 1, & \text{if } M = 1, \end{cases}$$

above result is sharp when $M = 1$. In this paper, we continue to investigate the Landau-Bloch type theorems of biharmonic mappings.

This paper is organized as follows. In Sect. 2, we should recall several lemmas, and establish four new lemmas, which play a key role in the proofs of our main results. In Sect. 3, by establishing Theorem 3.1, we first establish a new version of Landau-Bloch type theorem by adding a condition $\lambda_g(0) - 1 = 0$, and our result is sharp when $M_1 = M_2 = 1$. Then, by establishing Theorems 3.3 and 3.5, we establish two new versions of Landau-Bloch type theorems for biharmonic mappings with the coefficients condition (14), and obtain sharp results for $M_1 = 0, M_2 \ge 1$ or $M_1 = 1, M_2 \ge 1$ respectively. Finally, by establishing Theorem 3.6, we establish a new version of Landau-Bloch type theorem for biharmonic mappings $F(z) = |z|^2 g(z)$, with $g(z)$ being harmonic mapping and the Taylor expansion coefficients of $g(z)$ satisfying the condition (3), and obtain better result than that of Theorem D.

2. Preliminaries

In order to establish our main results, we need the following lemmas.

Lemma 2.1 ([9]) Suppose that $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping with $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$ being analytic in $\mathbb{U}$. If $|f(z)| \leq M$ for all $z \in \mathbb{U}$, then

$$\Lambda_f(z) \leq \frac{4M}{\pi(1-|z|^2)}.$$

Lemma 2.2 ([15, 25]) Suppose that $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$. If $\lambda_f(0) = 1$ and $|f(z)| \leq M$ for all $z \in \mathbb{U}$, then $M \geq 1$, and

$$|a_n| + |b_n| \leq \sqrt{2M^2 - 2}, \quad n = 2, 3, \cdots,$$
$$\Big(\sum_{n=2}^{\infty}(|a_n| + |b_n|)^2\Big)^{\frac{1}{2}} \leq \sqrt{2M^2 - 2}.$$

Lemma 2.3 ([8, 17]) Suppose that $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$. If $|f(z)| \leq M$ for all $z \in \mathbb{U}$, then $|a_0| \leq M$, and

$$|a_n| + |b_n| \leq \frac{4M}{\pi}, \quad n = 1, 2, 3, \cdots.$$

The result is sharp.

Lemma 2.4 ([11]) Let f be a harmonic mapping of the unit disk $\mathbb{U}$ with $f(0) = 0$ and $f(\mathbb{U}) \subset \mathbb{U}$. Then

$$|f(z)| \leq \frac{4}{\pi} \arctan |z| \leq \frac{4}{\pi}|z|, \ \text{ for } z \in \mathbb{U}.$$

In 1959, Heinz in his classical paper [11] proved the above result, which is called the Schwarz type Lemma of complex-valued harmonic functions with $f(0) = 0$. Later, Hethcote [12] removed the assumption $f(0) = 0$ and got the following sharp form

$$\left| f(z) - \frac{1-|z|^2}{1+|z|^2} f(0) \right| \leq \frac{4}{\pi} \arctan |z|,$$

where f is a complex-valued harmonic function from $\mathbb{U}$ into itself. The above inequality also was proved by Pavlović in [24, Theorem 3.6.1] independently. The related results also refer to [6, 21–23]. In particular, the sharp forms of the improvements of Hethcote's result are given in [22, 23] by M. Mateljević, M. Svetlik and A. Khalfallah.

Lemma 2.5 Suppose that $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $f_1(z) = \sum_{n=1}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$. If f satisfies $|f(z)| \leq M$ for all $z \in \mathbb{U}$ and $\lambda_f(0) = 1$, then $M \geq 1$, and

$$|a_1| + |b_1| \leq K_1(M) = \min\{\sqrt{2M^2 - 1}, 4M/\pi\}, \tag{8}$$

and $|a_n| + |b_n| \leq K_2(M)$ for $n = 2, 3, 4, \cdots$, where $K_2(M) = \min\{\sqrt{2M^2 - 2}, \frac{4M}{\pi}\}$. The inequality (8) is sharp for $M = 1$, with $f_0(z) = z$ being an extremal mapping.

Proof By Lemmas 2.2 and 2.3, we have $M \geq 1$ and $|a_n| + |b_n| \leq K_2(M)$ for $n = 2, 3, \cdots$. Now we prove that

$$|a_1| + |b_1| \leq \sqrt{2M^2 - 1}. \tag{9}$$

In fact, fix $r \in (0, 1)$ and set $z = re^{i\theta}, \theta \in [0, 2\pi]$. Then

$$f(re^{i\theta}) = \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{b_n} r^n e^{-in\theta}.$$

By Parseval's identity and the hypothesis of $|f(z)| \leq M$, we have

$$\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)r^{2n}=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2d\theta\leq M^2,$$

which implies that $(|a_1|^2+|b_1|^2)r^2 \leq M^2$.

Letting $r \to 1^-$, we have $|a_1|^2+|b_1|^2 \leq M^2$.

Since $\lambda_f(0) = ||a_1|-|b_1|| = 1$, we have $|a_1| = |b_1|+1$ or $|b_1| = |a_1|+1$. By the first equation, we have $2|b_1|^2+2|b_1|+1 \leq M^2$, and then

$$(|b_1|+\frac{1}{2})^2 \leq \frac{M^2-1}{2}+\frac{1}{4}=\frac{2M^2-1}{4}.$$

Hence $|a_1|+|b_1| = 2|b_1|+1 \leq \sqrt{2M^2-1}$.

By the second equation, we also have the same result. Thus, the inequality (9) holds. Hence the inequality (8) follows from (9) and Lemma 2.3, the proof is complete. □

Lemma 2.6 ([18]) Let $F_0(z) = a|z|^2z + b\overline{z}$ be a biharmonic mapping in the unit disk $\mathbb{U}$ with $|a| = |b| = 1$, then F_0 is univalent in the disk $\mathbb{U}_{\frac{\sqrt{3}}{3}}$, and $F_0(\mathbb{U}_{\frac{\sqrt{3}}{3}})$ contains a schlicht disk $\mathbb{U}_{\frac{2\sqrt{3}}{9}}$. This result is sharp.

Lemma 2.7 For $z_1, z_2 \in U_r, k \in \mathbb{N}_+ = \{1, 2, \cdots\}$, we have

$$||z_1|^2z_1^k-|z_2|^2z_2^k| \leq (k+2)r^{k+1}|z_1-z_2|.$$

Proof Since $z_1, z_2 \in U_r, k \in \mathbb{N}_+$, we have $|z_1| \leq r, |z_2| \leq r$, and

$$\begin{aligned} & ||z_1|^2z_1^k-|z_2|^2z_2^k| = ||z_1|^2z_1^k-|z_1|^2z_2^k+|z_1|^2z_2^k-|z_2|^2z_2^k| \\ \leq\ & |z_1|^2|z_1^k-z_2^k|+|z_2^k|||z_1|^2-|z_2|^2| \\ \leq\ & r^2|z_1-z_2||z_1^{k-1}+z_1^{k-2}z_2+\cdots+z_2^{k-1}|+r^k||z_1|-|z_2||(|z_1|+|z_2|) \\ \leq\ & kr^{k+1}|z_1-z_2|+2r^{k+1}|z_1-z_2| = (k+2)r^{k+1}|z_1-z_2|. \end{aligned}$$ □

Lemma 2.8 Suppose that $M_1 \geq 0$, $M_2 \geq 1$, and r_2 is the minimum positive root of the following equation

$$1-\frac{(M_2^2-1)(2M_2r-r^2)}{(M_2-r)^2}-\frac{4M_1}{\pi}\frac{3r^2-2r^4}{1-r^2}=0, \tag{10}$$

then $0 < r_2 \leq r_0 = \frac{1}{M_2+\sqrt{M_2^2-1}}$.

Proof Denote $f(r) = 1-\frac{(M_2^2-1)(2M_2r-r^2)}{(M_2-r)^2}$, $g(r) = \frac{4M_1}{\pi}\frac{3r^2-2r^4}{1-r^2}$. It is easy to verify that $f(r_0) = 0$.

We first prove that $f(r_0)-g(r_0) = -g(r_0) \leq 0$. In fact, since

$$g'(r)=\frac{4M_1}{\pi}\frac{2r(3-4r^2+2r^4)}{(1-r^2)^2}\geq 0$$

for $r \in (0,1)$, we obtain that $g(r)$ is increasing in $(0,1)$. Therefore, we have $g(r) \geq g(0) = 0$ for $r \in (0,1)$. Thus, $f(r_0)-g(r_0) = -g(r_0) \leq 0$.

Because $f(0)-g(0) = 1 > 0$, it follows from the intermediate value theorem that the minimum positive root r_2 of the equation (10) satisfies $0 < r_2 \leq r_0$. The proof is complete. □

Lemma 2.9 Suppose that $M_1 \geq 1$, $M_2 \geq 1$, $K_1(M) = \min\{\sqrt{2M^2-1}, 4M/\pi\}$, and r_3 is the minimum positive root of the following equation

$$M_2^2-\frac{M_2^2(M_2^2-1)}{(M_2-r)^2}-3K_1(M_1)r^2-\sqrt{2M_1^2-2}\cdot\frac{r^3\sqrt{16-23r^2+9r^4}}{(1-r^2)^{\frac{3}{2}}}=0, \tag{11}$$

then $0 < r_3 < r_0 = \frac{1}{M_2+\sqrt{M_2^2-1}}$.

Proof Denote $f(r) = M_2^2 - \frac{M_2^2(M_2^2-1)}{(M_2-r)^2}$,

$$g(r) = 3K_1(M_1)r^2 + \sqrt{2M_1^2-2}\cdot\frac{r^3\sqrt{16-23r^2+9r^4}}{(1-r^2)^{\frac{3}{2}}}.$$

It is easy to verify that $f(r_0) = 0$, and $g(r) > 0$ for $r \in (0,1)$. Thus we have

$$f(r_0) - g(r_0) = -g(r_0) < 0.$$

Because $f(0) - g(0) = 1 > 0$, it follows from the intermediate value theorem that the minimum positive root r_3 of the equation (11) satisfies $0 < r_3 < r_0$. This completes the proof. □

3. Main Results

We first establish a new version of Landau-Bloch type theorem for biharmonic mappings by adding a condition $\lambda_g(0) = 1$, and obtain a sharp result for $M_1 = M_2 = 1$.

Theorem 3.1 Let $F(z) = |z|^2g(z) + h(z)$ be a biharmonic mapping in the unit disk $\mathbb{U}$, where $g(z), h(z)$ are harmonic mappings in $\mathbb{U}$, and $g(0) = h(0) = 0, \lambda_F(0) = \lambda_g(0) = 1$, $|g(z)| \le M_1, |h(z)| \le M_2$ for $z \in \mathbb{U}$. Then $M_1, M_2 \ge 1$, and F is univalent in the disk $\mathbb{U}_{r_1}$, and F contains a schlicht disk $\mathbb{U}_{R_1}$, where $K_1(M_1) = \min\{\sqrt{2M_1^2-1}, \frac{4M_1}{\pi}\}$, r_1 is the minimum positive root of the following equation

$$1 - 3K_1(M_1)r^2 - \sqrt{2M_1^2-2}\cdot\frac{r^3\sqrt{16-23r^2+9r^4}}{(1-r^2)^{\frac{3}{2}}} - \sqrt{2M_2^2-2}\cdot\frac{r\sqrt{4-3r^2+r^4}}{(1-r^2)^{\frac{3}{2}}} = 0, \tag{12}$$

and

$$R_1 = r_1 - \sqrt{2M_2^2-2}\cdot\frac{r_1^2}{\sqrt{1-r_1^2}} - K_1(M_1)r_1^3 - \sqrt{2M_1^2-2}\cdot\frac{r_1^4}{\sqrt{1-r_1^2}}. \tag{13}$$

When $M_1 = M_2 = 1$, the radii $r_1 = \frac{\sqrt{3}}{3}$ and $R_1 = \frac{2\sqrt{3}}{9}$ are sharp.

Proof By Lemma 2.2, we see that $M_1 \ge 1, M_2 \ge 1$.
Let $g(z) = g_1(z) + \overline{g_2(z)}, h(z) = h_1(z) + \overline{h_2(z)}$ with

$$g_1(z) = \sum_{n=1}^{\infty} a_n z^n, g_2(z) = \sum_{n=1}^{\infty} b_n z^n, h_1(z) = \sum_{n=1}^{\infty} c_n z^n, h_2(z) = \sum_{n=1}^{\infty} d_n z^n,$$

where g_1, g_2, h_1 and h_2 are analytic in $\mathbb{U}$. Then, by the hypothesis of Theorem 3.1, we have

$$||c_1| - |d_1|| = \lambda_h(0) = \lambda_F(0) = 1.$$

By Lemmas 2.2 and 2.5, we have $|a_1| + |b_1| \le K_1(M_1)$, and

$$\Big(\sum_{n=2}^{\infty}(|a_n| + |b_n|)^2\Big)^{\frac{1}{2}} \le \sqrt{2M_1^2-2}, \quad \Big(\sum_{n=2}^{\infty}(|c_n| + |d_n|)^2\Big)^{\frac{1}{2}} \le \sqrt{2M_2^2-2}.$$

To prove F is univalent in $\mathbb{U}_{r_1}$, we choose two distinct points z_1, z_2 in $\mathbb{U}_r$ ($r < r_1$). By Lemma 2.7, we have

$$\begin{aligned}
|h(z_1) - h(z_2)| &\geq |c_1(z_1 - z_2) + \overline{d_1}(\overline{z_1} - \overline{z_2})| - \Big|\sum_{n=2}^{\infty} c_n(z_1^n - z_2^n) + \overline{d_n}(\overline{z_1^n} - \overline{z_2^n})\Big| \\
&\geq |z_1 - z_2|\Big[||c_1| - |d_1|| - \sum_{n=2}^{\infty}(|c_n| + |d_n|)nr^{n-1}\Big] \\
&\geq |z_1 - z_2|\Big[1 - \Big(\sum_{n=2}^{\infty}(|c_n| + |d_n|)^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{n=2}^{\infty} n^2 r^{2n-2}\Big)^{\frac{1}{2}}\Big] \\
&\geq |z_1 - z_2|\Big[1 - \sqrt{2M_2^2 - 2} \cdot \frac{r\sqrt{4 - 3r^2 + r^4}}{(1 - r^2)^{\frac{3}{2}}}\Big],
\end{aligned}$$

and

$$\begin{aligned}
||z_1|^2 g(z_1) - |z_2|^2 g(z_2)| &\leq \sum_{n=1}^{\infty}(|a_n| + |b_n|)\Big||z_1|^2 z_1^n - |z_2|^2 z_2^n\Big| \\
\leq\ & |z_1 - z_2|\Big[3(|a_1| + |b_1|)r^2 + \sum_{n=2}^{\infty}(|a_n| + |b_n|)(n + 2)r^{n+1}\Big] \\
\leq\ & |z_1 - z_2|\Big[3(|a_1| + |b_1|)r^2 + \Big(\sum_{n=2}^{\infty}(|a_n| + |b_n|)^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{n=2}^{\infty}(n + 2)^2 r^{2n+2}\Big)^{\frac{1}{2}}\Big] \\
\leq\ & |z_1 - z_2|\Big[3K_1(M_1)r^2 + \sqrt{2M_1^2 - 2} \cdot \frac{r^3\sqrt{16 - 23r^2 + 9r^4}}{\sqrt{(1 - r^2)^3}}\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
&|F(z_1) - F(z_2)| \geq |z_1 - z_2|\Big[1 - 3K_1(M_1)r^2 \\
&- \sqrt{2M_1^2 - 2} \cdot \frac{r^3\sqrt{16 - 23r^2 + 9r^4}}{(1 - r^2)^{\frac{3}{2}}} - \sqrt{2M_2^2 - 2} \cdot \frac{r\sqrt{4 - 3r^2 + r^4}}{(1 - r^2)^{\frac{3}{2}}}\Big] > 0.
\end{aligned}$$

This implies $F(z_1) \neq F(z_2)$, which shows that F is univalent in the disk $\mathbb{U}_{r_1}$.

Next, note that $F(0) = 0$, for each $z = r_1 e^{i\theta} \in \partial\mathbb{U}_{r_1}$, we have

$$\begin{aligned}
|F(z)| &\geq |c_1 z + \overline{d_1}\overline{z}| - \Big|\sum_{n=2}^{\infty}(c_n z^n + \overline{d_n}\overline{z}^n)\Big| - r_1^2|a_1 z + \overline{b_1}\overline{z}| - r_1^2\Big|\sum_{n=2}^{\infty}(a_n z^n + \overline{b_n}\overline{z}^n)\Big| \\
&\geq r_1\Big||c_1| - |d_1|\Big| - \sum_{n=2}^{\infty}(|c_n| + |d_n|)r_1^n - r_1^3(|a_1| + |b_1|) - r_1^2\sum_{n=2}^{\infty}(|a_n| + |b_n|)r_1^n \\
&\geq r_1\Big||c_1| - |d_1|\Big| - \Big(\sum_{n=2}^{\infty}(|c_n| + |d_n|)^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{n=2}^{\infty} r^{2n}\Big)^{\frac{1}{2}} - K_1(M_1)r_1^3 - r_1^2\Big(\sum_{n=2}^{\infty}(|a_n| + |b_n|)^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{n=2}^{\infty} r^{2n}\Big)^{\frac{1}{2}} \\
&\geq r_1 - \sqrt{2M_2^2 - 2} \cdot \frac{r_1^2}{\sqrt{1 - r_1^2}} - K_1(M_1)r_1^3 - \sqrt{2M_1^2 - 2} \cdot \frac{r_1^4}{\sqrt{1 - r_1^2}} = R_1.
\end{aligned}$$

Hence, $F(\mathbb{U}_{r_1}) \supset \mathbb{U}_{R_1}$.

Finally, we show that when $M_1 = M_2 = 1$, the radii $r_1 = \frac{\sqrt{3}}{3}, R_1 = \frac{2\sqrt{3}}{9}$ are sharp. In fact, by the hypothesis of Theorem 3.1 and Lemma 2.2, we get that

$$||a_1| - |b_1|| = ||c_1| - |d_1|| = 1, a_n = b_n = c_n = d_n = 0, n = 2, 3, \cdots.$$

By Lemma 2.5, we have $|a_1| + |b_1| = 1$. Thus, we have $|a_1| = 1, b_1 = 0, ||c_1| - |d_1|| = 1$, or $|a_1| = 0, |b_1| = 1, ||c_1| - |d_1|| = 1$.

Set $F(z) = a_1|z|^2z + \overline{d_1 z}$, with $|a_1| = |d_1| = 1$, or $F(z) = b_1|z|^2 z + \overline{c_1 z}$, with $|b_1| = |c_1| = 1$. By Lemma 2.6, we obtain that the radii $r_1 = \frac{\sqrt{3}}{3}, R_1 = \frac{2\sqrt{3}}{9}$ are sharp. This completes the proof of Theorem 3.1. □

Remark 3.2 The equation (12) and (6) cannot be solved explicitly. The Computer Algebra System Mathematica has calculated the numerical solutions to equations (12) and (6). Table 1 shows the approximate values of r_1, ρ_2 and R_1, σ_2 that correspond to different choice of the constants M_1 and M_2, which shows that $r_1 > \rho_2$ and $R_1 > \sigma_2$. That is, the result of Theorem 3.1 is better than that of Theorem C.

Table 1: The values of r_1, R_1 and ρ_2, σ_2 are in Theorem 3.1 and Theorem C

(M_1, M_2)	$(1,1)$	$(1.1, 1.3)$	$(1.5, 1.6)$	$(1.8, 2.3)$	$(2.5, 3.2)$
r_1	0.577350	0.268888	0.197637	0.140339	0.101949
ρ_2	0.387510	0.201729	0.144906	0.102401	0.072053
R_1	0.384900	0.154023	0.110351	0.074909	0.053310
σ_2	0.237346	0.108156	0.075924	0.052649	0.036698

Next, we establish two new versions of Landau-Bloch type theorems by changing the condition $|h(z)| \le M_2$ to the coefficients condition (14), and obtain some sharp results.

Theorem 3.3 Suppose that $M_1 \ge 0, M_2 \ge 1$. Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping in the unit disk $\mathbb{U}$, where $g(z)$ and $h(z) = \sum_{n=1}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \overline{d_n z^n}$ are harmonic in $\mathbb{U}$, and $\lambda_F(0) - 1 = 0, |g(z)| \le M_1$ in $\mathbb{U}$, and

$$\sum_{n=2}^{\infty} (|c_n| + |d_n|) n r^{n-1} \le \frac{(M_2^2 - 1)(2M_2 r - r^2)}{(M_2 - r)^2}, \quad 0 \le r < r_0 = \frac{1}{M_2 + \sqrt{M_2^2 - 1}}. \tag{14}$$

Then F is univalent in the disk $\mathbb{U}_{r_2}$ and $F(\mathbb{U}_{r_2})$ contains a schlicht disk $\mathbb{U}_{R_2}$, where r_2 is the minimum positive root in $(0, 1)$ of the following equation

$$1 - \frac{(M_2^2 - 1)(2M_2 r - r^2)}{(M_2 - r)^2} - \frac{4M_1}{\pi} \frac{3r^2 - 2r^4}{1 - r^2} = 0, \tag{15}$$

and

$$R_2 = r_2 - \frac{(M_2^2 - 1) r_2^2}{M_2 - r_2} - \frac{4M_1}{\pi} r_2^3. \tag{16}$$

When $M_1 = 0$ and $M_2 \ge 1$, the radii $r_2 = M_2 - \sqrt{M_2^2 - 1}$ and $R_2 = M_2 r_2^2$ are sharp.

Proof Since $|g(z)| \le M_1$ in $\mathbb{U}$, by Lemmas 2.1, 2.4 and 2.8 , we have

$$|g(z)| \le \frac{4}{\pi} M_1 |z|, \quad \Lambda_g(z) \le \frac{4M_1}{\pi(1 - |z|^2)}, \quad 0 < r_2 \le r_0 = \frac{1}{M_2 + \sqrt{M_2^2 - 1}}.$$

Since $\lambda_F(0) = 1$, we have $||c_1| - |d_1|| = \lambda_h(0) = \lambda_F(0) = 1$.

To prove F is univalent in the disk $\mathbb{U}_{r_2}$, we choose two different points $z_1, z_2 \in \mathbb{U}_r$ $(0 < r < r_2 \le r_0)$, we have

$$\begin{aligned}
||z_1|^2 g(z_1) - |z_2|^2 g(z_2)| &= \Big| \int_{[z_1,z_2]} (\overline{z} g(z) + |z|^2 g_z(z))dz + (zg(z) + |z|^2 g_{\overline{z}}(z))d\overline{z} \Big| \\
&\le \Big| \int_{[z_1,z_2]} g(z)(\overline{z}dz + zd\overline{z}) \Big| + \Big| \int_{[z_1,z_2]} |z|^2 (g_z(z)dz + g_{\overline{z}}(z)d\overline{z}) \Big| \\
&\le \int_{[z_1,z_2]} |g(z)|(|\overline{z}||dz| + |z||d\overline{z}|) + r^2 \int_{[z_1,z_2]} \Lambda_g(z)|dz| \\
&\le \Big[\frac{8M_1 r^2}{\pi} + \frac{4M_1 r^2}{\pi(1-r^2)}\Big]|z_1 - z_2| = \frac{4M_1(3r^2 - 2r^4)}{\pi(1-r^2)}|z_1 - z_2|,
\end{aligned}$$

Hence,

$$\begin{aligned}
|F(z_1) - F(z_2)| &\ge |z_1 - z_2|\Big[||c_1| - |d_1|| - \sum_{n=2}^{\infty}(|c_n| + |d_n|)nr^{n-1}\Big] - ||z_1|^2 g(z_1) - |z_2|^2 g(z_2)| \\
&\ge |z_1 - z_2|\Big[1 - \frac{(M_2^2 - 1)(2M_2 r - r^2)}{(M_2 - r)^2} - \frac{4M_1}{\pi}\frac{3r^2 - 2r^4}{1 - r^2}\Big] > 0,
\end{aligned}$$

this implies $F(z_1) \neq F(z_2)$, which shows that F is univalent in the disk $\mathbb{U}_{r_2}$.

Note that $F(0) = 0$, for any $z = r_2 e^{i\theta} \in \partial\mathbb{U}_{r_2}$, we have

$$\begin{aligned}
|F(z)| &\ge |c_1 z + d_1 \overline{z}| - r_2^2 |g(z)| - |\sum_{n=2}^{\infty}(c_n z^n + \overline{d_n} \overline{z}^n)| \\
&\ge r_2 - r_2^2 |g(z)| - \sum_{n=2}^{\infty}(|c_n| + |d_n|)r_2^n \\
&\ge r_2 - \frac{4M_1}{\pi} r_2^3 - \int_0^{r_2} \frac{(M_2^2 - 1)(2M_2 r - r^2)}{(M_2 - r)^2} dr \\
&= r_2 - \frac{4M_1}{\pi} r_2^3 - \frac{(M_2^2 - 1)r_2^2}{M_2 - r_2} = R_2.
\end{aligned}$$

Following the method of proof of [19, Theorem 3.5], we can easily obtain that when $M_1 = 0$ and $M_2 \ge 1$, the radii $r_2 = M_2 - \sqrt{M_2^2 - 1}$ and $R_2 = M_2 r_2^2$ are sharp. So, we omit the details. The proof is complete. □

In order to show the sharp result in Theorem 3.5, we recall an example as follows, which is a special form in [20, Example 3.6].

Example 3.4 *Let $F_0(z) = -|z|^2 z + M_2 z \frac{1 - M_2 z}{M_2 - z}$ be a biharmonic mapping of $\mathbb{U}$, where $M_2 \ge 1$. Then $F_0(z)$ is univalent in the disk $\mathbb{U}_{\gamma_0}$, where γ_0 is the unique positive root in $(0, 1)$ of the following equation*

$$M_2^2 - \frac{M_2^2(M_2^2 - 1)}{(M_2 - r)^2} - 3r^2 = 0, \tag{17}$$

and $F_0(\mathbb{U}_{\gamma_0})$ contains a schlicht disk $\mathbb{U}_{\tau_0}$, with

$$\tau_0 = M_2 \gamma_0 \frac{1 - M_2 \gamma_0}{M_2 - \gamma_0} - \gamma_0^3. \tag{18}$$

Both of γ_0 and τ_0 are sharp.

Theorem 3.5 Suppose that $M_1 \ge 0, M_2 \ge 1$. Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping in the unit disk $\mathbb{U}$, where $g(z)$ and $h(z) = \sum_{n=1}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \overline{d_n z^n}$ are harmonic in $\mathbb{U}$ with $\lambda_F(0) = \lambda_g(0) = 1$, $|g(z)| \le M_1$ in

$\mathbb{U}$, and c_n, d_n satisfying the inequality (14). Then $M_1 \geq 1$, F is univalent in the disk $\mathbb{U}_{r_3}$ and $F(\mathbb{U}_{r_3})$ contains a schlicht disk $\mathbb{U}_{R_3}$, where r_3 is the minimum positive root in $(0, 1)$ of the equation

$$M_2^2 - \frac{M_2^2(M_2^2-1)}{(M_2-r)^2} - 3K_1(M_1)r^2 - \sqrt{2M_1^2-2} \cdot \frac{r^3\sqrt{16-23r^2+9r^4}}{(1-r^2)^{\frac{3}{2}}} = 0 \tag{19}$$

and

$$R_3 = M_2 r_3 \frac{1-M_2 r_3}{M_2 - r_3} - K_1(M_1)r_3^3 - \sqrt{2M_1^2-2} \cdot \frac{r_3^4}{\sqrt{1-r_3^2}}. \tag{20}$$

When $M_1 = 1$ and $M_2 \geq 1$, the radii $r_3 = \gamma_0, R_3 = \tau_0$ are sharp, with $F_0(z)$ given in Example 3.4 being the extremal mapping.

Proof Since $\lambda_g(0) = 1$, $|g(z)| \leq M_1$ in $\mathbb{U}$, it follows from Lemma 2.2 that $M_1 \geq 1$.
By the hypothesis of Theorem 3.5 and Lemma 2.9, we have

$$||c_1| - |d_1|| = \lambda_h(0) = \lambda_F(0) = 1, \quad 0 < r_3 < r_0 = \frac{1}{M_2 + \sqrt{M_2^2 - 1}}.$$

Since $g(z)$ is harmonic in $\mathbb{U}$, we have that $g(z) = g_1(z) + \overline{g_2(z)}$ with

$$g_1(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g_2(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $\mathbb{U}$. Then, it follows from Lemmas 2.2 and 2.5 that $|a_1| + |b_1| \leq K_1(M_1)$, and

$$\Big(\sum_{n=2}^{\infty}(|a_n| + |b_n|)^2\Big)^{\frac{1}{2}} \leq \sqrt{2M_1^2 - 2}.$$

To prove F is univalent in $\mathbb{U}_{r_3}$, we choose two distinct points z_1, z_2 in $\mathbb{U}_r$ $(0 < r < r_3 \leq r_0)$. Then, we have

$$\begin{aligned} |h(z_1) - h(z_2)| &\geq |z_1 - z_2|[||c_1| - |d_1|| - \sum_{n=2}^{\infty}(|c_n| + |d_n|)nr^{n-1}] \\ &\geq |z_1 - z_2|\Big[1 - \frac{(M_2^2-1)(2M_2 r - r^2)}{(M_2-r)^2}\Big] = |z_1 - z_2|\Big[M_2^2 - \frac{M_2^2(M_2^2-1)}{(M_2-r)^2}\Big]. \end{aligned}$$

Since

$$\begin{aligned} ||z_1|^2 g(z_1) - |z_2|^2 g(z_2)| &= \Big|\sum_{n=1}^{\infty}[a_n(|z_1|^2 z_1^n - |z_2|^2 z_2^n) + \overline{b_n}(|z_1|^2\overline{z_1^n} - |z_2|^2\overline{z_2^n})]\Big| \\ &\leq \sum_{n=1}^{\infty}(|a_n| + |b_n|)\Big||z_1|^2 z_1^n - |z_2|^2 z_2^n\Big| \\ &\leq |z_1 - z_2|\Big[3(|a_1| + |b_1|)r^2 + \sum_{n=2}^{\infty}(|a_n| + |b_n|)(n+2)r^{n+1}\Big], \end{aligned}$$

we have,

$$||z_1|^2 g(z_1) - |z_2|^2 g(z_2)| \leq |z_1 - z_2|\Big[3K_1(M_1)r^2 + \sqrt{2M_1^2-2} \cdot \frac{r^3\sqrt{16-23r^2+9r^4}}{(1-r^2)^{\frac{3}{2}}}\Big].$$

Hence,

$$|F(z_1)-F(z_2)| \geq |z_1-z_2|\Big[M_2^2-\frac{M_2^2(M_2^2-1)}{(M_2-r)^2}-3K_1(M_1)r^2-\sqrt{2M_1^2-2}\cdot\frac{r^3\sqrt{16-23r^2+9r^4}}{(1-r^2)^{\frac{3}{2}}}\Big]>0.$$

This implies $F(z_1)\neq F(z_2)$.

Next, note that $F(0)=0$, for each $z=r_3e^{i\theta}\in\partial\mathbb{U}_{r_3}$, we have

$$\begin{aligned}
|F(z)| &= \Big||z|^2g(z)+h(z)\Big| = \Big|r_3^2\sum_{n=1}^{+\infty}(a_nz^n+\overline{b_n}\overline{z}^n)+\sum_{n=1}^{+\infty}(c_nz^n+\overline{d_n}\overline{z}^n)\Big| \\
&\geq |c_1z+\overline{d_1}\overline{z}|-\Big|\sum_{n=2}^{+\infty}(c_nz^n+\overline{d_n}\overline{z}^n)\Big|-r_3^2|a_1z+\overline{b_1}\overline{z}|-r_3^2\Big|\sum_{n=2}^{+\infty}(a_nz^n+\overline{b_n}\overline{z}^n)\Big| \\
&\geq r_3\Big||c_1|-|d_1|\Big|-\sum_{n=2}^{+\infty}(|c_n|+|d_n|)r_3^n-r_3^3(|a_1|+|b_1|)-r_3^2\sum_{n=2}^{+\infty}(|a_n|+|b_n|)r_3^n \\
&\geq M_2r_3\frac{1-M_2r_3}{M_2-r_3}-K_1(M_1)r_3^3-\sqrt{2M_1^2-2}\cdot\frac{r_3^4}{\sqrt{1-r_3^2}}=R_3.
\end{aligned}$$

Hence, $\mathbb{U}_{R_3}\subset F(\mathbb{U}_{r_3})$.

By applying Example 3.4, it is easy to prove that when $M_1=1, M_2\geq 1$, the radii $r_3=\gamma_0, R_3=\tau_0$ are sharp. The proof is complete. □

The Computer Algebra System Mathematica has calculated the numerical solutions to equations (12) and (19). From Table 2 as follow, it is easy to see that the result of Theorem 3.5 is better than that of Theorem 3.1.

Table 2: The values of r_3, R_3 and r_1, R_1 are in Theorems 3.5 and 3.1

(M_1,M_2)	$(1,1)$	$(1.1,1.3)$	$(1.5,1.6)$	$(1.8,2.3)$	$(2.5,3.2)$	$(3,3)$
r_3	0.577350	0.323005	0.237997	0.175912	0.131096	0.132967
r_1	0.577350	0.268888	0.197637	0.140339	0.101949	0.105393
R_3	0.384900	0.201709	0.142677	0.098878	0.071214	0.073392
R_1	0.384900	0.154023	0.110351	0.074909	0.053310	0.055745

Now, we establish a new version of Landau-Bloch type theorem for the biharmonic mapping of the form $F(z)=|z|^2g(z)$ with $g(z)$ satisfying a coefficients condition (21), which is different with that of Theorem 3.5, because $\lambda_F(0)=0$.

Theorem 3.6 Suppose $M\geq 1$. Let

$$g(z)=\sum_{n=1}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b_n}\overline{z}^n$$

be a harmonic mapping in the unit disk $\mathbb{U}$, with $\lambda_g(0)-1=0$ and a_n, $b_n (n=2,3,\cdots)$ satisfying the following inequality

$$\sum_{n=2}^{\infty}n(|a_n|+|b_n|)r^{n-1}\leq\frac{(M^2-1)(2Mr-r^2)}{(M-r)^2},\quad 0\leq r\leq\rho_0=M-\sqrt{M^2-1}. \tag{21}$$

Then, the biharmonic mapping $F(z)=|z|^2g(z)$ is univalent in $\mathbb{U}_{r_4}$, and $F(\mathbb{U}_{r_4})$ contains the schlicht disk $\mathbb{U}_{R_4}$, where r_4 is the minimum positive root of the following equation

$$(M-r)^2-(M^2-1)(4Mr-3r^2)=0, \tag{22}$$

and

$$R_4 = r_4^3 - r_4^4\frac{M^2-1}{M-r_4}.$$

This result is sharp when $M = 1$.

Proof By the hypothesis of Theorem 3.6, we see that $||a_1| - |b_1|| = \lambda_g(0) = 1$.

We first verify that $0 < r_4 \le \rho_0 = M - \sqrt{M^2-1}$. In fact, let

$$h(r) = (M-r)^2 - (M^2-1)(4Mr - 3r^2),$$

it is obvious that $h(r)$ is continuous in $[0,1]$, $h(0) = M > 0$, and

$$\begin{aligned} h(\rho_0) &= (M^2-1)\Big[1 - 4M(M-\sqrt{M^2-1}) + 3(M-\sqrt{M^2-1})^2\Big] \\ &= 2(M^2-1)\sqrt{M^2-1}(\sqrt{M^2-1} - M) \le 0, \end{aligned}$$

so that it follows from the intermediate value theorem that the minimum positive root r_4 of equation (22) satisfies $0 < r_4 \le \rho_0$.

Next, to prove that F is univalent in $\mathbb{U}_{r_4}$, we choose two distinct points $z_1, z_2 \in \mathbb{U}_r$ $(0 < r < r_4 \le \rho_0)$. Let $[z_1, z_2]$ denote by the line segment between z_1 and z_2, and let $z = (1-t)z_1 + tz_2$ $(t \in [0,1])$. Then, we have

$$\begin{aligned} ||z_1|^2 z_1 - |z_2|^2 z_2| &= \Big|\int_{[z_1,z_2]} 2z\overline{z}dz + z^2 d\overline{z}\Big| = \Big|\int_0^1 2|z|^2(z_2 - z_1)dt + \int_0^1 z^2(\overline{z_2} - \overline{z_1})dt\Big| \\ &\ge |z_1 - z_2|\int_0^1 |z|^2 dt, \end{aligned}$$

and

$$\begin{aligned} ||z_1|^2 z_1^n - |z_2|^2 z_2^n| &= \Big|\int_{[z_1,z_2]} (n+1)z^n\overline{z}dz + z^{n+1}d\overline{z}\Big| \\ &\le \int_{[z_1,z_2]} (n+1)|z|^2|z|^{n-1}|dz| + \int_{[z_1,z_2]} |z|^2|z|^{n-1}|d\overline{z}| \\ &\le (n+2)r^{n-1}\int_{[z_1,z_2]} |z|^2|dz| = (n+2)r^{n-1}|z_1 - z_2|\int_0^1 |z|^2 dt. \end{aligned}$$

Hence, it follows from the above two inequalities that

$$\begin{aligned} & |F(z_1) - F(z_2)| = ||z_1|^2 g(z_1) - |z_2|^2 g(z_2)| = \Big||z_1|^2\sum_{n=1}^{\infty}(a_n z_1^n + \overline{b_n z_1^n}) - |z_2|^2\sum_{n=1}^{\infty}(a_n z_2^n + \overline{b_n z_2^n})\Big| \\ \ge\ & |a_1(|z_1|^2 z_1 - |z_2|^2 z_2) + b_1(|z_1|^2\overline{z_1} - |z_2|^2\overline{z_2})| - \Big|\sum_{n=2}^{\infty}[a_n(|z_1|^2 z_1^n - |z_2|^2 z_2^n) + b_n(|z_1|^2\overline{z_1^n} - |z_2|^2\overline{z_2^n})]\Big| \\ \ge\ & ||a_1| - |b_1|| \cdot ||z_1|^2 z_1 - |z_2|^2 z_2| - \sum_{n=2}^{\infty}(|a_n| + |b_n|)||z_1|^2 z_1^n - |z_2|^2 z_2^n| \\ \ge\ & |z_1 - z_2|\int_0^1 |z|^2 dt[||a_1| - |b_1|| - \sum_{n=2}^{\infty}(n+2)(|a_n| + |b_n|)r^{n-1}] \\ \ge\ & |z_1 - z_2|\int_0^1 |z|^2 dt\Big[1 - \frac{2}{r}\int_0^r \frac{(M^2-1)(2Mt - t^2)}{(M-t)^2}dt - \frac{(M^2-1)(2Mr - r^2)}{(M-r)^2}\Big] \\ =\ & |z_1 - z_2|\int_0^1 |z|^2 dt\Big[1 - \frac{(M^2-1)(4Mr - 3r^2)}{(M-r)^2}\Big] \\ =\ & |z_1 - z_2|\int_0^1 |z|^2 dt \cdot \frac{(M-r)^2 - (M^2-1)(4Mr - 3r^2)}{(M-r)^2} > 0. \end{aligned}$$

Hence F is univalent in the disk $\mathbb{U}_{r_4}$.

Finally, for each $z = r_4e^{i\theta} \in \partial\mathbb{U}_{r_4}$, we have

$$\begin{aligned}|F(z)| &= r_4^2|g(z)| = r_4^2|\sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z}^n| \\ &\geq r_4^2\Big[||a_1| - |b_1||r_4 - \sum_{n=2}^{\infty}(|a_n| + |b_n|)r_4^n\Big] \\ &\geq r_4^2\Big[r_4 - \frac{(M^2-1)r_4^2}{M - r_4}\Big] = R_4.\end{aligned}$$

That is, $F(\mathbb{U}_{r_4}) \supseteq \mathbb{U}_{R_4}$.

When $M = 1$, it is obvious that $r_4 = R_4 = 1$ are sharp. This completes the proof. □

Table 3: The values of ρ_3, σ_3 and r_4, R_4 are in Theorems D and Theorem 3.6.

	$M = 1.2$	$M = 1.3$	$M = 1.5$	$M = 1.8$	$M = 2$
r_4	0.402414	0.325756	0.240438	0.176412	0.151000
ρ_3	0.204705	0.158362	0.107320	0.069879	0.055554
R_4	0.050699	0.026593	0.010583	0.004154	0.002599
σ_3	0.006507	0.003094	0.001001	0.000287	0.000147

Remark 3.7 By solving Equation (22), we have $r_4 = \frac{1}{2M - \frac{1}{M} + \sqrt{4M^2 - 7 + \frac{3}{M^2}}}$. Table 3 shows the approximate values of $r_4, \rho_3, R_4, \sigma_3$ that correspond to different choice of the constants M , which shows that $r_4 > \rho_3$ and $R_4 > \sigma_3$, that is, Theorem 3.6 is an improvement of Theorem D.

Finally, we give several examples of harmonic mappings satisfying the conditions (14) or (21). Note that for every integer $k \geq 2$ and $M \geq 1$, we have

$$\sum_{n=2}^{k} \frac{M^2-1}{M^{n-1}}\, n\, r^{n-1} \leq \sum_{n=2}^{\infty} \frac{M^2-1}{M^{n-1}}\, n\, r^{n-1} = \frac{(M^2-1)(2Mr - r^2)}{(M-r)^2}, \quad 0 \leq r < 1,$$

it is easy to verify the following facts.

Example 3.8 Suppose that $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| + |\beta| \leq 1$, $M \geq 1$ and $M_2 \geq 1$.

(1) Let $k \geq 2$ be an integer, and

$$g_0(z) = \alpha\frac{M^k z - M^{k+1}z^2 + (M^2-1)z^{k+1}}{M^{k-1}(M-z)} + (1+|\alpha|)\overline{z} = \alpha\Big(z - \sum_{n=2}^{k}\frac{M^2-1}{M^{n-1}}z^n\Big) + (1+|\alpha|)\overline{z}.$$

Then $g_0(z)$ is a harmonic mapping of $\mathbb{U}$ with $\lambda_{g_0}(0) = 1$, and it satisfies the inequality (21).

(2) Let

$$\begin{aligned}g_1(z) &= (1+|\beta|)z + \alpha\frac{(M^2-1)z^2}{M-z} + \beta M\overline{z}\frac{1-M\overline{z}}{M-\overline{z}} \\ &= (1+|\beta|)z + \alpha\sum_{n=2}^{\infty}\frac{M^2-1}{M^{n-1}}z^n + \overline{\beta\Big(z - \sum_{n=2}^{+\infty}\frac{M^2-1}{M^{n-1}}z^n\Big)}.\end{aligned}$$

Then $g_1(z)$ is a harmonic mapping of $\mathbb{U}$ with $\lambda_{g_1}(0) = 1$, and it satisfies the inequality (21).

(3) Let $h_0(z) = \alpha M_2 z \frac{1-M_2 z}{M_2 - z} + (1+|\alpha|)\overline{z}$. Then $h_0(z)$ is a harmonic mapping of $\mathbb{U}$ with $\lambda_{h_0}(0) = 1$, and it satisfies the inequality (14).

(4) Let $h_1(z) = \alpha M_2 z \frac{1-M_2 z}{M_2 - z} + (\gamma - \beta)\overline{z} + M_2 \beta\, \overline{z} \frac{1-M_2 \overline{z}}{M_2 - \overline{z}}$ with $||\alpha| - |\gamma|| = 1$. Then $h_1(z)$ is a harmonic mapping of $\mathbb{U}$ with $\lambda_{h_1}(0) = 1$, and it satisfies the inequality (14).

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