Filomat 37:2 (2023), 417–425
https://doi.org/10.2298/FIL2302417G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# A study on entire functions of hyper-order sharing a finite set with their high-order difference operators

**Hongxiao Guo[a], Feng Lü[a], Weiran Lü[a]**

[a] *College of Science, China University of Petroleum, Qingdao, Shandong, 266580, P.R. China*

**Abstract.** In this paper, due to the Borel lemma and Clunie lemma, we will deduce the relationship between an entire function $f$ of hyper-order less than 1 and its $n$-th difference operator $\Delta_c^n f(z)$ if they share a finite set and $f$ has a Borel exceptional value 0, where the set consists of two entire functions of smaller orders. Moreover, the exact form of $f$ is given and an example is provided to show the sharpness of the condition.

## 1. Introduction and main results

The paper is mainly devoted to studying the relationship between an entire function $f$ of hyper-order less than 1 and its $n$-th difference operator $\Delta_c^n f(z)$ when they share a finite set CM, which is inspired by the famous problem of unique range set (URSE) in Nevanlinna theory. A set is called unique range set (URSE) for a certain class of entire functions if each inverse image of the set uniquely determines a function from the given class. Let $f$ be an entire function, and let $S$ be a finite set such that all elements of $S$ are entire functions. Then, a set $E(f, S)$ is defined as

$$E(f, S) = \{(z, m) \in \mathbb{C} \times Z,\ f(z) - a(z) = 0 \text{ with multiplicity } m,\ a \in S\}.$$

Assume that $g$ is another entire function. We say that $f$ and $g$ share $S$ CM whenever $E(f, S) = E(g, S)$. A set $S$ is called URSE if two entire functions $f$, $g$ satisfy $E(f, S) = E(g, S)$, then $f = g$.

It was Gross and Yang [7] who gave the first example of a unique range set $\{z : z + e^z = 0\}$. It is seen that there exists infinitely many elements in the set. Later on, Gross in [6] posed the question: Does there exist a finite unique range set? In 1995, the problem was solved by Yi in [19]. Since then, the problem of unique range set has been studied in various settings, see e.g., [4, 5, 16].

As we all know, the Nevanlinna's theory is an important part of the theory of meromorphic functions. Recently, the difference analogues to Nevanlinna's theory was established by Halburd and Korhonen [10, 11], Chiang and Feng [1], independently, which have been a powerful theoretical tool to study the

2020 *Mathematics Subject Classification*. Primary 30D35; Secondary 30D20, 30D30.
*Keywords*. URSE; Entire function; Difference operator; Sharing set.
Received: 30 October 2020; Accepted: 17 April 2021
Communicated by Miodrag Mateljević
Corresponding author: Feng Lü
Research supported by Natural Science Foundation of Shandong Province (ZR2022MA014)
*Email addresses:* s180900013@s.upc.edu.cn (Hongxiao Guo), lvfeng18@gmail.com (Feng Lü), uplvwr@163.com (Weiran Lü)

uniqueness problems of meromorphic functions taking into account their shifts or difference operators. With help of this tool, there is another study direction on the unique range set of an entire function $f$, that is to seek the set $S$ such that if $E(f,S) = E(\Delta_c f, S)$, then $f = \Delta_c f$, where $\Delta_c f(z) = f(z+c) - f(z)$ is the first difference operator. Recently, Liu [17] firstly paid attentions to this direction and obtained the following result.

**Theorem A.** Suppose that $a$ is a nonzero complex number, $f$ is a transcendental entire function with finite order. If $f$ and $\Delta_c f$ share $\{a, -a\}$ CM, then $\Delta_c f(z) = f(z)$ for all $z \in \mathbb{C}$.

In the same paper, Liu [17] posed the following question: Let $a$ and $b$ are two small function of $f$ with period $c$. When a transcendental entire function $f$ of finite order and its difference operator $\Delta_c f$ share the set $\{a, b\}$ CM, what can we say about the relationship between $f$ and $\Delta_c f$? The follow-up research on this aspect was done by Li in [15]. In fact, Li proved the following theorem.

**Theorem B.** Suppose that $a,\ b$ are two distinct entire functions, and $f$ is a nonconstant entire function with $\rho(f) \neq 1$ and $\lambda(f) < \rho(f) < \infty$ such that $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If $f$ and $\Delta_c f$ share $\{a, b\}$ CM, then $\Delta_c f(z) = f(z)$ for all $z \in \mathbb{C}$.

Here, the order $\rho(f)$ and the exponent of convergence of zeros $\lambda(f)$ are defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},\ \ \lambda(f) = \limsup_{r\to\infty} \frac{\log N(r,\frac{1}{f})}{\log r}.$$

Very recently, Qi-Wang-Gu [13] further studied the above problem and showed that the theorem B still holds without the condition $\rho(f) \neq 1$. More precisely, they gave the specific form of $f$ in a simple way.

**Theorem C.** Suppose that $a,\ b$ are two distinct entire functions, and $f$ is a nonconstant entire function of finite order with $\lambda(f) < \rho(f) < \infty$ such that $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If $f$ and $\Delta_c f$ share $\{a,\ b\}$ CM, then $f(z) = Ae^{\mu z}$, where $A,\ \mu$ are two nonzero constants satisfying $e^{\mu c} = 2$. Furthermore, $\Delta_c f(z) = f(z)$.

Note that the function $f$ is finite order in the theorems A-C. It is known that the difference analogues to Nevanlinna theory given by Halburd and Korhonen , Chiang and Feng has been improved by Halburd, Korhonen, and Tohge [9] from the finite order of meromorphic functions to infinite order (hyper-order strictly less than 1). So, one may ask the question:

**Question 1.** Whether the theorem C still holds or not if the entire function $f$ is of hyper-order strictly less than 1?

After studying Theorem C carefully, we also have the follow question:

**Question 2.** Whether the first difference operator $\Delta_c f$ would be generalized to the $n$-th difference operator $\Delta_c^n f$ in Theorem C, where

$$\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(z+ic),$$

$c$ is a nonzero complex number and $n$ is a integer. (When $n = 1$, $\Delta_c^n f(z)$ reduces to the first difference operator $\Delta_c f(z)$.)

Unfortunately, the answer to the latter question is negative, which is showed by the following example.

**Example 1.** Consider $f(z) = Ae^{\lambda z}$ with a nonzero constant $A$ and $e^{\lambda} = 1 + i$. A calculation yields $\Delta_1^2 f(z) = -Ae^{\lambda z} = -f(z)$. Assume $a$ is an arbitrary entire function of order less than 1. Then $\Delta_1^2 f(z)$ and $f(z)$

share the set $\{a, -a\}$. However, it does not satisfy the conclusion of Theorem C.

From **Example 1** and Theorem C, we see that $\Delta_c^n f(z) = f(z)$ or $\Delta_c^n f(z) = -f(z)$ for case $n = 1$ or $n = 2$. This observation leads us to ask the next question:

**Question 3.** Whether the conclusion is $\Delta_c^n f(z) = \pm f(z)$ in Theorem C if $\Delta_c f$ is replaced by $\Delta_c^n f(z)$.

In the present paper, we consider the above questions 1 and 3, and give affirmative answers to them. More specifically, we have the following.

**Main Theorem.** Suppose that $a$, $b$ are two distinct entire functions, and $f$ is an entire function of hyper-order strictly less than 1 such that $\lambda(f) < \rho(f)$, $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$. If $f$ and $\Delta_c^n f(z)(\not\equiv 0)$ share the set $\{a, b\}$ CM, then $f(z) = Ae^{\lambda z}$, where $A$, $\lambda$ are two nonzero constants with $(e^{\lambda c} - 1)^n = \pm 1$. Furthermore,
(1) if $(e^{\lambda c} - 1)^n = 1$, then $\Delta_c^n f(z) = f(z)$;
(2) if $(e^{\lambda c} - 1)^n = -1$, then $\Delta_c^n f(z) = -f(z)$ and $b = -a$.

**Remark.** It is easy to see that the case $(e^{\lambda c} - 1)^n = -1$ cannot occur if $n = 1$, since $e^{\lambda c} \neq 0$. So, the main theorem is a generalization of Theorem C. We also point out both $\Delta_c^n f(z) = f(z)$ and $\Delta_c^n f(z) = -f(z)$ may happen if $n \geq 2$.

We give an example to show that the condition $\lambda(f) < \rho(f)$ is sharp.

**Example 2.** Consider $f(z) = A2^z(e^{2\pi i z} + B)$, where $A$, $B$ are two nonzero constants. Obviously, $\Delta_1 f(z) = f(z)$. Assume that $a$, $b$ are two arbitrary entire functions of order less than 1. Then $\Delta_1 f(z)$ and $f(z)$ share the set $\{a, b\}$ CM. Therefore, the example satisfies all the assumptions of Main theorem except for $\lambda(f) < \rho(f)$, since $\lambda(f) = \rho(f) = 1$. And the form of $f$ does not satisfy the conclusion of Main theorem.

Before to proceed, we assume that the reader is familiar with the basic results of Nevanlinna theory and its standard notation (see, e.g., [8, 20]) such as the proximity function $m(r, f)$, the integrated counting function $N(r, f)$ and the characteristic function $T(r, f)$, and we say a nonconstant meromorphic function $a(z)$ is a small function of $f$ if $T(r, a) = S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, where $f$ is a meromorphic function in the complex plane. And the lower order $\mu(f)$, the hyper-order $\rho_2(f)$ are defined as

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r, f)}{\log r}.$$

## 2. Proof of Main theorem

In this section, we shall prove our theorem. Before to its proof, we first give the following results, where the first one is Theorem 5.1 of Halburd-Korhonen-Tohge in [9], the second one is Lemma 3.3 of Bergweiler and Langley in [18], the last one is the Borel lemma in [20].

**Lemma 2.1**. Let $f$ be a nonconstant meromorphic function and $c \in \mathbb{C}$. If $f$ is a finite order, then

$$m(r, \frac{f(z + c)}{f(z)}) = O(\frac{\log\ r}{r} T(r, f)),$$

for all $r$ outside of a set $E$ with zero logarithmic density. If the hyper-order $\rho_2$ of $f$ is less than one, then for each $\varepsilon > 0$, we have

$$m(r, \frac{f(z + c)}{f(z)}) = o(\frac{T(r, f)}{r^{1-\rho_2-\varepsilon}}),$$

for all $r$ outside of a set of finite logarithmic measure.

**Lemma 2.2**. Let $g$ be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an $\epsilon$-set $E$ such that

$$\frac{g(z+\eta)}{g(z)} \to 1, as\ z \to \infty\ in\ \mathbb{C} \setminus E,$$

uniformly in $\eta$ for $|\eta| \leq h$.

**Lemma 2.3**. Suppose that $f_j(z)$ $(j = 1, 2, ..., n)$ $(n \geq 2)$ is a meromorphic function, $g_j(z)$ $(j = 1, 2, ..., n)$ is an entire function satisfying the following conditions:
(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0$;
(2) $g_j - g_k$ are not constant for $1 \leq j < k \leq n$;
(3) For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_k})), r \to \infty, r \notin E$, where $E \subset (1, \infty)$ is of finite linear measure.
Then, $f_j(z) = 0$ for $j = 1, 2, ..., n$.

*Proof.* [Proof of Main theorem] Based on the idea of [13], we will give the proof of the main theorem. For the convenience of the reader, we present our proof in all detail.

Since $f$ and $\Delta_c^n f$ share the set $\{a, b\}$ CM, one can assume

$$\frac{(\Delta_c^n f - a)(\Delta_c^n f - b)}{(f-a)(f-b)} = e^{\alpha}, \tag{2.1}$$

in which $\alpha$ is an entire function.

By the assumption $\lambda(f) < \rho(f)$ and Hadamard factorization theorem, we suppose that $f(z) = h(z)e^{\beta(z)}$, where $h(z)(\not\equiv 0)$ and $\beta$ are two entire functions satisfying

$$\lambda(f) = \rho(h) < \rho(f), \ \ \rho(e^{\beta}) = \rho(f), \ \ \rho(\beta) = \rho_2(f) < 1.$$

In particular, the function $\beta$ is a polynomial if $\rho(f)$ is finite.

We know that

$$\Delta_c^n f = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(z+ic) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} h(z+ic)e^{\beta(z+ic)}.$$

Substitute the forms of $f$ and $\Delta_c^n f$ into (2.1) yields that

$$\begin{aligned} &\left( [\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} h(z+ic)e^{\beta(z+ic)-\beta(z)}]e^{\beta(z)} - a \right) \\ &\left( [\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} h(z+ic)e^{\beta(z+ic)-\beta(z)}]e^{\beta(z)} - b \right) \\ &= e^{\alpha}(h(z)e^{\beta(z)} - a)(h(z)e^{\beta(z)} - b). \end{aligned} \tag{2.2}$$

Set $\omega_1 = \frac{\Delta_c^n f}{e^{\beta}} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} h(z+ic)e^{\beta(z+ic)-\beta(z)}$. Obviously, $\omega_1 \not\equiv 0$. Using Lemma 2.1, we know for any $1 \leq i \leq n$ that

$$m(r, e^{\beta(z+ic)-\beta(z)}) = m(r, \frac{e^{\beta(z+ic)}}{e^{\beta(z)}}) = S(r, e^{\beta(z)}), \tag{2.3}$$

and

$$T(r,\omega_1)\leq\sum_{i=0}^{n}T(r,e^{\beta(z+ic)-\beta(z)})+\sum_{i=0}^{n}T(r,h(z+ic))+S(r,e^{\beta(z)})$$
$$=\sum_{i=0}^{n}m(r,e^{\beta(z+ic)-\beta(z)})+S(r,e^{\beta(z)})=S(r,e^{\beta(z)}).$$

Therefore $\omega_1$ is a small function of $e^{\beta}$. Rewrite (2.2) as

$$e^{\alpha}=\frac{\omega_1^2[e^{\beta}-\frac{a}{\omega_1}][e^{\beta}-\frac{b}{\omega_1}]}{h^2[e^{\beta}-\frac{a}{h}][e^{\beta}-\frac{b}{h}]}. \tag{2.4}$$

Note that $a\not\equiv b$. Without loss of generality, we suppose that $a\not\equiv 0$. Aussme that $z_0$ is a zero of $e^{\beta}-\frac{a}{h}$, but not a zero of $\omega_1$. It follows from (2.4) and the assumption about sharing set $\{a,b\}$ that $z_0$ is a zero of $e^{\beta}-\frac{a}{\omega_1}$ or $e^{\beta}-\frac{b}{\omega_1}$. Below, we denote by $N_1(r,e^{\beta})$ the reduced counting function of those common zeros of $e^{\beta}-\frac{a}{h}$ and $e^{\beta}-\frac{a}{\omega_1}$. Similarly, denote by $N_2(r,e^{\beta})$ the reduced counting function of those common zeros of $e^{\beta}-\frac{a}{h}$ and $e^{\beta}-\frac{b}{\omega_1}$. Note that $h$ is a small function respect to $e^{\beta}$; applying the second fundamental theorem to $e^{\beta}$ and the first fundamental theorem to $e^{\beta}-\frac{a}{h}$ yields that

$$T(r,e^{\beta})=\bar{N}(r,\frac{1}{e^{\beta(z)}-\frac{a}{h}})+S(r,e^{\beta})=N_1(r,e^{\beta})+N_2(r,e^{\beta})+S(r,e^{\beta}), \tag{2.5}$$

which implies that either $N_1(r,e^{\beta})\neq S(r,e^{\beta})$ or $N_2(r,e^{\beta})\neq S(r,e^{\beta})$. Next, we consider two cases.

**Case 1.** $N_1(r,e^{\beta})\neq S(r,e^{\beta})$.

Let $a_0$ be a zero of $e^{\beta}-\frac{a}{h}$ and $e^{\beta}-\frac{a}{\omega_1}$. It is clear that $a_0$ is a zero of $\frac{a}{h}-\frac{a}{\omega_1}$. If $\frac{a}{h}-\frac{a}{\omega_1}\not\equiv 0$, then

$$N_1(r,e^{\beta})\leq N(r,\frac{1}{\frac{a}{h}-\frac{a}{\omega_1}})\leq T(r,\frac{a}{h}-\frac{a}{\omega_1})=S(r,e^{\beta}), \tag{2.6}$$

a contradiction. Thus $h=\omega_1$. Note that $\omega_1=\frac{\Delta_c^n f}{e^{\beta}}$ and $f=he^{\beta}$. Then, we have the desired result $f=\Delta_c^n f$.

Next, we give the form of $f$. The equation $f=\Delta_c^n f$ leads to

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{h(z+ic)}{h(z)}e^{\beta(z+ic)-\beta(z)}=1. \tag{2.7}$$

Set $C_i=(-1)^{n-i}\binom{n}{i}\frac{h(z+ic)}{h(z)}$. Then,

$$\sum_{i=1}^{n}C_ie^{\beta(z+ic)-\beta(z)}=1-(-1)^n. \tag{2.8}$$

Suppose that $\rho(e^{\beta(z+jc)-\beta(z+ic)})=\infty$ for any $0\leq i\neq j\leq n$. Note that $C_i$ is a meromorphic function with finite order and

$$\mu(e^{\beta(z+jc)-\beta(z+ic)})=\rho(e^{\beta(z+jc)-\beta(z+ic)})=\infty,$$

which shows that $C_i$ is a small function of $e^{\beta(z+jc)-\beta(z+ic)}$ for any $0\leq i\neq j\leq n$. Further, applying Borel Lemma (Lemma 2.3) to the equation (2.8), one get $C_i\equiv 0$ and $1-(-1)^n=0$, which is impossible. So, there exists indexs $(i,j)$ such that $\rho(e^{\beta(z+jc)-\beta(z+ic)})<\infty$, which implies that $\beta(z+ic)-\beta(z+jc)$ is a polynomial. We next show that $\beta$ is a polynomial.

Without loss of generality, assume $\deg(\beta(z+ic)-\beta(z+jc)) = s$. Differentiating the function $\beta(z+ic)-\beta(z+jc)$ $(s+1)$-times, one has $\beta^{(s+1)}(z+ic)-\beta^{(s+1)}(z+jc) = 0$, which yields that $\beta^{(s+1)}(z+ic)$ is a periodic function with period $(j-i)c$. If $\beta^{(s+1)}(z+ic)$ is not constant, we know the fact that the order of $\beta^{(s+1)}(z+ic)$ is not less than 1. Further, $\rho(\beta(z)) = \rho(\beta(z+ic)) = \rho(\beta^{(s+1)}(z+ic)) \geq 1$. However, $\rho(\beta(z)) = \rho_2(f) < 1$, a contradiction. Thus, $\beta^{(s+1)}(z+ic)$ is constant and $\beta(z+ic)$ is a polynomial. Therefore, the above discussion yields that $\beta(z)$ is a polynomial.

Set $\beta(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_0$, where $a_m \not\equiv 0, a_{m-1}, ..., a_0$ are constants and $m$ is a positive integer. We will show that $m = 1$. On the contrary, suppose that $m \geq 2$.

Then for $1 \leq i \leq n$, we have

$$\beta(z+ic) - \beta(z) = icma_m z^{m-1} + B_i(z),$$

where $B_i(z)$ is a polynomial with degree at most $m-2$. Set $g = e^{cma_m z^{m-1}}$. Clearly, $e^{B_i(z)}$ is a small function of $g$.

Rewriting (2.7), we have

$$\begin{aligned} 1-(-1)^n &= \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\frac{h(z+ic)}{h(z)}e^{icma_m z^{m-1}}e^{B_i(z)} \\ &= \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\frac{h(z+ic)}{h(z)}e^{B_i(z)}(e^{cma_m z^{m-1}})^i \\ &= \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\frac{h(z+ic)}{h(z)}e^{B_i(z)}g^i. \end{aligned}$$

Then,

$$e^{B_n(z)}\frac{h(z+nc)}{h(z)}g^n = -\sum_{i=1}^{n-1}(-1)^{n-i}\binom{n}{i}\frac{h(z+ic)}{h(z)}e^{B_i(z)}g^i + 1-(-1)^n. \tag{2.9}$$

Rewrite (2.9) as

$$g^n = -e^{-B_n(z)}\sum_{i=1}^{n-1}(-1)^{n-i}\binom{n}{i}\frac{h(z+ic)}{h(z+nc)}g^i e^{B_i(z)} + e^{-B_n(z)}\frac{h(z)}{h(z+nc)}(1-(-1)^n). \tag{2.10}$$

For any $\varepsilon > 0$ and $0 \leq i \leq n-1$, by the first part of Lemma 2.1, we have

$$m(r, \frac{h(z+ic)}{h(z+nc)}) = O(r^{\rho(h)-1+\varepsilon}).$$

(The above fact can also be deduced by the Collorary 2.6 of Chiang and Feng in [1].) Noting that $\rho(h) < \rho(f) = \rho(e^\beta) = m$, we can take $\varepsilon$ small enough such that $\rho(h)-1+\varepsilon < \rho(e^\beta)-1 = m-1 = \rho(g)$. Then

$$m(r, \frac{h(z+ic)}{h(z+nc)}) = S(r, g).$$

We may rewrite (2.10) as

$$g^{n-1}g = \sum_{i=1}^{n-1} b_i g^i + b_0,$$

where $b_i = (-1)^{n-i+1}\binom{n}{i}\frac{h(z+ic)}{h(z+nc)}e^{B_i(z)-B_n(z)}$ and $b_0 = (1-(-1)^n)e^{-B_n(z)}\frac{h(z)}{h(z+nc)}$. Clearly,

$$m(r, b_i) = S(r, g), \quad m(r, b_0) = S(r, g).$$

Now, we employ well-known Clunie lemma in Nevanlinna theory, which is stated as follows. It can be found in [3] and [12, Lemma 2.4.2 and Proposition 9.2.3].

***Clunie Lemma.*** *Let $f$ be a transcendental meromorphic solution of the difference equation $f^n A(z, f) = B(z, f)$, where $A(z, f)$, $B(z, f)$ are differential polynomials in $f$ having coefficients $a_j^*$ such that $m(r, a_j^*) = S(r, f)$. If the total degree of $B(z, f)$ is at most $n$, then $m(r, A(z, f)) = S(r, f)$.*

Using the above Clunie lemma, we have $m(r, g) = S(r, g)$, a contradiction. So $m = 1$. And we can set $f = h(z)e^{\lambda z}$, where $\lambda$ is a nonzero constant.

We substitute the form of $f$ into (2.7) to find

$$1 = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{h(z+ic)}{h(z)}(e^{\lambda c})^i. \tag{2.11}$$

Note that $\rho(h) < \rho(f) = \rho(e^{\lambda z}) = 1$. Then by Lemma 2.2, we know that there exists an $\varepsilon$-set $E$, as $z \notin E$ and $z \to \infty$, such that $\frac{h(z+ic)}{h(z)} \to 1$. It leads to

$$1 = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(e^{\lambda c})^i = (e^{\lambda c}-1)^n. \qquad (2.12).$$

We rewrite (2.7) as

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}h(z+ic)e^{\lambda ci} + ((-1)^n - 1)h(z) = 0. \tag{2.13}$$

Below, we need a property of the meromorphic solutions of linear difference equation, which is due to Lü etc in [14].

**Lemma 2.4.** *Let $a_0, a_1, ..., a_n$ be constants satisfying $a_0 a_n \neq 0$. If $f$ is a nonconstant meromorphic solution of difference equation*

$$a_n f(z+c) + ... + a_1 f(z+1) + a_0 f(z) = P(z),$$

*where $P$ is a polynomial, then either $\rho(f) \geq 1$ or $f$ is a polynomial. In particular, if $a_n \neq \pm a_0$, then $\rho(f) \geq 1$.*

By $\rho(h) < 1$ and Lemma 2.4, we obtain that $h(z)$ is a polynomial. Set $h(z) = c_k z^k + c_{k-1}z^{k-1} + ... + c_0$, where $k$ is an integer and ($c_k \neq 0$). Suppose that $k \geq 1$. Due to the idea in [2], we will finish this case. Compare the coefficient of $z^k$ of both side of (2.13) yields

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}e^{\lambda ci} + ((-1)^n - 1) = (e^{\lambda c}-1)^n - 1 = 0. \tag{2.14}$$

Moreover, compare the coefficient of $z^{k-1}$ of both side of (2.13) yields

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}e^{\lambda ci}(kic + a_{k-1}) + ((-1)^n - 1)a_{k-1} = 0. \tag{2.15}$$

Substitute (2.14) into (2.15) yields

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}ie^{\lambda ci}=0.$$

On the other hand

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}ie^{\lambda ci}=\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}e^{\lambda c(i-1)}ne^{\lambda c}=ne^{\lambda c}(e^{\lambda c}-1)^{n-1}\neq 0,$$

a contradiction. So $k=0$ and $h$ is a nonzero constant, say $A$. Therefore, we derive the desire result $f(z)=Ae^{\lambda z}$.

**Case 2.** $N_2(r,e^{\beta})\neq S(r,e^{\beta})$.

Let $b_0$ is a zero of $e^{\beta}-\frac{a}{h}$ and $e^{\beta}-\frac{b}{\omega_1}$. Then it is clear that $b_0$ is a zero of $\frac{a}{h}-\frac{b}{\omega_1}$. If $\frac{a}{h}-\frac{b}{\omega_1}\not\equiv 0$, then

$$S(r,e^{\beta})\neq N_2(r,e^{\beta})\leq N(r,\frac{1}{\frac{a}{h}-\frac{b}{\omega_1}})\leq T(r,\frac{a}{h}-\frac{b}{\omega_1})=S(r,e^{\beta}),$$

a contradiction. Thus

$$\frac{a}{h}-\frac{b}{\omega_1}=0. \tag{2.16}$$

If $b=0$, then $\frac{a}{h}=0$, a contradiction. Thus $b\neq 0$.

We assume that $c_0$ is a zero of $e^{\beta}-\frac{b}{h}$, but not a zero of $\omega_1$. It follows from (2.4) and the assumption about sharing that $c_0$ is a zero of $e^{\beta}-\frac{a}{\omega_1}$ or $e^{\beta}-\frac{b}{\omega_1}$. We denote by $N_3(r,e^{\beta})$ the reduced counting function of those common zeros of $e^{\beta}-\frac{b}{h}$ and $e^{\beta}-\frac{a}{\omega_1}$. Similarly, We denote by $N_4(r,e^{\beta})$ the reduced counting function of those common zeros of $e^{\beta}-\frac{b}{h}$ and $e^{\beta}-\frac{b}{\omega_1}$. Note that $h$ is a small function respect to $e^{\beta}$; applying the second fundamental theorem to $e^{\beta}$ and the first fundamental theorem to $e^{\beta}-\frac{a}{h}$ yields

$$T(r,e^{\beta})=\bar{N}(r,\frac{1}{e^{\beta(z)}-\frac{b}{h}})+S(r,e^{\beta})=N_3(r,e^{\beta})+N_4(r,e^{\beta})+S(r,e^{\beta}), \tag{2.17}$$

which implies that either $N_3(r,e^{\beta})\neq S(r,e^{\beta})$ or $N_4(r,e^{\beta})\neq S(r,e^{\beta})$. If $N_4(r,e^{\beta})\neq S(r,e^{\beta})$, similar to Case 1, we get the desire result. So we assume that $N_3(r,e^{\beta})\neq S(r,e^{\beta})$. Below, similar to Case 2, we can deduce that

$$\frac{b}{h}-\frac{a}{\omega_1}=0. \tag{2.18}$$

It follows from (2.16) and (2.18) that $a^2=b^2$. Note that $a\neq b$. Thus, $a=-b$. Again by (2.18), one has $\omega_1=-h$. We can rewrite it as

$$\omega_1=\frac{\Delta_c^n f}{e^{\beta}}=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}h(z+ic)e^{\beta(z+ic)-\beta(z)}=-h(z), \tag{2.19}$$

It follows from (2.19) that $\Delta_c^n f=-f$. Furthermore, the same argument as in Case 1 yields that $f(z)=Ae^{\lambda z}$ with $(e^{\lambda c}-1)^n=-1$.

Therefore, the proof is finished. □

**Acknowledgements**. The authors would like to thank the referee for helpful suggestions.

## References


[1] Y.M. Chiang and S.J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, Ramanujan J. 16(2008), 105-129.
[2] C.X. Chen and Z.X. Chen, Entire functions and their higher order differences, Taiwanese J. Math. 18(2014), 677-685.
[3] J. Clunie, On integral and meromorphic functions, J. London Math. Soc. 37(1962), 17-27.
[4] G. Frank and M. Reinders, A unique range set for meromorphic functions with 11 elements, Complex Variables Theory Appl. 37(1998), 185-193.
[5] H. Fujimoto, On uniqueness polynomials for meromorphic functions, Nagoya Math. J. 170(2003), 33-46.
[6] F. Gross, Factorizationo fmeromorphicfunctionsandsom eopenproblems, Complex Analysis(Proc. Conf. Univ. Kentucky, Lexington, Ky.1976), pp.51-69, Lecture Notes in Math. Vol.599, Springer, Berlin, 1977.
[7] F. Gross and C.C. Yang, On preimage range sets of meromorphic functions, Proc. Japan Acad. Ser. A. 58(1982), 17-20.
[8] W.K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[9] R.G. Halburd, R.J. Korhonen and K. Tohge, Holomorphic curves with shift-invariant hyperplane preimages, Trans. Amer. Math. Soc. 366(2014), 4267-4298.
[10] R.G. Halburd and R.J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math. 31(2006), 463-487.
[11] R.G. Halburd and R.J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, J. Math. Anal. Appl. 314(2006), 477-487.
[12] I. Laine, Nevanlinna Theory and Complex Differential Equations, Studies in math, Vol 15, de Guryter, Berlin, 1993.
[13] J.M. Qi, Y.F. Wang and Y. Gu, A note on ertire functions sharing a finite set with their difference operators, Adv Differ Equ. (2019) 2019: 114.
[14] F. Lü, W.R Lü, C.P. Li and J.F. Xu, Growth and uniqueness related to complex differential and difference equations, Results Math. 74(2019), Art. 30, 18 pp.
[15] X.M. Li, Entire functions sharing a finite set with their difference operators, Comput Meth Funct Theory. 12(2012), 307-328.
[16] P. Li and C.C. Yang, Some further results on the unique range sets of meromorphic functions, Kodai Math. J. 18(1995), 437-450.
[17] K. Liu, Meromorphic functions sharing a set with applications to difference equations, J. Math. Anal. Appl. 359(2009), 384-393.
[18] W. Bergweiler and J.K. Langley, Zeros of differences of meromorphic functions, Math. Proc. Cambridge Philos. Soc. 142(2007), 133-147.
[19] H.X. Yi, A question of gross and the uniqueness of entire function, Nagoya Math. J. 138(1995), 169-177.
[20] C.C. Yang and H.X. Yi, Uniqueness Theory of Meromorphic Functions, Science Press, Beijing/New York, 2003.