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Sharp estimates for the unique solution for a class of fractional differential equations

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Abstract. In this paper, we investigated the sharp estimate for the condition of the given interval which guarantees for the unique solution of a Reimman-Liouville-type fractional differential equations with boundary conditions. The method of analysis is obtained by the principle of contraction mapping through using the maximum value of the integral of the Green's function. Besides, we also concluded a sharper lower bound of the eigenvalues for an eigenvalue problem. Finally, two examples are presented to clarify the principle results.

1. Introduction and preliminaries

Recently, the differential equations of the fractional order have received immense attention from researchers because of their massive development and diverse applications in real phenomena, and in multiple fields of physics, biology, chemistry, control theory, electrical circuits, wave propagation, blood flow phenomena, signal and image processing...etc. We refer the reader to the papers [1–8] and the references cited therein for more details.

There are several excellent results about the existence of the solutions for the differential equations of the fractional order by the use fixed point theorems. For example, see [9–14] and the references therein. The search for sufficient conditions that must be met to ensure the existence and uniqueness of the solution is very important in the study of differential equations, whether for the fractional order or the integer one. In the past years we have noticed that there is a method of study considered as distinct from the others of the results' accuracy, in which it relies by using the maximum value of the integral equation of a Green's function, meanwhile, it is worth to note that there are only a few papers around this analysis method, and before we take a look at this study and avoid repetitive writing, we assume in the whole paper that:
(H). $g : \mathbb{R} \times [a, b] \to \mathbb{R}$ is a continuous function and satisfies a uniform Lipschitz condition with respect to the first variable on $\mathbb{R} \times [a, b]$ with Lipschitz constant $K > 0$, that is,

$$\left|g(\mu, t) - g(\overline{\mu}, t)\right| \leq K \left|\mu - \overline{\mu}\right| \text{ for all } (\mu, t), (\overline{\mu}, t) \in \mathbb{R} \times [a, b], \text{ with } a, b \in \mathbb{R}.$$

A result had been concluded through Theorem 7.7 in the book [15], can be stated as follows: If the hypothesis (H) holds, then the following classical differential equation with two-point boundary conditions

$$u''(t) = g(u(t), t),\ a < t < b,\ \ u(a) = \zeta,\ u(b) = \overline{\zeta},\ \zeta, \overline{\zeta} \in \mathbb{R}, \tag{1}$$

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has a unique solution on $[a, b]$ if

$$b - a < 2\sqrt{\frac{2}{K}}.$$

In 2016, Ferreira [16] discussed the existence and uniqueness of solutions for a fractional boundary value problems:

$$^{\mathcal{RL}}\mathfrak{D}_a^{\theta} u(t) = -g(u(t), t),\ a < t < b,\ 1 < \theta \leq 2,\ \ u(a) = 0,\ u(b) = \overline{\zeta},\ \overline{\zeta} \in \mathbb{R}, \tag{2}$$

where $^{\mathcal{RL}}\mathfrak{D}_a^{\theta}$ denotes the Reimman-Liouville fractional derivative of order θ, and he came to the following result: If the hypothesis (H) holds, and if

$$b - a < \left(\frac{\theta^{(\theta+1)}\Gamma(\theta)}{K(\theta-1)^{(\theta-1)}}\right)^{\frac{1}{\theta}}.$$

Then the boundary value problem (2) has a unique solution.

Ferreira in 2019 [17] corrected a recent uniqueness result [18] for the following two-point fractional boundary value problem:

$$^{\mathcal{LC}}\mathfrak{D}_a^{\theta} u(t) = -g(u(t), t),\ a < t < b,\ \ u(a) = \zeta,\ u(b) = \overline{\zeta},\ \zeta, \overline{\zeta} \in \mathbb{R}, \tag{3}$$

where $^{\mathcal{LC}}\mathfrak{D}_a^{\theta}$ denotes the Liouville-Caputo fractional derivative of order $\theta \in (1, 2]$. Although the result obtained (see Theorem 2.1 in [17]) is very complex and does not meet the desired purpose, but listed by the author to show the error contained in the paper [18]. See [16, 17] and the references therein for more details.

In 2021, Laadjal and Adjeroud [19] established the sharp estimate for the unique solution of the following fractional differential equation:

$$^{\mathcal{H}}\mathfrak{D}_a^{\theta} u(t) = -g(u(t), t),\ 0 < a < t < b,\ \ u(a) = 0,\ u(b) = \overline{\zeta},\ \overline{\zeta} \in \mathbb{R}, \tag{4}$$

where $^{\mathcal{H}}\mathfrak{D}_a^{\theta}$ denotes the Hadamard fractional derivative of order $\theta \in (1, 2]$.

Motivated by the above mentioned works, in this paper, we investigated the sharp estimate for the unique solution of the following boundary value problem with Reimman-Liouville fractional derivative:

$$^{\mathcal{RL}}\mathfrak{D}_a^{\theta} u(t) = -g(u(t), t),\ a < t < b,\ \theta \in (2, 3),\ \ u(a) = pu'(a),\ u'(b) = \overline{\zeta},\ \overline{\zeta} \in \mathbb{R}, p \in \mathbb{R}\backslash\{0\}. \tag{5}$$

Further, we will also obtain a result for the following third order nonlinear differential equation with boundary conditions:

$$u'''(t) = -g(u(t), t),\ a < t < b,\ \ u(a) = u'(a) = 0,\ u'(b) = \overline{\zeta},\ \overline{\zeta} \in \mathbb{R}. \tag{6}$$

Now, we present some definitions and properties that are helpful for our discussion

Definition 1.1. *Let $h \in L^1[a, b]$, $-\infty < a \leq b < +\infty$, and $\kappa \in [0, +\infty)$. We have the following definitions: The Riemann-Liouville fractional integral of order κ for a function h is defined by*

$$\mathcal{I}_a^0 h(t) = h(t) \text{ and } \mathcal{I}_a^{\kappa} h(t) = \frac{1}{\Gamma(\kappa)} \int_a^t \frac{h(\tau)}{(t-\tau)^{1-\kappa}} d\tau \text{ for } \kappa \in (0, +\infty);$$

The Riemann-Liouville fractional derivative of order κ for a function $h \in L^1[a, b]$ is defined by

$$^{\mathcal{RL}}\mathfrak{D}_a^0 h(t) = h(t) \text{ and } {}^{\mathcal{RL}}\mathfrak{D}_a^{\kappa} h(t) = \frac{d^n}{dt^n} \left(\mathcal{I}_a^{n-\kappa} h\right)(t),\ (n - 1 < \kappa \leq n, n \in \mathbb{N}) \text{ for } \kappa \in (0, +\infty).$$

Remark 1.2. *Let $\delta, \kappa \in [0, +\infty)$, and $a, b \in \mathbb{R}$ with $a < b$. For all $h \in L^1[a, b]$, we have*
(i). $\mathcal{I}_a^{\delta}(\mathcal{I}_a^{\kappa}h)(t) = \mathcal{I}_a^{\kappa}(\mathcal{I}_a^{\delta}h)(t) = \mathcal{I}_a^{\delta+\kappa}h(t)$;
(ii). ${}^{\mathcal{RL}}\mathfrak{D}_a^{\kappa}\, \mathcal{I}_a^{\kappa}h(t) = h(t)$.

Proposition 1.3. *Let $\kappa \in [0, +\infty)$ where $n - 1 < \kappa \leq n$, ($n \in \mathbb{N}$). Then the differential equation ${}^{\mathcal{RL}}\mathfrak{D}_a^{\kappa}h(t) = 0$, has this general solution*

$$h(t) = q_1 (t-a)^{\kappa-1} + q_2 (t-a)^{\kappa-2} + ... + q_n (t-a)^{\kappa-n}, \tag{7}$$

where $q_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Lemma 1.4. *In view of Proposition 1.3 and (ii) in Remark 1.2, we have the following property:*

$$\mathcal{I}_a^{\kappa}\, {}^{\mathcal{RL}}\mathfrak{D}_a^{\kappa}h(t) = h(t) + q_1 (t-a)^{\kappa-1} + q_2 (t-a)^{\kappa-2} + ... + q_n (t-a)^{\kappa-n}, \tag{8}$$

where $q_i \in \mathbb{R}$, with $i = 1, 2, ..., n$.

For more details on fractional calculus, the reader is referred to the book [20], also we urge those interested to refer to the paper [21] as well, as it contains important and new additions.

2. Main results

This section is designated to prove the essential results of the problem (5), and then derive the result of the problem (6).

Lemma 2.1. *Let $\phi \in C([a, b], \mathbb{R}) \cap L^1([a, b], \mathbb{R})$, the solution of the following problem*

$${}^{\mathcal{RL}}\mathfrak{D}_a^{\theta}u(t) = -\phi(t),\ a < t < b,\ \theta \in (2, 3),\ \ u(a) = pu'(a),\ u'(b) = \overline{\zeta},\ \overline{\zeta} \in \mathbb{R},\ p \in \mathbb{R}\backslash\{0\}, \tag{9}$$

is given by the integral equation

$$u(t) = \int_a^b G(t, \tau)\phi(\tau)d\tau + \frac{\overline{\zeta}}{(\theta - 1)(b - a)^{\theta-2}} (t-a)^{\theta-1}, \tag{10}$$

where

$$G(t, \tau) = \frac{1}{\Gamma(\theta)} \begin{cases} h_1(t, \tau) = \frac{(t-a)^{\theta-1}}{(b-a)^{\theta-2}} (b-\tau)^{\theta-2} - (t-\tau)^{\theta-1}, & a \leq \tau \leq t \leq b, \\ h_2(t, \tau) = \frac{(t-a)^{\theta-1}}{(b-a)^{\theta-2}} (b-\tau)^{\theta-2}, & a \leq t \leq \tau \leq b. \end{cases} \tag{11}$$

Proof. The general solution of the differential equation ${}^{\mathcal{RL}}\mathfrak{D}_a^{\theta}u(t) = -\phi(t)$ is given by

$$u(t) = -\mathcal{I}_a^{\theta}\phi(t) + q_1 (t-a)^{\theta-1} + q_2 (t-a)^{\theta-2} + q_3 (t-a)^{\theta-3}, \tag{12}$$

where $q_1, q_2, q_3 \in \mathbb{R}$.
Using the boundary condition $u(a) = pu'(a)$, we get $q_2 = q_3 = 0$, yields

$$u(t) = -\mathcal{I}_a^{\theta}\phi(t) + q_1 (t-a)^{\theta-1}. \tag{13}$$

Next by the condition $u'(b) = \overline{\zeta}$, we get

$$q_1 = \frac{\overline{\zeta}}{(\theta - 1)(b - a)^{\theta-2}} + \frac{1}{\Gamma(\theta)(b - a)^{\theta-2}} \int_a^b (b-\tau)^{\theta-2} \phi(\tau)d\tau.$$

Substituting the value of q_1 in (13), we obtain the integral equation (10). The proof is completed. □

Lemma 2.2. *The Green's function G defined in Lemma 2.1, has the following properties:*
i). $G(t,\tau) \geq 0$. For all $(t,\tau) \in [a,b] \times [a,b]$;
ii). $\max_{t \in [a,b]} \int_a^b G(t,\tau)\, d\tau = \frac{(b-a)^\theta}{(\theta-1)\Gamma(\theta+1)}$.

Proof. *i*). Obviously, $h_2(t,\tau) \geq 0$ for $a \leq t \leq \tau \leq b$; and $h_1(t,a) = 0$ for all $t \in [a,b]$.
Now, we start by fixing an arbitrary $\tau \in (a,b]$. Differentiating the function h_1 with respect to t.
For $a < \tau \leq t \leq b$ we have

$$\begin{aligned} \partial_t h_1(t,\tau) &= (\theta-1)\frac{(t-a)^{\theta-2}}{(b-a)^{\theta-2}}(b-\tau)^{\theta-2} - (\theta-1)(t-\tau)^{\theta-2} \\ &= (\theta-1)(t-a)^{\theta-2}\left[\left(\frac{b-\tau}{b-a}\right)^{\theta-2} - \left(\frac{t-\tau}{t-a}\right)^{\theta-2}\right]. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \left(\frac{b-\tau}{b-a}\right)^{\theta-2} &= \left(1 - \frac{\tau-a}{b-a}\right)^{\theta-2} \\ &\geq \left(1 - \frac{\tau-a}{t-a}\right)^{\theta-2} \\ &= \left(\frac{t-\tau}{t-a}\right)^{\theta-2}, \end{aligned}$$

which yields

$$\partial_t h_1(t,\tau) \geq 0, \tag{14}$$

Thus, $h_1(t,\tau)$ is an increasing function of $t \in [a,b]$, when $t \geq \tau$. So, we get

$$h_1(t,\tau) \geq h_1(\tau,\tau) \geq 0. \tag{15}$$

Hence, we conclude that $G(t,\tau) \geq 0$ for all $(t,\tau) \in [a,b] \times [a,b]$.
ii). From Green's function (11), we have

$$\begin{aligned} \Gamma(\theta)\int_a^b G(t,\tau)\, d\tau &= \int_a^t \left[\frac{(t-a)^{\theta-1}}{(b-a)^{\theta-2}}(b-\tau)^{\theta-2} - (t-\tau)^{\theta-1}\right] d\tau \\ &\quad + \int_t^b \frac{(t-a)^{\theta-1}}{(b-a)^{\theta-2}}(b-\tau)^{\theta-2}\, d\tau \\ &= \frac{(t-a)^{\theta-1}}{(b-a)^{\theta-2}}\int_a^b (b-\tau)^{\theta-2}\, d\tau - \int_a^t (t-\tau)^{\theta-1}\, d\tau \\ &= \frac{(t-a)^{\theta-1}(b-a)}{\theta-1} - \frac{(t-a)^\theta}{\theta}, \end{aligned}$$

which yields

$$\int_a^b G(t,\tau)\, d\tau = \frac{\Theta(t)}{\Gamma(\theta)}, \tag{16}$$

where

$$\Theta(t) = \frac{(b-a)(t-a)^{\theta-1}}{\theta-1} - \frac{(t-a)^\theta}{\theta},\ t \in [a,b]. \tag{17}$$

Observe that $\Theta(a) = 0$ and $\Theta(b) = \frac{(b-a)^\theta}{\theta(\theta-1)}$. If $t \in (a, b)$, we differentiate $\Theta(t)$ to get

$$\begin{aligned} \Theta'(t) &= (b-a)(t-a)^{\theta-2} - (t-a)^{\theta-1} \\ &= (t-a)^{\theta-2}(b-t) > 0, \end{aligned}$$

Since Θ is a continuous and an increasing function on the interval [a,b], we obtain

$$\max_{t\in[a,b]} \Theta(t) = \Theta(b) = \frac{(b-a)^\theta}{\theta(\theta-1)}. \tag{18}$$

By (16) and (18), we get the second property in Lemma 2.2. The proof is completed. □

Theorem 2.3. *Assume that the hypothesis* (H) *holds. If*

$$b - a < \left(\frac{(\theta-1)\Gamma(\theta+1)}{K}\right)^{\frac{1}{\theta}}. \tag{19}$$

Then the fractional boundary value problem (5) has a unique solution for any values of p and $\overline{\zeta}$.

Proof. Let $\Sigma = C([a, b], \mathbb{R})$ be the Banach space endowed with the norm

$$\|u\|_\Sigma = \sup\{|u(t)| : t \in [a, b]\},$$

and we define the operator $\Psi : \Sigma \to \Sigma$ by

$$\Psi u(t) = \int_a^b G(t,\tau) g(u(\tau),\tau) d\tau + \frac{\overline{\zeta}}{(\theta-1)(b-a)^{\theta-2}} (t-a)^{\theta-1}, \tag{20}$$

where the function G is given by (11).
Noting that, the problem (5) has a solution $u \in \Sigma$ if and only if u is a fixed point of the operator Ψ.
For all $(u, t), (v, t) \in \Sigma \times [a, b]$, we have

$$\begin{aligned} |\Psi u(t) - \Psi v(t)| &\le \int_a^b G(t,\tau) \left|g(u(\tau),\tau) - g(v(\tau),\tau)\right| d\tau \\ &\le \int_a^b K G(t,\tau) |u(\tau) - v(\tau)| d\tau \\ &\le K \int_a^b G(t,\tau) d\tau \, \|u - v\|_\Sigma, \end{aligned}$$

using the second property in Lemma 2.2, yields

$$\|\Psi u - \Psi v\|_\Sigma \le \frac{K(b-a)^\theta}{(\theta-1)\Gamma(\theta+1)} \|u - v\|_\Sigma. \tag{21}$$

Can be easily check that the assumption (19) leads to the principle of contraction mapping. Hence, Ψ is contraction mapping, we conclude that the problem (5) has a unique solution. □

In our previous study we note that $2 < \theta < 3$. Now, for $\theta = 3$, we created the problem (6). The same previous way can be easily check that u is a solution of the problem (6) if and only if u is a fixed point of the operator Ψ, where Ψ is given by (20). This leads us to the following result:

Corollary 2.4. *Assume that the hypothesis* (H) *holds. If*

$$b - a < \sqrt[3]{\frac{12}{K}}. \tag{22}$$

Then the boundary value problem (6) has a unique solution.

Now, we present a lower bound of the eigenvalues for an eigenvalue problem.

Theorem 2.5. *If the following eigenvalue problem*

$$^{\mathcal{RL}}\mathfrak{D}_a^{\theta} u(t) = \lambda u(t),\ a < t < b,\ \theta \in (2,3),\ \ u(a) = pu'(a),\ u'(b) = 0, p \in \mathbb{R}\backslash\{0\}, \tag{23}$$

has a non-trivial continuous solution, then

$$|\lambda| \geq \frac{(\theta - 1)\,\Gamma(\theta + 1)}{(b - a)^{\theta}}. \tag{24}$$

Proof. From Lemma 2.1, the problem (23) is equivalent to the following integral equation

$$u\,(t) = -\int_a^b \lambda G\,(t, \tau)\,u\,(\tau)\,d\tau.$$

Which yields

$$\|u\|_{\Sigma} \leq \sup_{t \in [a,b]} \left\{ |\lambda| \int_a^b |G\,(t, \tau)|\,|u\,(\tau)|\,d\tau \right\} \leq |\lambda|\,\|u\|_{\Sigma}\,\frac{(b - a)^{\theta}}{(\theta - 1)\,\Gamma(\theta + 1)}.$$

Since u is non-trivial, then $\|u\|_{\Sigma} \neq 0$, we get

$$1 \leq |\lambda|\,\frac{(b - a)^{\theta}}{(\theta - 1)\,\Gamma(\theta + 1)},$$

we obtain the equality (24). The proof is ended. □

Corollary 2.6. *If*

$$|\lambda| < \frac{(\theta - 1)\,\Gamma(\theta + 1)}{(b - a)^{\theta}}, \tag{25}$$

Then the boundary value problem (23) has no non-trivial solution.

In order to confirm the ability of applying the attained results, we provide two examples with the application of the two Theorems 2.3 and 2.5.

Example 2.7. *Consider the following problem*

$$^{\mathcal{RL}}\mathfrak{D}^{5/2} u(t) = 1 - t^2 + 4\tan^{-1} u(t),\ \ 0 < t < 1,\ \ u(0) = pu'(0),\ u'(1) = \overline{\zeta},\ \ \overline{\zeta} \in \mathbb{R},\ p \in \mathbb{R}\backslash\{0\}. \tag{26}$$

Here $\theta = \frac{5}{2}, a = 0, b = 1$ *and* $g(u,t) = 1 - t^2 + 4\tan^{-1} u$. *For all* $(u_1, t), (u_2, t) \in \mathbb{R} \times [0, 1]$ *we have* $\left|g(u_2, t) - g(u_1, t)\right| \leq K\,|u_2 - u_1|$ *with* $K = 4$. *So,*

$$\left(\frac{(\theta - 1)\Gamma(\theta + 1)}{K}\right)^{\frac{1}{\theta}} = \left(\frac{45}{64}\sqrt{\pi}\right)^{\frac{2}{5}} \approx 1.0921 > 1.$$

Therefore, the inequality (19) is satisfied. Hence, from Theorem 2.3, we educe that the fractional boundary value problem (26) has a unique solution on $[0, 1]$.

Example 2.8. *Consider the following eigenvalue problem*

$$ {}^{\mathcal{RL}}\mathfrak{D}_a^{5/2}u(t) = \lambda u(t),\ 0 < t < 1,\ \ u(a) = pu'(a),\ u'(b) = 0, p \in \mathbb{R}\backslash\{0\}, \tag{27} $$

Here $\theta = \frac{5}{2}$, *and* $[a, b] = [0, 1]$. *So, we obtain*

$$ \frac{(\theta - 1)\,\Gamma(\theta + 1)}{(b - a)^{\theta}} = \frac{45}{16}\sqrt{\pi}. \tag{28} $$

From Theorem 2.5, we deduce that: If λ *is an eigenvalue of the problem (27), we should have* $|\lambda| \geq \frac{45}{16}\sqrt{\pi}$.

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