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Continuity of the scattering function and Levinson type formula for Klein-Gordon s-wave equation with boundary conditon depends on spectral parameter

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Abstract. In this article, we consider inverse problem of scattering theory for Klein-Gordon s-wave equation with boundary condition depends on spectral parameter. We define the scattering data and we also prove that the continuity of the scattering function discussing the scattering solutions. Furthermore, we obtain Levinson type formula.

1. Introduction

Scattering problems and analysis have been significant roles in mathematical physics. Studies of scattering problems first begin for Schrödinger equations in [1,2]. In [2], Marchenko has investigated the properties of eigenvalues of Sturm-Liouville boundary value problem given by

$$-y'' + q(x)y = \lambda^2 y\ , \quad 0 \le x < \infty \tag{1.1}$$

$$y(0) = 0 \tag{1.2}$$

for a real valued function q, and then he has obtained the Jost function of (1.1) defined by

$$e(\lambda) = 1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\ , \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C},\ \operatorname{Im}\lambda \ge 0\}\,.$$

which has a finite number of simple zeros in $\mathbb{C}_+$. The scattering data of (1.1)-(1.2) is the following set

$$\{S(\lambda), i\lambda_k, m_k : k = 1, 2, ..., n\} \tag{1.3}$$

where $i\lambda_k$ are the zeros of Jost function, m_k^{-1} are the norm of the zeros of Jost function for $\lambda = i\lambda_k$ in $L_2(0,\infty)$ and $S(\lambda)$ is scattering function of (1.1)-(1.2) defined by

$$S(\lambda) := \frac{\overline{e(\lambda)}}{e(\lambda)}\ , \quad \lambda \in (-\infty, \infty)$$

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([1]). As the potential function q is given, the problem of getting scattering data written (1.3) and investigating the properties of scattering data is called the direct problem for scattering theory. Oppositely, finding the potential function q according to the scattering data is known inverse problem of scattering theory. There are a lot of papers about direct and inverse scattering problems for Sturm-Liouville, Schrödinger and Dirac equations. Some of them are given by [3-9]. Furthermore, the spectral properties with eigenvalues and spectral singularities of Klein-Gordon s-wave equation was given in [10], and the continuity of the scattering function and Levinson Formula of Sturm-Liouville problem depends on spectral parameter was examined in [11-12]. But scattering theory of Klein-Gordon s-wave equation with quadratic spectral parameter dependent boundary condition has not been investigated yet. In this work, we will consider the operator L_μ generated by the Klein-Gordon s-wave equation of second order

$$y'' + [\lambda - q(x)]^2 y = 0\,, \quad 0 \le x < \infty \tag{1.4}$$

with boundary condition depends on spectral parameter

$$y'(0,\lambda) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y(0,\lambda) = 0, \tag{1.5}$$

for a complex parameter $\lambda = \mu^2$ where α_i are real numbers for $i = 0,1,2$, $\alpha_1 \le 0$, $\alpha_2 > 0$, $\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 \neq 0$ and q is a non-negative real valued function satisfying the following condition

$$\int_0^\infty x\left[\left|q(x)\right| + \left|q'(x)\right|\right]dx < \infty. \tag{1.6}$$

When the condition (1.6) is satisfied, equation (1.4) has the following solutions $f^{(1)}(x,\mu) = f(x,\mu^2)$ and $\overline{f^{(1)}(x,\mu)} = \overline{f(x,\mu^2)}$ for $\mu \in \mathbb{R}_1 := \{\mu : \operatorname{Re}\mu \ge 0,\ \operatorname{Im}\mu = 0\}$. Moreover, they have analytic continuation to $\overline{\mathbb{C}_1^+} := \{\mu \in \mathbb{C} : \operatorname{Re}\mu \ge 0,\ \operatorname{Im}\mu \ge 0\}$ and $\overline{\mathbb{C}_1^-} := \{\mu \in \mathbb{C} : \operatorname{Re}\mu \ge 0,\ \operatorname{Im}\mu \le 0\}$, respectively and the asymptotic behaviour

$$\begin{aligned} &f^{(1)}(x,\mu) = e^{i\mu^2 x}\left[1 + o(1)\right]\,,\ x \to \infty\,,\\ &f_x^{(1)}(x,\mu) = e^{i\mu^2 x}\left[i\mu^2 + o(1)\right]\,,\ x \to \infty \end{aligned} \tag{1.7}$$

is valid. The solutions $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$, called Jost solutions of L_λ, are respectively analytic in $\mathbb{C}_1^+ := \{\mu \in \mathbb{C} : \operatorname{Re}\mu > 0,\ \operatorname{Im}\mu > 0\}$ and $\mathbb{C}_1^- := \{\mu \in \mathbb{C} : \operatorname{Re}\mu > 0,\ \operatorname{Im}\mu < 0\}$, and they are continuous on real and imaginary axes with respect to μ ([10]). The Jost solutions can be expressed as

$$\begin{aligned} f^{(1)}(x,\mu) &= f(x,\mu^2) = e^{i\left[\alpha(x)+\mu^2 x\right]} + \int_x^\infty K(x,t)e^{i\mu^2 t}dt,\\ \overline{f^{(1)}(x,\mu)} &= \overline{f(x,\mu^2)} = e^{-i\left[\alpha(x)+\mu^2 x\right]} + \int_x^\infty K(x,t)e^{-i\mu^2 t}dt \end{aligned} \tag{1.8}$$

where $\alpha(x) = \int_x^\infty q(t)dt$ and $K(x,t)$ are solutions of integral equations of Volterra type and are continuously differentiable with respect to their arguments.

Moreover, $|K(x,t)|$, $|K_x(x,t)|$, $|K_t(x,t)|$ satisfy the following inequalities:

$$|K(x,t)| \le c\omega\left(\frac{x+t}{2}\right)\exp\left(\gamma(x)\right), \tag{1.9}$$

$$|K_x(x,t)|\,, |K_t(x,t)| \le c\left[\omega^2\left(\frac{x+t}{2}\right) + \theta\left(\frac{x+t}{2}\right)\right] \tag{1.10}$$

where

$$\begin{aligned}\omega(x) &= \int_x^{\infty}\left[\left|q(t)\right|^2 + \left|q'(t)\right|\right]dt,\\ \gamma(x) &= \int_x^{\infty} t\left[\left|q(t)\right|^2 + 2\left|q(t)\right|\right]dt,\\ \theta(x) &= \tfrac{1}{4}\left[2\left|q(x)\right|^2 + \left|q'(x)\right|\right]\end{aligned}$$

and $c > 0$ is a constant ([13]).

From (1.7) and (1.8), the Wronskian of the solutions of $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ is

$$W\left[f^{(1)}(x,\mu),\overline{f^{(1)}(x,\mu)}\right] = \lim_{x\to\infty} W\left[f^{(1)}(x,\mu),\overline{f^{(1)}(x,\mu)}\right] = -2i\mu^2$$

for $\mu \in \mathbb{R}_1$. Hence $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are the fundamental solutions of (1.4) for $\mu \in \mathbb{R}_1^* = \mathbb{R}_1 \setminus \{0\}$.

Let $\varphi^{(1)}(x,\mu) = \varphi(x,\mu^2)$ denote the solution of (1.4) satisfying the initial conditions

$$\begin{aligned}\varphi^{(1)}(0,\mu) &= \varphi(0,\mu^2) = 1,\\ \varphi_x^{(1)}(0,\mu) &= \varphi_x(0,\mu^2) - (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4).\end{aligned}$$

compatible with (1.5). We must give the following lemma which could have been proved as like [14]:

Lemma 1.1. *The identity*

$$\frac{2i\mu^2\varphi^{(1)}(x,\mu)}{f_x^{(1)}(0,\mu) + (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4)f^{(1)}(0,\mu)} = \overline{f^{(1)}(x,\mu)} - S_1(\mu)f^{(1)}(x,\mu) \tag{1.11}$$

holds for all $\mu \in \mathbb{R}_1^*$, where

$$S_1(\mu) = S(\mu^2) = \frac{\overline{F(\mu^2)}}{F(\mu^2)} = \frac{\overline{F_1(\mu)}}{F_1(\mu)} \tag{1.12}$$

and

$$\overline{S_1(\mu)} = \left[S_1(\mu)\right]^{-1}. \tag{1.13}$$

The functions $F_1(\mu) = f_x^{(1)}(0,\mu) + (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4)f^{(1)}(0,\mu)$ and $S_1(\mu)$ are respectively called Jost function and scattering function of L_μ. $S_1(\mu)$ is meromorphic function in $\mathbb{C}_1^+$ and the poles councide wtih the zeros of $F_1(\mu)$. After examining the properties of zeros of $F_1(\mu)$ then we can say that zeros of $F_1(\mu)$ may have only a finite number on the imaginary axis and they are all simple. Also, $S_1(\mu) = 1 + O\left(\frac{1}{\mu^2}\right)$ for $\left|\mu\right| \to \infty$ under the condition (1.6). These relations were obtained in [14].

To find the unique kernel $K(x,y)$ of the solution $f^{(1)}(x,\mu)$ and the potential $q(x) = -\frac{1}{2}\frac{d}{dx}K(x,x)$, it suffices to find the Levinson formula for L_μ.

2. The Continuity of the Scattering Function $S_1(\mu)$

Lemma 2.1. *If the function*

$$F_{S_1}(x) = \frac{1}{\pi}\int_0^\infty \mu\left[1 - S_1(\mu)\right] e^{i\mu^2 x} d\mu \tag{2.1}$$

is Fourier transformation of $\mu\left[1 - S_1(\mu)\right]$ *for all* $x \geq 0$, *it belongs to the* $L_2(0, \infty)$ *space.*

Proof. We can easily verify that

$$\mu\left[1 - S_1(\mu)\right] \approx O\left(\frac{1}{\mu}\right), \quad |\mu| \to \infty.$$

It follows that $\mu\left[1 - S_1(\mu)\right] \in L_2(0, \infty)$ and hence the function $F_{S_1}(x)$ also belongs to the space $L_2(0, \infty)$. □

Definition 2.2. *For* $k = 1, 2, ..., n$,

$$m_k^{-1} = \frac{\left[f^{(1)}(0, \mu_k)\right]^2}{\mu_k^2}\left\{\frac{1}{\left|f^{(1)}(0, \mu_k)\right|^2}\int_0^\infty \left[q(x) - \mu_k^2\right]\left|f^{(1)}(x, \mu_k)\right|^2 dx - \frac{\alpha_1 + 2\alpha_2\mu_k^2}{2}\right\}$$

where μ_k *are zeros of Jost function on the upper imaginary axis.*

Lemma 2.3. *The kernel function* $K(x, t)$ *satisfies the following equation*

$$e^{i\alpha(x)}G(x + y) + K(x, y) + \int_x^\infty K(x, t)G(t + y)dt = 0\,, \quad (x < y) \tag{2.2}$$

where

$$G(x) = \sum_{k=1}^n m_k e^{i\mu_k^2 x} + F_{S_1}(x). \tag{2.3}$$

Proof. Lets rewrite (1.11) as follows

$$\frac{2i\mu^2\varphi^{(1)}(x, \mu)}{F_1(\mu)} = \overline{f^{(1)}(x, \mu)} - S_1(\mu)f^{(1)}(x, \mu),$$

and substitute $f^{(1)}(x, \mu)$ in this by its expressions (1.8), we get that

$$\frac{2i\mu^2\varphi^{(1)}(x, \mu)}{F_1(\mu)} = e^{-i\left[\alpha(x) + \mu^2 x\right]} + \int_x^\infty K(x, t)e^{-i\mu^2 t}dt - S_1(\mu)\left[e^{i\left[\alpha(x) + \mu^2 x\right]} + \int_x^\infty K(x, t)e^{i\mu^2 t}dt\right].$$

Also, as necessary arrangements are made and equation (2.1) is used, we reach

$$\frac{2i}{\pi}\int_0^\infty \frac{\mu^3\varphi^{(1)}(x, \mu)e^{i\mu^2 y}}{F_1(\mu)}d\mu = e^{i\alpha(x)}F_{S_1}(x + y) + K(x, y) + \int_x^\infty K(x, t)F_{S_1}(t + y)dt. \tag{2.4}$$

By using Jordan Lemma and Residue Theorem,

$$\begin{aligned} \frac{2i}{\pi}\int_0^\infty \frac{\mu^3\varphi^{(1)}(x,\mu)e^{i\mu^2 y}}{F_1(\mu)}d\mu &= 2\pi i\frac{2i}{\pi}\sum_{k=1}^n \operatorname{Res}(F_1,\mu_k) \\ &= -\sum_{k=1}^n \frac{4\mu_k^3\varphi^{(1)}(x,\mu_k)e^{i\mu_k^2 y}}{(\dot{F_1})(\mu_k)} \end{aligned}$$

and then

$$\frac{2i}{\pi}\int_0^\infty \frac{\mu^3\varphi^{(1)}(x,\mu)}{F_1(\mu)}e^{i\mu^2 y}d\mu = \sum_{k=1}^n m_k f^{(1)}(x,\mu_k)e^{i\mu_k^2 y}$$

because of the fact that $\varphi^{(1)}(x,\mu_k)$ and $f^{(1)}(x,\mu_k)$ are linearly dependent with $\varphi^{(1)}(x,\mu_k) = \frac{f^{(1)}(x,\mu_k)}{f^{(1)}(0,\mu_k)}$ since $F_1(\mu_k) = 0$. If we consider the last equation and (2.4) together, we get

$$\sum_{k=1}^n m_k\left[f^{(1)}(x,\mu_k)e^{i\mu_k^2 y}\right] = e^{i\alpha(x)}F_{S_1}(x+y) + K(x,y) + \int_x^\infty K(x,t)F_{S_1}(t+y)dt,$$

and from (2.3), we obtain (2.2). □

Lemma 2.4. *For $F_1(0) = 0$,*

$$-K(0,0) + \int_0^\infty K_x(0,t)dt + \left[\alpha_0 + i\alpha'(0)\right]e^{i\alpha(0)} + \alpha_0\int_0^\infty K(0,t)dt = 0. \tag{2.5}$$

Proof. From (1.8),

$$\begin{aligned} f^{(1)}(0,0) &= e^{i\alpha(0)} + \int_0^\infty K(0,t)dt, \\ f_x^{(1)}(0,0) &= i\alpha'(0)e^{i\alpha(0)} - K(0,0) + \int_0^\infty K_x(0,t)dt. \end{aligned}$$

As last equations are substituted in $F_1(0) = 0$, we hold (2.5). □

Lemma 2.5. *Following integral equation*

$$K_1(z) = \int_z^\infty \left[K_x(0,y) + \alpha_0 e^{i\alpha(0)}K(0,y)\right]dy \tag{2.6}$$

belongs to $L_1(0,\infty)$ space and $K_1(z)$ is bounded.

Proof. The equation (2.6) can be proved belongs $L_1(0,\infty)$ space easily using equations (1.9) and (1.10).

Substituting $x = 0$ into the main equation (2.3) and using partial integrate respect to x, we get

$$\begin{aligned} 0 &= i\alpha'(0)e^{i\alpha(0)}G(y) + e^{i\alpha(0)}G_x(y) + K_x(0,y) \\ &\quad - K(0,0)G(y) + \int_0^\infty K_x(0,t)G(t+y)dt. \end{aligned} \tag{2.7}$$

If we multiple both sides of the main equation (2.2) by α_0 and we add to (2.7) then we get

$$0 = \int_z^\infty [K_x(0,y) + \alpha_0 K(0,y)]\, dy + \left[(\alpha_0 + i\alpha'(0))\, e^{i\alpha(0)} - K(0,0)\right] \int_z^\infty G(y)dy$$
$$- e^{i\alpha(0)} G(z) + \int_0^\infty [K_x(0,t) + \alpha_0 K(0,t)]\, dt \int_{z+t}^\infty G(\xi)d\xi \tag{2.8}$$

including

$$-d\left\{\int_t^\infty [K_x(0,y) + \alpha_0 K(0,y)]\, dy\right\} = K_x(0,t) + \alpha_0 K(0,t). \tag{2.9}$$

If (2.9) is used in integral equation (2.8), we get

$$0 = K_1(z) + \left[(\alpha_0 + i\alpha'(0))\, e^{i\alpha(0)} - K(0,0)\right] \int_z^\infty G(y)dy$$
$$- e^{i\alpha(0)} G(z) - \int_0^\infty \left(\int_{z+t}^\infty G(\xi)d\xi\right) dK_1(t)dt$$
$$= K_1(z) + \left[(\alpha_0 + i\alpha'(0))\, e^{i\alpha(0)} - K(0,0) + K_1(0)\right] \int_z^\infty G(y)dy$$
$$- e^{i\alpha(0)} G(z) - \int_0^\infty K_1(t) G(z+t)dt. \tag{2.10}$$

From (2.8),

$$i\alpha'(0)e^{i\alpha(0)} + \alpha_0 e^{i\alpha(0)} - K(0,0) = -\alpha_0 \int_0^\infty K(0,t)dt - \int_0^\infty K_x(0,t)dt. \tag{2.11}$$

As $K_1(0)$ is added to both sides of (2.11), we obtain

$$i\alpha'(0)e^{i\alpha(0)} + \alpha_0 e^{i\alpha(0)} - K(0,0) + K_1(0) = 0. \tag{2.12}$$

If last equation is used in (2.10), we get

$$K_1(z) - \int_0^\infty K_1(t)G(z+t)dt = e^{i\alpha(0)} G(z).$$

Finally, we have shown that the function $K_1(z)$ is a bounded solution to L_μ. □

Theorem 2.6. *Under the condition (1.6), the scattering function $S_1(\mu)$ is continuous in $\mathbb{R}_1$ when α_1 is not equal to* $[-\sin\alpha(0)]^{-1}\left[\cos\alpha(0) + \int_0^\infty K_1(t)dt\right]$ *or* $\sin\alpha(0)\left[\cos\alpha(0) + \int_0^\infty K(0,t)dt\right]^{-1}$.

Proof. It is clear that $F_1(\mu)$ does not have zero in $\mathbb{R}_1^*$. Moreover, if $F_1(0) \neq 0$ then the function $S_1(\mu) = \frac{\overline{F_1(\mu)}}{F_1(\mu)}$ is continuous at $\mu = 0$.

Now we shall prove the continuity of the function $S_1(\mu)$ for $\mu = 0$ in case of $F_1(0) = 0$.

The Jost function $F_1(\mu)$ can be written via (1.8) that

$$\begin{aligned} F_1(\mu) &= i\left(\alpha'(0) + \mu^2\right)e^{i\alpha(0)} - K(0,0) + \int_0^\infty K_x(0,t)e^{i\mu^2 t}dt \\ &\quad + \alpha_0\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right] + \alpha_1\mu^2\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right] \\ &\quad + \alpha_2\mu^4\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right]. \end{aligned} \tag{2.13}$$

If we define I_1 as

$$I_1 = -K(0,0) + \int_0^\infty K_x(0,t)e^{i\mu^2 t}dt + \alpha_0\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right], \tag{2.14}$$

we get that $F_1(\mu)$ as follows

$$\begin{aligned} F_1(\mu) &= I_1 + \alpha_1\mu^2\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right] \\ &\quad + \alpha_2\mu^4\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right] + i\left(\alpha'(0) + \mu^2\right)e^{i\alpha(0)}. \end{aligned} \tag{2.15}$$

If we apply the partial integration to I_1, then we hold

$$\begin{aligned} I_1 &= -K(0,0) + \alpha_0 e^{i\alpha(0)} + \int_0^\infty K_x(0,y)dy + \alpha_0\int_0^\infty K(0,y)dy \\ &\quad + i\mu^2\int_0^\infty\int_t^\infty K_x(0,y)e^{i\mu^2 t}dydt + i\mu^2\alpha_0\int_0^\infty\int_t^\infty K(0,y)e^{i\mu^2 t}dydt. \end{aligned} \tag{2.16}$$

From Lemma 2.4, we get

$$\begin{aligned} I_1 &= -i\alpha'(0)e^{i\alpha(0)} + i\mu^2\int_0^\infty\int_t^\infty K_x(0,y)e^{i\mu^2 t}dydt + i\mu^2\alpha_0\int_0^\infty\int_t^\infty K(0,y)e^{i\mu^2 t}dydt \\ &= -i\alpha'(0)e^{i\alpha(0)} + i\mu^2\int_0^\infty\int_t^\infty \left[K_x(0,y) + \alpha_0 K(0,y)\right]e^{i\mu^2 t}dydt. \end{aligned}$$

By using Lemma 2.5, we obtain that

$$I_1 = -i\alpha'(0)e^{i\alpha(0)} + i\mu^2\int_0^\infty K_1(t)e^{i\mu^2 t}dt.$$

If I_1 is written in the integral equation (2.15), we get

$$\begin{aligned} F_1(\mu) &= -i\alpha'(0)e^{i\alpha(0)} + i\left(\alpha'(0)+\mu^2\right)e^{i\alpha(0)} + i\mu^2\int_0^\infty K_1(t)e^{i\mu^2 t}dt \\ &\quad +\alpha_1\mu^2\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right] + \alpha_2\mu^4\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right] \\ &= i\mu^2\left\{e^{i\alpha(0)} + \int_0^\infty K_1(t)e^{i\mu^2 t}dt\right. \\ &\quad \left. -i\alpha_1\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right] - i\alpha_2\mu^2\left[e^{i\alpha(0)} + \int_0^\infty K(0,t)e^{i\mu^2 t}dt\right]\right\}. \end{aligned}$$

From $K_1(t) \in L_1(0,\infty)$ and $K(0,t) \in L_1(0,\infty)$ and definition of $\alpha(x)$, we can rewrite $F_1(\mu)$ as follows

$$\begin{aligned} F_1(\mu) &= i\mu^2\widetilde{K}(\mu), \\ \overline{F_1(\mu)} &= -i\mu^2\overline{\widetilde{K}(\mu)} \end{aligned}$$

where

$$\begin{aligned} \widetilde{K}(\mu) &= e^{i\alpha(0)}\left(1 - i\alpha_1 - i\alpha_2\mu^2\right) + \int_0^\infty K_1(t)e^{i\mu^2 t}dt \\ &\quad - i\left(\alpha_1 + \alpha_2\mu^2\right)\int_0^\infty K(0,t)e^{i\mu^2 t}dt, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \overline{\widetilde{K}(\mu)} &= e^{-i\alpha(0)}\left(1 + i\alpha_1 + i\alpha_2\mu^2\right) + \int_0^\infty K_1(t)e^{-i\mu^2 t}dt \\ &\quad + i\left(\alpha_1 + \alpha_2\mu^2\right)\int_0^\infty K(0,t)e^{-i\mu^2 t}dt. \end{aligned}$$

From definition of scattering function, we obtained following result

$$S_1(\mu) = \frac{-i\mu^2\overline{\widetilde{K}(\mu)}}{i\mu^2\widetilde{K}(\mu)} = -\frac{\overline{\widetilde{K}(\mu)}}{\widetilde{K}(\mu)}.$$

and for $\mu = 0$,

$$S_1(0) = -\frac{\overline{\widetilde{K}(0)}}{\widetilde{K}(0)}$$

where $\operatorname{Re}\widetilde{K}(0) = \alpha_1\sin\alpha(0) + \cos\alpha(0) + \int_0^\infty K_1(t)dt$ and $\operatorname{Im}\widetilde{K}(0) = \sin\alpha(0) - \alpha_1\left[\cos\alpha(0) + \int_0^\infty K(0,t)dt\right]$. So, the scattering function $S_1(\mu)$ is continuous at $\mu = 0$ because $\operatorname{Re}\widetilde{K}(0) \neq 0$ or $\operatorname{Im}\widetilde{K}(0) \neq 0$. As a result, $S_1(\mu)$ is continuous in $\mathbb{R}_1$. $\square$

3. The Levinson Formula of L_μ

Lemma 3.1. *Let the following identity*

$$F_1(\mu) = re^{i\theta(\mu)} \tag{3.1}$$

holds where $\arg F_1(\mu) = \theta(\mu)$.

Theorem 3.2. *The following formula*

$$\frac{\theta(\infty) - \theta_i(\infty)}{2\pi} + C(\alpha_1) + T(F_1) = n, \tag{3.2}$$

is valid where

$$\theta_i(\tau) = \theta(i\tau),$$

$$C(\alpha_1) = \begin{cases} \frac{1}{2}, & \text{if } \alpha_1 \neq 0 \\ 1, & \text{if } \alpha_1 = 0 \end{cases},$$

$$T(F_1) = \begin{cases} 0, & \text{if } F_1(0) \neq 0 \\ -\frac{1}{2}, & \text{if } F_1(0) = 0 \end{cases}$$

and n is the number of the zeros of the function $F_1(\mu)$ on the first quarter of complex plane. The equation (3.2) is called the Levinson Type Formula.

Proof. We define the following equation for sufficiently large $R > 0$ and sufficiently little $\varepsilon > 0$,

$$\Gamma^+_{R,\varepsilon} = C^+_R \cup C^-_\varepsilon \cup C_{\mathrm{Im}} \cup C_{\mathrm{Re}}$$

where C^+_R and C^-_ε are quarter circles with centers in origin and corresponding radius of R and ε, respectively. C^+_R is oriented by opposite of clockwise and C^-_ε is oriented by clockwise. Also, C_{Re} and C_{Im} are line segments oriented points ε to R and iR to $i\varepsilon$, respectively (see Figure 3.1).

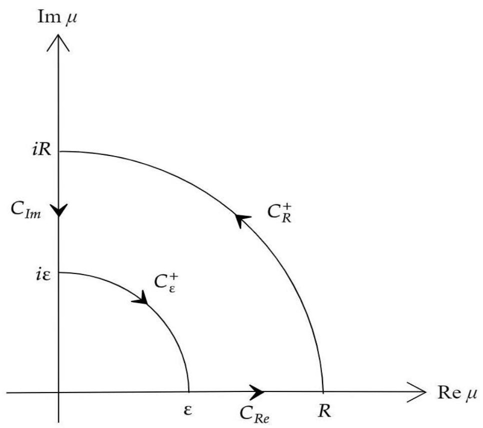

Figure 3.1

The function $F_1(\mu)$ is analytic inside the curve $\Gamma^+_{R,\varepsilon}$, and is continuous on the boundary of $\Gamma^+_{R,\varepsilon}$. Furthermore, $\Gamma^+_{R,\varepsilon}$ does not include the finite zeros of $F_1(\mu)$ defined as μ_k, $(k = 1, 2, ..., n)$. By the argument principle, we can write

$$\begin{aligned} n &= \frac{1}{2\pi}\Delta\Gamma^+_{R,\varepsilon} \arg F_1(\mu) = \frac{1}{2\pi}\Delta\Gamma^+_{R,\varepsilon}\theta(\mu) \\ &= \frac{1}{2\pi}\Delta C^+_R\theta(\mu) + \frac{1}{2\pi}\Delta C^-_\varepsilon\theta(\mu) \\ &+ \frac{1}{2\pi}\Delta C_{\mathrm{Re}}\theta(\mu) + \frac{1}{2\pi}\Delta C_{\mathrm{Im}}\theta(\mu). \end{aligned} \tag{3.3}$$

If we use the asymptotic equation for Jost function in [14] as

$$F_1(\mu) \approx \begin{cases} (\alpha_1 + i)e^{i\alpha(0)}\mu^2 & , \ \alpha_1 \neq 0, \ |\mu| \to \infty \\ \alpha_2\mu^4 & , \ \alpha_1 = 0, \ |\mu| \to \infty \end{cases}$$

and following equalities

$$\begin{aligned} \arg F_1(\mu) &= \theta(\mu) = \arg\left[(\alpha_1 + i)e^{i\alpha(0)}\right] + 2\arg\mu, \ \ \alpha_1 \neq 0, \\ \arg F_1(\mu) &= \theta(\mu) = \arg(\alpha_2) + 4\arg\mu, \ \ \alpha_1 = 0, \end{aligned}$$

then we can obtain

$$\frac{1}{2\pi}\lim_{R\to\infty}\Delta C^+_R\theta(\mu) = \frac{1}{2\pi}\begin{cases} \pi \ , & \alpha_1 \neq 0 \\ 2\pi \ , & \alpha_1 = 0 \end{cases}$$

and

$$C(\alpha_1) = \begin{cases} \frac{1}{2} \ , \text{ if } \alpha_1 \neq 0 \\ 1 \ , \text{ if } \alpha_1 = 0 \end{cases} , \tag{3.4}$$

because

$$\lim_{R\to\infty}\Delta C^+_R\left[(\alpha_1 + i)e^{i\alpha(0)}\right] = \lim_{R\to\infty}\Delta C^+_R\left[\arg(\alpha_2)\right] = 0$$

and

$$\lim_{R\to\infty}\Delta C^+_R\left[\arg\mu\right] = \frac{\pi}{2}.$$

Moreover, we can write $F_1(\mu)$ by using (2.17) as follows

$$F_1(\mu) \approx \begin{cases} F_1(0) & , \ \text{if } F_1(0) \neq 0 \\ i\mu^2\widetilde{K}(0) , & \text{if } F_1(0) = 0 \end{cases}$$

for $\operatorname{Re}\mu \geq 0$, $\operatorname{Im}\mu \geq 0$, $|\mu| \to 0$.

Thus, we hold

$$T(F_1) = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\Delta C^-_\varepsilon\theta(\mu) = \frac{1}{2\pi}\begin{cases} 0 \ , \text{ if } F_1(0) \neq 0 \\ -\pi \ , \text{ if } F_1(0) = 0 \end{cases} = \begin{cases} 0 \ , \text{ if } F_1(0) \neq 0 \\ -\frac{1}{2} \ , \text{ if } F_1(0) = 0 \end{cases} . \tag{3.5}$$

Furthermore, we get following equality

$$\begin{aligned} \frac{1}{2\pi}\lim_{\substack{R\to\infty \\ \varepsilon\to 0}}\Delta\left(C_{\mathrm{Re}} \cup C_{\mathrm{Im}}\right)\theta(\mu) &= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}}\frac{1}{2\pi}\left[\theta(i\varepsilon) - \theta(iR) + \theta(R) - \theta(\varepsilon)\right] \\ &= \frac{1}{2\pi}\left[\theta(\infty) - \theta_i(\infty)\right]. \end{aligned} \tag{3.6}$$

Taking into account (3.4)-(3.6) in (3.3), we can find (3.2). □

Corollary 3.3. *Under the condition (1.6), the Levinson Formula for L_μ has also following representation*

$$\frac{\ln S_1^i(\infty) - \ln S_1(\infty)}{4\pi i} = n - C(\alpha_1) - T(F_1) \tag{3.7}$$

where

$$S_1^i(\tau) = S_1(i\tau).$$

Proof. From definition of scattering function, we can write

$$S_1(\mu) = \frac{\overline{F_1(\mu)}}{F_1(\mu)} = \frac{re^{-i\theta(\mu)}}{re^{i\theta(\mu)}} = e^{-2i\theta(\mu)}.$$

By using properties of logarithm function,

$$\theta(\mu) = -\frac{\ln S_1(\mu)}{2i}.$$

Also, we can find

$$\frac{\theta(\infty) - \theta_i(\infty)}{2\pi} = \frac{\ln S_1^i(\infty) - \ln S_1(\infty)}{4\pi i} \tag{3.8}$$

from (3.2), and then (3.7) can be obtained from (3.8). □

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