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Functional differential inclusions with maximal monotone operators and nonconvex perturbations

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Abstract. In this paper, we study the existence of solutions for functional differential inclusions governed by time-dependent maximal monotone operators with nonconvex perturbations depending on all the variables.

1. Introduction

Let H be a separable Hilbert space with the norm $\|\cdot\|$ and the scalar product $\langle\cdot,\cdot\rangle$, and I be a closed bounded interval in $\mathbb{R}$. We denote by $C(I,H)$ the Banach space of continuous functions from I to H equipped with the norm

$$\|x\|_\infty := \sup\{\|x(t)\|; t \in I\}.$$

For a a positive number, we put $C_a := C([-a,0],H)$ and for any $t \in [0,b]$, we define the operator $T(t)$ from $C([-a,b],H)$ to C_a with

$$(T(t)(x(\cdot)))(s) := (T(t)x)(s) := x(t+s),\ s \in [-a,0].$$

In this paper, we are mainly interested in the evolution problem :

$$\begin{cases} \dot{x}(t) \in -A(t)x(t) + G(t,T(t)x), \text{ a.e. } t \in [0,\tau]; \\ x(t) = \varphi(t), \quad \forall t \in [-a,0]; \\ x(t) \in D(A(t)), \quad \forall t \in [0,\tau], \end{cases} \tag{1}$$

where $A(t)$ is a maximal monotone operator with domain $D(A(t))$, φ is a continuous function, and $G : [0,b] \times C_a \longrightarrow 2^H$ is a multivalued mapping with non-empty, closed and non-convex values.

This kind of problems has been studied by several authors in the last few years. Azzam et al. [1] have studied the problem

$$-\frac{du}{dr}(t) \in A(t)u(t) + f(t,u(t)),$$

where du is the Stieltjes or the differential measure of u and dr is a positive measure. They have assumed that the mapping $t \mapsto A(t)$ has continuous bounded or Lipschitz variation on $[0,\tau]$, in the sense of Vladimirov's pseudo-distance, and the perturbation f is separately integrable on $[0,\tau]$ and separately Lipschitz on H.

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Tolstonogov, in [11], has considered the same last problem with set-valued perturbation. The author has assumed that the mapping $t \mapsto A(t)$ has Lipschitz variation on $[0, \tau]$, in the sense of Vladimirov's pseudo-distance, and the set-valued perturbation F is separately measurable on $[0, \tau]$, separately Lipschitz on H, has non-convex values and satisfies

$$d(0_H, F(t, 0_H)) \leq b(t) \text{ a.e. or } \|F(t, 0_H)\| := \sup\{\|x\|; x \in F(t, 0_H)\} \leq b(t) \text{ a.e.}, \tag{2}$$

where $b \in L^1([0, \tau], \mathbb{R}^+)$ and 0_H is the zero element of H.

In [2], Azzam et al. have considered the problem

$$-\dot{u}(t) \in A(t)u(t) + f(t, u(t)), \tag{3}$$

in the case where the mapping $t \mapsto A(t)$ has absolutely continuous variation on $[0, \tau]$. This case has been investigated by several authors, see for example [3, 7].

Vilches and Nguyen [12] have presented the existence of solutions for (3), where $D(A(t)) = H$, for all $t \in [0, \tau]$, but without assumptions concerning time regularity on A.

In [1, 2], the authors have solved the problem (1) with and without memory. The perturbation G has convex and weakly compact values and $G(t, .)$ is scalarly upper semicontinuous. The set $\bigcup_t D(A(t))$ is ball compact, i.e., its intersection with any closed ball of H is compact. The mapping $t \mapsto A(t)$ has Lipschitz variation in the functional case, and absolutely continuous variation in the ordinary case. For Lipschitz perturbation, we refer the reader to [8].

The main objective of the present work is to develop the existence theory of functional differential inclusion dependent of maximal monotone operators. We are interested in proving the existence results for (1), when G is a Lipschitz multivalued mapping with non-empty, closed and non-convex values. The main results of this work extend, to the functional case, some existence results in [1, 2, 8, 11, 12] and in the literature related to this kind of problems. We consider weaker hypothesis on A and G. The set $\bigcup_t D(A(t))$ is not necessary ball compact in the first result. The perturbation G satisfies a weaker growth condition in the second result. Some remarks deserve mentioning: the methods used in this work are different from those used in the papers [1, 2, 8, 11, 12] and it is not trivial to obtain the existence of solutions for (1) with the assumptions (2) and the method applied in [11].

The paper is organized as follows. In the next section, we introduce some notations that will be used in the sequel and we recall several properties of maximal monotone operators analysis which are involved throughout the paper. We establish in Section 3 the first existence result for the problem (1). In the last section, we give the second result for (1).

2. Preliminaries

For $x \in H$ and $r > 0$, let $B(x, r) := \{y \in H; \|y - x\| < r\}$ be the open ball of center x with radius r, $\overline{B}(x, r)$ be its closure and put $B = B(0, 1)$. For $\varphi \in C_a$ and $r > 0$, let $B_a(\varphi, r) := \{\psi \in C_a; \|\psi - \varphi\|_\infty < r\}$ be the open ball of C_a centered at φ with radius r and $\overline{B}_a(\varphi, r)$ be its closure. For closed subsets A and B of H, the Hausdorff distance between A and B is defined by

$$d_H(A, B) = \max\left\{e(A, B), e(B, A)\right\},$$

where

$$e(A, B) := \sup\left\{d(a, B); a \in A\right\} \text{ and } d(a, B) = \inf_{x \in B} \|a - x\|.$$

Now, we shortly review the definition and some useful properties of maximal monotone operators. We refer the reader to [4, 6, 14] for their basic theory and more details.

A set-valued operator A from H to H is a mapping from H into 2^H. The domain, the range and the graph of A are respectively the following sets:

$$D(A) = \{x \in H : Ax \neq \emptyset\}, \ R(A) = \{y \in H : \exists x \in D(A), y \in Ax\}$$

and

$$\text{gph}(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}.$$

We say that $A : D(A) \subset H \to 2^H$ is monotone, if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0,$$

whenever $(x_i, y_i) \in \text{gph}(A),\ i = 1, 2$. It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator. In this case, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where I_H stands for the identity mapping of H. If A is a maximal monotone operator, then, for every $x \in D(A)$, Ax is nonempty, closed and convex. So that, the projection $A^0(x)$ of the origin into Ax, exists and is unique. We define the resolvent of a maximal monotone operator A by $J_\lambda^A = (I_H + \lambda A)^{-1}$ where $\lambda > 0$. This operator is single valued and defined on all of H. Now, let $A : D(A) \subset H \longrightarrow 2^H$ and $B : D(B) \subset H \to 2^H$ be two maximal monotone operators. We denote by $\text{dis}(A, B)$ the pseudo-distance between A and B defined by

$$\text{dis}(A, B) = \sup \left\{ \frac{\langle y - \bar{y}, \bar{x} - x \rangle}{1 + \|y\| + \|\bar{y}\|} : (x, y) \in \text{gph}(A), (\bar{x}, \bar{y}) \in \text{gph}(B) \right\}. \tag{4}$$

We'll need the following Lemmas to prove our results.

Lemma 2.1. *[13] Let A and B be two maximal monotone operators. Then,*

1. *for $\lambda > 0$ and $x \in D(A)$*

$$\|x - J_\lambda^B(x)\| \leq \lambda \|A^0(x)\| + \text{dis}(A, B) + \sqrt{\lambda(1 + \|A^0(x)\|)\,\text{dis}(A, B)},$$

2. *J_λ^A is nonexpansive, that is, for $\lambda > 0$ and $x, \bar{x} \in H$*

$$\|J_\lambda^A(x) - J_\lambda^A(\bar{x})\| \leq \|x - \bar{x}\|.$$

Lemma 2.2. *[13] Let A be a maximal monotone operator. If $x \in \overline{D(A)}$ and $y \in H$ are such that*

$$\langle A^0(z) - y, z - x \rangle \geq 0, \quad \forall z \in D(A),$$

then $x \in D(A)$ and $y \in A(x)$.

Lemma 2.3. *[13] Let A_n, $n \in \mathbb{N}$, and A be maximal monotone operators such that* $\text{dis}(A_n, A) \longrightarrow 0$. *Suppose that $x_n \in D(A_n)$ with $x_n \longrightarrow x$ and $y_n \in A_n(x_n)$ with $y_n \longrightarrow y$ weakly for some $x, y \in H$. Then $x \in D(A)$ and $y \in A(x)$.*

Lemma 2.4. *[13] Let A_n, $n \in \mathbb{N}$, and A be maximal monotone operators such that* $\text{dis}(A_n, A) \to 0$. *Suppose further that $z_n \in D(A_n)$ with $z_n \to z$ for some $z \in D(A)$ and that $l = \sup_{n \in \mathbb{N}} \|A_n^0(z_n)\| < \infty$. Then there exists a sequence $(\zeta_n)_n$ such that*

$$\zeta_n \in D(A_n),\ \zeta_n \to z \text{ and } A_n^0(\zeta_n) \to A^0(z). \tag{5}$$

We can take $\zeta_n = J_{\lambda_n}^{A_n}(z)$ with $\lambda_n = (\|z_n - z\| + \text{dis}(A_n, A))^{\frac{1}{2}}$. In particular, if $\text{dis}(A_n, A) \to 0$ *and $\|A_n^0(x)\| \leq c(1 + \|x\|)$ for some $c > 0$, all $n \in \mathbb{N}$ and $x \in D(A_n)$, then for every $z \in D(A)$, there exists a sequence $(\zeta_n)_n$ such that* (5) *is satisfied.*

Let us recall the following lemmas that will be used in the sequel. We recall that a multifunction is said to be measurable if its graph is measurable.

Lemma 2.5. *[15]. Let Ω be a nonempty set in H. Assume that $F : [a, b] \times \Omega \to 2^H$ is a multifunction with nonempty closed values satisfying:*

- *for every $x \in \Omega$, $F(., x)$ is measurable on $[a, b]$;*

- *for every $t \in [a,b]$, $F(t,.)$ is (Hausdorff) continuous on Ω.*

Then for any measurable function $x(.) : [a,b] \to \Omega$, the multifunction $F(.,x(.))$ is measurable on $[a,b]$.

Lemma 2.6. *[15]. Let $G : [a,b] \to 2^H$ be a measurable multifunction and $y(.) : [a,b] \to H$ a measurable function. Then for any positive measurable function $r(.) : [a,b] \to \mathbb{R}^+$, there exists a measurable selection $g(.)$ of G such that for almost all $t \in [a,b]$*

$$\|g(t) - y(t)\| \quad \leq \quad d\Big(y(t), G(t)\Big) + r(t).$$

Lemma 2.7. *[5] Let $\preceq$ be a given preorder on the nonempty set $\mathcal{B}$ and let $\phi : \mathcal{B} \to \mathbb{R} \cup \{+\infty\}$ be an increasing function. Suppose that each increasing sequence in $\mathcal{B}$ is majorated in $\mathcal{B}$. Then, for each $x_0 \in \mathcal{B}$, there exists $x_1 \in \mathcal{B}$ such that $x_0 \preceq x_1$ and $\phi(x_1) = \phi(x)$ if $x_1 \preceq x$.*

Definition 2.8. *If D is a bounded set of H, then the Kuratowski's measure of noncompactness of D, $\beta(D)$ is defined by*

$$\beta(D) = \inf\Big\{d > 0 : D \text{ is covered by a finite number of sets with diameter less than } d\Big\}.$$

In the following lemma, we recall some useful properties for the measure of noncompactness β.

Lemma 2.9. *[9] Let X be an infinite dimensional real Banach space and D_1, D_2 be two bounded subsets of X. Then*

(i) $\beta(D_1) = 0 \Leftrightarrow D_1$ *is relatively compact,*
(ii) $\beta(\lambda D_1) = |\lambda|\beta(D_1)$, $\lambda \in \mathbb{R}$,
(iii) $D_1 \subseteq D_2 \Rightarrow \beta(D_1) \leq \beta(D_2)$,
(iv) $\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$,
(v) *if $x_0 \in X$ and r is a positive real number then* $\beta(B(x_0, r)) = 2r$,
(vi) $\beta(D_1 \cup D_2) = \max\{\beta(D_1), \beta(D_2)\}$.

Definition 2.10. *The variation of a function $u : [0,T] \to H$ is defined as*

$$\operatorname{var}(u) \quad = \quad \sup\left\{ \sum_{i=0}^{N-1} \|u(t_{i+1}) - u(t_i)\| : 0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T \right. \\ \left. \text{is a partition of } [0,T] \right\}.$$

u is called of bounded variation if $\operatorname{var}(u) < \infty$.

In the sequel, we will apply the following theorem.

Theorem 2.11. *[10] Let H be a Hilbert space and $(u_n)_{n\in\mathbb{N}}$ a sequence of functions $u_n : [0,T] \to H$ that is bounded uniformly in norm and variation, i.e.,*

$$\|u_n(t)\| \leq M_1, \ \forall n \in \mathbb{N}, \ \forall t \in [0,T], \text{ and } \operatorname{var}(u_n) \leq M_2, \ \forall n \in \mathbb{N}$$

for some constants $M_1, M_2 > 0$ independent of $n \in \mathbb{N}$ and $t \in [0,T]$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function u: $[0,T] \to H$ such that $\operatorname{var}(u) \leq M_2$ and $u_{n_k}(t) \to u(t)$ weakly in H for all $t \in [0,T]$, i.e., $\langle u_{n_k}(t), z\rangle \to \langle u(t), z\rangle$, for all $z \in H$, as $k \to \infty$.

3. First Result

In this section, we state and prove the first result of our paper. First, let us introduce the following hypotheses which will be used throughout this section.

(H1) Let $A(t) : D(A(t)) \subset H \longrightarrow 2^H$, for all $t \in [0, b]$, be a maximal monotone operator satisfying:

(a) there exists $K > 0$ such that

$$\mathrm{dis}(A(t), A(s)) \leq K|t - s|, \quad \forall t, s \in [0, b],$$

(b) there exists $c \geq 0$ such that

$$\|A(t)^0 x\| \leq c(1 + \|x\|) \text{ for all } t \in [0, b] \text{ and } x \in D(A(t)),$$

(H2) $G : [0, b] \times C_a \to 2^H$ is a set-valued map with nonempty closed values satisfying:

(i) for each $\psi \in C_a$, $t \mapsto G(t, \psi)$ is measurable,

(ii) there exists a function $m \in L^1([0, b], \mathbb{R}^+)$ such that for all $t \in [0, b]$ and for all $\psi_1, \psi_2 \in C_a$

$$d_H\Big(G(t, \psi_1), G(t, \psi_2)\Big) \ \leq \ m(t)\|\psi_1 - \psi_2\|_\infty,$$

(iii) for all $\varphi \in C_a$, there exist $r > 0$ and $M > 0$ such that for all $t \in [0, b]$ and for all $\psi \in \overline{B}_a(\varphi, r)$

$$\|G(t, \psi)\| := \sup_{y \in G(t,\psi)} \|y\| \ \leq \ M(1 + \|\psi\|_\infty),$$

Now, we are able to state the first result for (1).

Theorem 3.1. *If assumptions (H1) and (H2) are satisfied, for all $\varphi \in C_a$ such that $\varphi(0) \in D(A(0))$, there exist $\tau > 0$ and a continuous function $x : [-a, \tau] \longrightarrow H$, that is absolutely continuous on $[0, \tau]$, such that x is solution of the evolution problem (1).*

Proof. Fix $\varphi \in C_a$ such that $x_0 := \varphi(0) \in D(A(0))$. There exist $r > 0$ and $M > 0$ such that

$$\|G(t, \psi)\| \leq M(1 + \|\psi\|_\infty), \quad \forall (t, \psi) \in [0, b] \times \overline{B}_a(\varphi, r).$$

Let $\tau_1 > 0$ be such that

$$\tau_1 < \frac{r}{2L},$$

where

$$\begin{aligned} L_1 &= 2(M + M(r + \|\varphi\|_\infty)) + \frac{1}{2} + \frac{3}{2}c(1 + \|\varphi(0)\| + r) + \frac{3}{2}K \\ &\text{and } L = L_1 + M + M(r + \|\varphi\|_\infty). \end{aligned} \tag{6}$$

Let $\tau_2 > 0$ be such that

$$\int_0^{\tau_2} m(s)ds < \frac{1}{2}.$$

For $\varepsilon > 0$, set

$$\eta(\varepsilon) = \sup\Big\{\gamma \in]0, \varepsilon] : \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon, \ \text{ if } |t_1 - t_2| < \gamma\Big\}.$$

Note here that $\eta(\varepsilon)$ is well defined because φ is uniformly continuous on $[-a, 0]$. Put $\tau = \min\{\frac{1}{2}\eta(\frac{r}{2}), \tau_1, \tau_2, a, b\}$. On the other hand, for all $0 < \varepsilon < a$ and $y \in L^1([0, b], H)$, set $\mathcal{B}(\varepsilon, y)$ the set of all $(d, g, x, \theta, \overline{\theta})$, where $d \in]0, \tau]$, $g \in L^1([0, d[, H)$, $x : [-a, d] \to H$ is a continuous function and θ, $\overline{\theta} : [0, d[\to [0, d]$ are step functions such that:

1. $x \equiv \varphi$ on $[-a, 0]$, $x(\overline{\theta}(t)) \in D(A(\overline{\theta}(t)))$, for all $t \in [0, d[$,
2. $x(t) \in \overline{B}(\varphi(0), r)$, $T(\overline{\theta}(t))x \in \overline{B}_a(\varphi, r)$, for all $t \in [0, d[$,
3. $x(d) \in D(A(d))$, $T(d)x \in \overline{B}_a(\varphi, r)$,
4. $\|x(t) - x(s)\| \leq (t - s)L$, for all $t, s \in [0, d]$ with $s \leq t$,
5. $g(t) \in G(t, T(\theta(t))x)$, $0 \leq t - \theta(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leq \overline{\theta}(t) - t \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $\forall t \in [0, d[$,
6. $\|g(t) - y(t)\| \leq d(y(t), G(t, T(\theta(t))x)) + \varepsilon$, for almost all $t \in [0, d[$,
7. $\|\dot{x}(t) - g(t)\| \leq L_1$ and $\|\dot{x}(t)\| \leq L$, for almost all $t \in [0, d]$,
8. $\dot{x}(t) - g(t) \in -A(\overline{\theta}(t))x(\overline{\theta}(t))$, for almost all $t \in [0, d]$.

The result in the following lemma is also needed in our proof.

Lemma 3.2. *If assumptions* (H1) *and* (H2) *are satisfied, then for all* $0 < \varepsilon < a$ *and* $y \in L^1([0, b], H)$, *there exists at least one* $(\tau, g, x, \theta, \overline{\theta}) \in \mathcal{B}(\varepsilon, y)$.

Proof. Let $0 < \varepsilon < a$ and $y \in L^1([0, b], H)$. Set $x(s) = \varphi(s)$ for all $s \in [-a, 0]$. Put $x_0 = \varphi(0) \in D(A(0))$ and fix $t_1 \in]0, \inf\{\tau, \frac{1}{4}\eta(\frac{\varepsilon}{4})\}]$. The set-valued map $t \longmapsto G(t, T(0)x)$ is measurable, in view of Lemma 2.6, there exists a function $g_0 \in L^1([0, t_1], H)$ such that $g_0(t) \in G(t, T(0)x)$ for all $t \in [0, t_1]$ and

$$\|g_0(t) - y(t)\| \leq d(y(t), G(t, T(0)x)) + \varepsilon, \text{ for almost all } t \in [0, t_1].$$

Put

$$x_1 = J_1\left(x_0 + \int_0^{t_1} g_0(s)\mathrm{d}s\right),$$

where $J_1(x) := (I + h_1A(t_1))^{-1}(x)$ for all $x \in H$ and $h_1 = t_1$. Notice that, by construction, $x_1 \in D(A(t_1))$ and

$$x_1 - x_0 - \int_0^{t_1} g_0(s)\mathrm{d}s \in -h_1A(t_1)x_1.$$

Hence, by (H1) and Lemma 2.1, we have

$$\begin{aligned}
\|x_1 - x_0\| &\leq \left\|J_1\left(x_0 + \int_0^{t_1} g_0(s)\mathrm{d}s\right) - J_1(x_0)\right\| + \|J_1(x_0) - x_0\| \\
&\leq \int_0^{t_1} \|g_0(s)\|\mathrm{d}s + h_1\|A(0)^0x_0\| + \mathrm{dis}(A(t_1), A(0)) \\
&\quad + \sqrt{h_1(1 + \|A(0)^0x_0\|)\,\mathrm{dis}(A(t_1), A(0))} \\
&\leq \int_0^{t_1} \|g_0(s)\|\mathrm{d}s + t_1c(1 + \|x_0\|) + Kt_1 \\
&\quad + \frac{1}{2}t_1(1 + c(1 + \|x_0\|)) + \frac{1}{2}Kt_1 \\
&\leq t_1\Big(M + M\|\varphi\|_\infty + \frac{1}{2} + \frac{3}{2}c(1 + \|x_0\|) + \frac{3}{2}K\Big).
\end{aligned}$$

Then, using this last relation, we get

$$\left\|x_1 - \left(x_0 + \int_0^{t_1} g_0(s)ds\right)\right\| \leq t_1L_1.$$

Now, set $d_0 = t_1$ and

$$x_0(t) = x_0 + \frac{t}{d_0}\left(x_1 - x_0 - \int_0^{d_0} g_0(s)ds\right) + \int_0^t g_0(s)ds, \tag{7}$$

for all $t \in [0, d_0]$. Observe that

$$\|x_0(t) - x_0(s)\| \leq (t-s)L, \text{ for all } t, s \in [0, d_0] \text{ with } s \leq t.$$

Also, for almost all $t \in [0, d_0]$

$$\dot{x}_0(t) = \frac{1}{d_0}\left(x_1 - x_0 - \int_0^{d_0} g_0(s)ds\right) + g_0(t).$$

Next, set $\theta_0(t) = 0$ and $\overline{\theta}_0(t) = d_0$ for all $t \in [0, d_0[$. We conclude that

$$\dot{x}_0(t) - g_0(t) \in -A(\overline{\theta}_0(t))x_0(\overline{\theta}_0(t)), \ \|\dot{x}_0(t) - g_0(t)\| \leq L_1 \text{ and } \|\dot{x}_0(t)\| \leq L,$$

for almost all $t \in [0, d_0]$. On the other hand, from (H2), (6) and (7), we obtain

$$\|x_0(t) - \varphi(0)\| \leq tL_1 + \int_0^t M(1 + \|\varphi\|_\infty)ds \leq tL \leq \frac{r}{2}.$$

Which is equivalent to $x_0(t) \in \overline{B}(\varphi(0), \frac{r}{2})$ for all $t \in [0, d_0]$. Now, we have to estimate $\|(T(d_0)x_0)(s) - \varphi(s)\|$ for each $s \in [-a, 0]$. If $-d_0 \leq s \leq 0$, then $(d_0 + s) \in [0, d_0]$. Thus, by the fact that $|s| \leq d_0 \leq \tau < \eta(\frac{r}{2})$, we get

$$\begin{aligned} \|(T(d_0)x_0)(s) - \varphi(s)\| &= \|x_0(d_0 + s) - \varphi(s)\| \\ &\leq \|x_0(d_0 + s) - \varphi(0)\| + \|\varphi(s) - \varphi(0)\| \\ &< r. \end{aligned}$$

For $-a \leq s \leq -d_0$, we have

$$\|(T(d_0)x_0)(s) - \varphi(s)\| = \|\varphi(d_0 + s) - \varphi(s)\| \qquad < r.$$

Therefore, $T(d_0)x \in \overline{B}_a(\varphi, r)$. Next, it is clear that $(d_0, g_0, x_0, \theta_0, \overline{\theta}_0) \in \mathcal{B}(\varepsilon, y)$. Hence $\mathcal{B}(\varepsilon, y) \neq \emptyset$. Now, consider the following preorder:

$$\begin{aligned} (d_1, g_1, x_1, \theta_1, \overline{\theta}_1) \leq (d_2, g_2, x_2, \theta_2, \overline{\theta}_2) \quad \Leftrightarrow \quad & d_1 \leq d_2, \ g_1 = g_2|_{[0,d_1[}, \ x_1 = x_2|_{[0,d_1]}, \\ & \theta_1 = \theta_2|_{[0,d_1[} \text{ and } \overline{\theta}_1 = \overline{\theta}_2|_{[0,d_1[}. \end{aligned}$$

Let $\phi : \mathcal{B}(\varepsilon, y) \to \mathbb{R}$ be the function defined by

$$\phi((d, g, x, \theta, \overline{\theta})) = d, \quad \forall (d, g, x, \theta, \overline{\theta}) \in \mathcal{B}(\varepsilon, y).$$

By definition, ϕ is increasing on $\mathcal{B}(\varepsilon, y)$. Also, if $((d_i, g_i, x_i, \theta_i, \overline{\theta}_i))_{i \in \mathbb{N}}$ is an increasing sequence in $\mathcal{B}(\varepsilon, y)$, we construct a majorant $(d, g, x, \theta, \overline{\theta})$ of $((d_i, g_i, x_i, \theta_i, \overline{\theta}_i))_{i \in \mathbb{N}}$ as follows: $d = \lim_{i \to +\infty} d_i$, $g(t) = g_i(t)$, $\theta(t) = \theta_i(t)$, $\overline{\theta}(t) = \overline{\theta}_i(t)$, for all $t \in [0, d_i[$, $x \equiv \varphi$ on $[-a, 0]$, $x(t) = x_i(t)$ for all $t \in [0, d_i]$. We claim that $(d, g, x, \theta, \overline{\theta}) \in \mathcal{B}(\varepsilon, y)$. Indeed, for all $i \in \mathbb{N}$, we have $x(d_i) = x_i(d_i) \in D(A(d_i))$. Then $x(d_i) \in D(A(d_i)) \cap \overline{B}(x_0, r)$, for all $i \in \mathbb{N}$. By Lemma 2.3, we conclude that $x(d) \in D(A(d)) \cap \overline{B}(x_0, r)$. The rest is self-evident. After that, we'll need the claim below to apply Lemma 2.7.

Claim 3.3. *For all $(d, g, x, \theta, \overline{\theta}) \in \mathcal{B}(\varepsilon, y)$ with $d < \tau$, there exists $(d_1, g_1, x_1, \theta_1, \overline{\theta}_1) \in \mathcal{B}(\varepsilon, y)$ such that $(d, g, x, \theta, \overline{\theta}) \leq (d_1, g_1, x_1, \theta_1, \overline{\theta}_1)$ and $\phi((d, g, x, \theta, \overline{\theta})) < \phi((d_1, g_1, x_1, \theta_1, \overline{\theta}_1))$.*

Proof. Let $(d, g, x, \theta, \overline{\theta}) \in \mathcal{B}(\varepsilon, y)$ with $d < \tau$. Fix $h \in]0, \inf\{\tau - d, \frac{1}{4}\eta(\frac{\varepsilon}{4})\}]$. There exists a function $\tilde{g} \in L^1([d, d+h], H)$ such that $\tilde{g}(t) \in G(t, T(d)x)$ for all $t \in [d, d+h]$ and

$$\|\tilde{g}(t) - y(t)\| \leq d(y(t), G(t, T(d)x)) + \varepsilon, \text{ for almost all } t \in [d, d+h].$$

Set

$$\tilde{x} = J_h\left(x(d) + \int_d^{d+h} \tilde{g}(s)\mathrm{d}s\right) \text{ where } J_h(x) := (I + hA(d+h))^{-1}(x),$$

for all $x \in H$. Notice that, by construction, $\tilde{x} \in D(A(d+h))$ and

$$\tilde{x} - x(d) - \int_d^{d+h} \tilde{g}(s)\mathrm{d}s \in -hA(d+h)\tilde{x}.$$

Next, set $\bar{d} = d + h$ and

$$\tilde{x}(t) = x(d) + \frac{t-d}{\bar{d}-d}\left(\tilde{x} - x(d) - \int_d^{\bar{d}} \tilde{g}(s)ds\right) + \int_d^t \tilde{g}(s)ds, \quad \forall t \in [d, \bar{d}].$$

Now, for all $t \in [d, \bar{d}]$, we have

$$\begin{aligned}
\|\tilde{x} - x(d)\| &\leq \left\| J_h\left(x(d) + \int_d^{\bar{d}} \tilde{g}(s)\mathrm{d}s\right) - J_h(x(d)) \right\| + \|J_h(x(d)) - x(d)\| \\
&\leq \int_d^{\bar{d}} \|\tilde{g}(s)\|\mathrm{d}s + h\|A(d)^0 x(d)\| + \mathrm{dis}(A(\bar{d}), A(d)) \\
&\qquad + \sqrt{h(1 + \|A(d)^0 x(d)\|)\,\mathrm{dis}(A(\bar{d}), A(d))} \\
&\leq \int_d^{\bar{d}} \|\tilde{g}(s)\|\mathrm{d}s + (\bar{d} - d)c(1 + \|x(d)\|) + (\bar{d} - d)K \\
&\qquad + \frac{1}{2}(\bar{d} - d)(1 + c(1 + \|x(d)\|)) + \frac{1}{2}(\bar{d} - d)K \\
&\leq (\bar{d} - d)\Big[M + M(r + \|\varphi\|_\infty) + \frac{1}{2} + \frac{3}{2}c(1 + r + \|\varphi(0)\|) + \frac{3}{2}K\Big].
\end{aligned}$$

Then, by (6), we get

$$\left\| \tilde{x} - \left(x(d) + \int_d^{\bar{d}} g(s)ds\right) \right\| \leq (\bar{d} - d)L_1,$$

and

$$\|\tilde{x}(t) - x(d)\| \leq (t - d)L, \quad \forall t \in [d, \bar{d}].$$

We claim that $\tilde{x}$ is absolutely continuous. Indeed, for all t and s in $[d, \bar{d}]$, $s < t$, one has

$$\tilde{x}(t) - \tilde{x}(s) = \frac{t-s}{\bar{d}-d}\left(\tilde{x} - x(d) - \int_d^{\bar{d}} \tilde{g}(\tau)d\tau\right) + \int_s^t \tilde{g}(\tau)d\tau.$$

Then, we get

$$\|\tilde{x}(t) - \tilde{x}(s)\| \quad \leq \quad (t - s)\Big[L_1 + M + M(\|\varphi\|_\infty + r)\Big] = (t - s)L.$$

Hence, $\tilde{x}$ is absolutely continuous. Remark that, for almost all t in $[d, \bar{d}]$,

$$\dot{\tilde{x}}(t) = \frac{1}{\bar{d}-d}\left(\tilde{x} - x(d) - \int_d^{\bar{d}} \tilde{g}(s)ds\right) + \tilde{g}(t).$$

So, for almost all $t \in [d, \bar{d}]$,

$$\|\dot{\tilde{x}}(t) - \tilde{g}(t)\| \leq L_1, \quad \|\dot{\tilde{x}}(t)\| \leq L \text{ and } \dot{\tilde{x}}(t) - \tilde{g}(t) \in -A(\bar{d})\tilde{x}(\bar{d}),$$

Similarly, for all $t \in [d, \bar{d}]$, we have

$$\begin{aligned} \|\tilde{x}(t) - \varphi(0)\| &\leq \|\tilde{x}(t) - x(d)\| + \|x(d) - \varphi(0)\| \\ &\leq (t-d)L + dL \\ &\leq \frac{r}{2}. \end{aligned}$$

Then, $\tilde{x}(t) \in \overline{B}(\varphi(0), \frac{r}{2})$ for all $t \in [d, \bar{d}]$. Next, we define g_1, θ_1, $\overline{\theta}_1$ and x_1 as follows: $g_1(t) = g(t)$, $\theta_1(t) = \theta(t)$, $\overline{\theta}_1(t) = \overline{\theta}(t)$ for all $t \in [0, d[$ and $g_1(t) = \tilde{g}(t)$, $\theta_1(t) = d$, $\overline{\theta}_1(t) = \bar{d}$ for all $t \in [d, \bar{d}[$, $x_1 \equiv \varphi$ on $[-a, 0]$, $x_1(t) = x(t)$ for all $t \in [0, d]$ and $x_1(t) = \tilde{x}(t)$ for all $t \in [d, \bar{d}]$. Now, let $s \in [-a, 0]$. If $s \leq -\bar{d}$, we have

$$\|T(\bar{d})x_1(s) - \varphi(s)\| = \|x_1(s + \bar{d}) - \varphi(s)\| = \|\varphi(s + \bar{d}) - \varphi(s)\| \leq r.$$

If $s \geq -\bar{d}$, one has

$$\|T(\bar{d})x_1(s) - \varphi(s)\| = \|x_1(s + \bar{d}) - \varphi(s)\| \leq \|x_1(s + \bar{d}) - \varphi(0)\| + \|\varphi(s) - \varphi(0)\| \leq r.$$

Consequently, we obtain $T(\bar{d})x_1 \in \overline{B}_a(\varphi, r)$. Finally, we conclude that $(\bar{d}, g_1, x_1, \theta_1, \overline{\theta}_1) \in \mathcal{B}(\varepsilon, y)$, $(d, g, x, \theta, \overline{\theta}) \leq (\bar{d}, g_1, x_1, \theta_1, \overline{\theta}_1)$ and $\phi((d, g, x, \theta, \overline{\theta})) < \phi((\bar{d}, g_1, x_1, \theta_1, \overline{\theta}_1))$. □

Now, we are ready to complete the proof of Lemma 3.2. From Lemma 2.7, there exists $(d, g, x, \theta, \overline{\theta}) \in \mathcal{B}(\varepsilon, y)$ such that $\phi((d, g, x, \theta, \overline{\theta})) = \phi((d_1, g_1, x_1, \theta_1, \overline{\theta}_1))$ and $(d, g, x, \theta, \overline{\theta}) \leq (d_1, g_1, x_1, \theta_1, \overline{\theta}_1)$ for all $(d_1, g_1, x_1, \theta_1, \overline{\theta}_1) \in \mathcal{B}(\varepsilon, y)$. Moreover, if $\phi((d, g, x, \theta, \overline{\theta})) < \tau$, by Claim 3.3, there exists $(d_1, g_1, x_1, \theta_1, \overline{\theta}_1) \in \mathcal{B}(\varepsilon, y)$ such that $(d, g, x, \theta, \overline{\theta}) \leq (d_1, g_1, x_1, \theta_1, \overline{\theta}_1)$ and $\phi((d, g, x, \theta, \overline{\theta})) < \phi((d_1, g_1, x_1, \theta_1, \overline{\theta}_1))$. Hence, $\phi((d, g, x, \theta, \overline{\theta})) = \tau$. □

In the rest, we will finish the proof of Theorem 3.1. Let $(\varepsilon_n)_{n\geq 1}$ be a strictly decreasing sequence of positive scalars such that $0 < \varepsilon_n < a$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. In view of Lemma 3.2, we can define inductively sequences $(g_n)_{n\geq 1} \subset L^1([0, \tau[, H)$, $(x_n)_{n\geq 1} \subset C([-a, \tau], H)$ and $(\theta_n)_{n\geq 1}$, $(\overline{\theta}_n)_{n\geq 1} \subset S([0, \tau[, [0, \tau])$, where $S([0, \tau[, [0, \tau])$ denotes the space of step functions from $[0, \tau[$ into $[0, \tau]$, such that

$$x_n \equiv \varphi \text{ on } [-a, 0], \; x_n(\overline{\theta}_n(t)) \in D(A(\overline{\theta}_n(t))), \; \forall t \in [0, \tau[, \tag{8}$$

$$x_n(t) \in \overline{B}(\varphi(0), r), \; \forall t \in [0, \tau], \; T(\theta_n(t))x_n \in \overline{B}_a(\varphi, r), \; \forall t \in [0, \tau[, \tag{9}$$

$$\|x_n(t) - x_n(s)\| \leq (t-s)L, \quad \forall s, t \in [0, \tau], \; s \leq t, \tag{10}$$

$$\|\dot{x}_n(t) - g_n(t)\| \leq L_1, \; \|\dot{x}_n(t)\| \leq L, \text{ for almost all } t \in [0, \tau], \tag{11}$$

$$g_n(t) \in G(t, T(\theta_n(t))x_n), \; \forall t \in [0, \tau[, \tag{12}$$

$$0 \leq t - \theta_n(t) \leq \tfrac{1}{4}\eta(\tfrac{\varepsilon_n}{4}), \; 0 \leq \overline{\theta}_n(t) - t \leq \tfrac{1}{4}\eta(\tfrac{\varepsilon_n}{4}), \; \forall t \in [0, \tau[,$$

$$\|g_{n+1}(t) - g_n(t)\| \leq d(g_n(t), G(t, T(\theta_{n+1}(t))x_{n+1})) + \varepsilon_{n+1}, \text{ a.e. on } [0, \tau], \tag{13}$$

$$\dot{x}_n(t) - g_n(t) \in -A(\overline{\theta}_n(t))x_n(\overline{\theta}_n(t)), \text{ for almost all } t \in [0, \tau].$$

Step 1: *Convergence of $(x_n)_n$ to some absolutely continuous mapping x.* The sequence $(x_n)_n$ is bounded in norm and in variation. Indeed, by (9), we have

$$\sup_{n\in\mathbb{N}^*} \|x_n\|_\infty \leq r + \|\varphi(0)\| := R_1,$$

and by (10)

$$\sup_{n\in\mathbb{N}^*} \mathrm{var}(x_n) \leq \tau L.$$

Hence, by Theorem 2.11, there is a continuous map with bounded variation $x : [0,\tau] \to H$ such that $x_n(t) \to x(t)$ weakly for all $t \in [0,\tau]$. By taking lim inf as $n \to \infty$ in (10), we obtain

$$\|x(t) - x(s)\| \leq (t-s)L, \text{ for all } 0 \leq s \leq t \leq \tau.$$

So, x is absolutely continuous satisfying $x(0) = x_0$. Next, by (11)

$$\|\dot{x}_n\|_{L^2}^2 = \int_0^\tau \|\dot{x}_n(s)\|^2 \,\mathrm{d}s \leq \tau L^2.$$

Then, the sequence $(\dot{x}_n)_n$ is bounded in $L^2([0,\tau],H)$. Hence, we can extract a subsequence, still denoted $(\dot{x}_n)_n$, which converges weakly in $L^2([0,\tau],H)$ to some mapping $w \in L^2([0,\tau],H)$. Then, for all $t \in [0,\tau]$ and $u \in H$, one has

$$\begin{aligned}\langle u, x(t) - x(0)\rangle &= \lim_{n\to\infty}\langle u, x_n(t) - x_n(0)\rangle \\ &= \lim_{n\to\infty}\left\langle u, \int_{]0,t]} \dot{x}_n(s)\mathrm{d}s\right\rangle = \lim_{n\to\infty}\int_{[0,\tau]}\langle \chi_{]0,t]}(s)u, \dot{x}_n(s)\rangle \mathrm{d}s \\ &= \int_{[0,\tau]}\langle \chi_{]0,t]}(s)u, w(s)\rangle\mathrm{d}s = \left\langle u, \int_{]0,t]} w(s)\mathrm{d}s\right\rangle\end{aligned}$$

where $\chi_{]0,t]}$ is the characteristic function of $]0,t]$. Then

$$x(t) - x(0) = \int_0^t w(s)\mathrm{d}s, \text{ for all } t \in [0,\tau],$$

which means that $\dot{x} \equiv w$, a.e. on $[0,\tau]$. So $(\dot{x}_n)_n$ converges weakly to $\dot{x}$ in $L^2([0,\tau],H)$. On the other side, using (4), (11) and (12), for almost all $t \in [0,\tau]$, we get

$$\begin{aligned}&\left\langle x_n(\overline{\theta}_n(t)) - x_{n+1}(\overline{\theta}_{n+1}(t)), \dot{x}_n(t) - g_n(t) - (\dot{x}_{n+1}(t) - g_{n+1}(t))\right\rangle \\ \leq\ & \Big(1 + \|\dot{x}_n(t) - g_n(t)\| + \|\dot{x}_{n+1}(t) - g_{n+1}(t)\|\Big)\operatorname{dis}\Big(A(\overline{\theta}_n(t)), A(\overline{\theta}_{n+1}(t))\Big) \\ \leq\ & \Big(1 + 2L_1\Big)\Big((\overline{\theta}_n(t) - t)K + (\overline{\theta}_{n+1}(t) - t)K\Big) \\ \leq\ & \Big(1 + 2L_1\Big)K\frac{\varepsilon_n}{2}.\end{aligned}$$

Also

$$\begin{aligned}&\left\langle x_n(\overline{\theta}_n(t)) - x_{n+1}(\overline{\theta}_{n+1}(t)), g_n(t) - g_{n+1}(t)\right\rangle \\ \leq\ & \|x_n(\overline{\theta}_n(t)) - x_{n+1}(\overline{\theta}_{n+1}(t))\|\,\|g_n(t) - g_{n+1}(t)\|.\end{aligned}$$

Next, by (H2) and (13), for almost all $t \in [0,\tau]$, we have

$$\begin{aligned}\|g_{n+1}(t) - g_n(t)\| &\leq d_H\Big(G(t, T(\theta_n(t))x_n), G(t, T(\theta_{n+1}(t))x_{n+1})\Big) + \varepsilon_{n+1} \\ &\leq m(t)\|T(\theta_{n+1}(t))x_{n+1} - T(\theta_n(t))x_n\|_\infty + \varepsilon_{n+1}.\end{aligned}$$

It is easy to verify that $\|x_n(t) - x_n(s)\| \leq L\varepsilon_n$, for all $n \in \mathbb{N}^*$ and $t,s \in [-a,\tau]$ such that $|s-t| < \frac{1}{2}\eta(\frac{\varepsilon_n}{4})$. In addition, for all $t \in [0, \tau]$, we see that

$$\begin{aligned}
& \|T(\theta_{n+1}(t))x_{n+1} - T(\theta_n(t))x_n\|_\infty \\
\leq \ & \sup_{-a\leq s\leq 0} \|x_{n+1}(\theta_{n+1}(t)+s) - x_{n+1}(t+s)\| + \sup_{-a\leq s\leq 0} \|x_{n+1}(t+s) - x_n(t+s)\| \\
& \quad + \sup_{-a\leq s\leq 0} \|x_n(t+s) - x_n(\theta_n(t)+s)\| \\
\leq \ & L\varepsilon_{n+1} + \|x_{n+1} - x_n\|_\infty + L\varepsilon_n \\
\leq \ & 2L\varepsilon_n + \|x_{n+1} - x_n\|_\infty.
\end{aligned}$$

This yields, for almost all $t \in [0, \tau]$,

$$\|g_{n+1}(t) - g_n(t)\| \leq m(t)\|x_{n+1} - x_n\|_\infty + 2L\varepsilon_n m(t) + \varepsilon_{n+1}. \tag{14}$$

Since, for all $t \in [0, \tau]$,

$$\begin{aligned}
& \|x_n(\overline{\theta}_n(t)) - x_{n+1}(\overline{\theta}_{n+1}(t))\| \\
\leq \ & \|x_n(\overline{\theta}_n(t)) - x_n(t)\| + \|x_n(t) - x_{n+1}(t)\| + \|x_{n+1}(t) - x_{n+1}(\overline{\theta}_{n+1}(t))\| \\
\leq \ & \|x_{n+1} - x_n\|_\infty + 2L\varepsilon_n,
\end{aligned}$$

we obtain, for almost all $t \in [0, \tau]$,

$$\begin{aligned}
& \|x_n(\overline{\theta}_n(t)) - x_{n+1}(\overline{\theta}_{n+1}(t))\| \, \|g_n(t) - g_{n+1}(t)\| \\
\leq \ & m(t)\|x_{n+1} - x_n\|_\infty^2 + \varepsilon_n\Big[4Lm(t)\|x_{n+1} - x_n\|_\infty \\
& +4L^2 m(t)\varepsilon_n + \|x_{n+1} - x_n\|_\infty + 2L\varepsilon_n\Big] \\
\leq \ & m(t)\|x_{n+1} - x_n\|_\infty^2 + \varepsilon_n\alpha(t),
\end{aligned}$$

where $\alpha(t) = (8LR_1 + 4L^2a)m(t) + 2R_1 + 2La$. Moreover, using (10) and (11), we obtain, for almost all $t \in [0, \tau]$,

$$\begin{aligned}
& \Big\langle x_n(t) - x_n(\overline{\theta}_n(t)) - x_{n+1}(t) + x_{n+1}(\overline{\theta}_{n+1}(t)), \dot{x}_n(t) - \dot{x}_{n+1}(t)\Big\rangle \\
& \quad \leq (\|\dot{x}_n(t)\| + \|\dot{x}_{n+1}(t)\|)\Big(\|x_n(t) - x_n(\overline{\theta}_n(t))\| + \|x_{n+1}(\overline{\theta}_{n+1}(t)) - x_{n+1}(t)\|\Big) \\
& \quad \leq 2L(L\varepsilon_n + L\varepsilon_{n+1}) \\
& \quad \leq 4L^2\varepsilon_n.
\end{aligned}$$

As a result of the above, for almost all $t \in [0, \tau]$, we deduce that

$$\begin{aligned}
& \langle x_n(t) - x_{n+1}(t), \dot{x}_n(t) - \dot{x}_{n+1}(t)\rangle \\
& = \Big\langle x_n(t) - x_n(\overline{\theta}_n(t)) - x_{n+1}(t) + x_{n+1}(\overline{\theta}_{n+1}(t)), \dot{x}_n(t) - \dot{x}_{n+1}(t)\Big\rangle \\
& + \Big\langle x_n(\overline{\theta}_n(t)) - x_{n+1}(\overline{\theta}_{n+1}(t)), \dot{x}_n(t) - g_n(t) - (\dot{x}_{n+1}(t) - g_{n+1}(t))\Big\rangle \\
& + \Big\langle x_n(\overline{\theta}_n(t)) - x_{n+1}(\overline{\theta}_{n+1}(t)), g_n(t) - g_{n+1}(t)\Big\rangle \\
& \leq 4L^2\varepsilon_n + (1 + 2L_1)\varepsilon_n + m(t)\|x_{n+1} - x_n\|_\infty^2 + \varepsilon_n\alpha(t) \\
& = m(t)\|x_{n+1} - x_n\|_\infty^2 + \varepsilon_n\beta(t),
\end{aligned}$$

where $\beta(t) = 4L^2 + 1 + 2L_1 + \alpha(t)$. Since, $x_n(0) = x_{n+1}(0) = x_0$, for all $t \in [0, \tau]$, we get

$$\begin{aligned}
\frac{1}{2}\|x_n - x_{n+1}\|_\infty^2 \quad & = \int_0^t \Big\langle x_n(s) - x_{n+1}(s), \dot{x}_n(s) - \dot{x}_{n+1}(s)\Big\rangle \mathrm{d}s \\
& \leq \|x_{n+1} - x_n\|_\infty^2 \int_0^\tau m(s)ds + \varepsilon_n \int_0^\tau \beta(s)ds.
\end{aligned}$$

Hence

$$\|x_n - x_{n+1}\|_\infty^2 \leq \frac{\int_0^\tau \beta(s)ds}{\frac{1}{2} - \int_0^\tau m(s)ds}\varepsilon_n.$$

So, for $n < m$,

$$\|x_n - x_m\|_\infty^2 \leq \frac{\int_0^\tau \beta(s)ds}{\frac{1}{2} - \int_0^\tau m(s)ds}\sum_{j=n}^{m-1}\varepsilon_j.$$

Consequently, the sequence $(x_n)_n$ is a Cauchy sequence in $C([0,\tau], H)$. Therefore $(x_n)_n$ converges uniformly on $[0,\tau]$ to a function x. Since all functions x_n agree with φ on $[-a, 0]$, we can obviously say that x_n converges uniformly to x on $[-a,\tau]$, if we extend x in such a way that $x \equiv \varphi$ on $[-a, 0]$.
Step 2: *$T(\theta_n(t))x_n$ converges to $T(t)x$ in C_a.* For all $t \in [0,\tau[$, one has

$$\begin{aligned}
&\|T(\theta_n(t))x_n - T(t)x\|_\infty \\
= \;& \sup_{-a\leq s\leq 0}\|x_n(\theta_n(t)+s) - x(t+s)\| \\
\leq \;& \sup_{-a\leq s\leq 0}\|x_n(\theta_n(t)+s) - x(\theta_n(t)+s)\| + \sup_{-a\leq s\leq 0}\|x(\theta_n(t)+s) - x(t+s)\| \\
\leq \;& \|x_n - x\|_\infty + \sup_{-a\leq s\leq 0}\|x(\theta_n(t)+s) - x(t+s)\|.
\end{aligned}$$

Since the right term of the above relation converges to 0, $T(\theta_n(t))x_n$ converges to $T(t)x$ in C_a. Similarly

$$\|T(\overline{\theta}_n(t))x_n - T(t)x\| \to 0, \quad \text{as } n \to \infty.$$

Step 3: *x is a solution of (1).* By (14), for almost all $t \in [0,\tau]$, $(g_n(t))_{n\geq 1}$ is a Cauchy sequence and $(g_n(t))_{n\geq 1}$ converges to $g(t)$. Moreover, observe that

$$\begin{aligned}
d(g(t), G(t, T(t)x)) &\leq \|g(t) - g_n(t)\| + d_H(G(t, T(\theta_n(t))x_n), G(t, T(t)x)) \\
&\leq \|g(t) - g_n(t)\| + m(t)\|T(\theta_n(t))x_n - T(t)x\|_\infty.
\end{aligned}$$

So, $g(t) \in G(t, T(t)x)$ for almost all $t \in [0,\tau[$. Let us prove that $x(t) \in D(A(t))$ on $[0,\tau[$. By (8), $x_n(\overline{\theta}_n(t)) \in D(A(\overline{\theta}_n(t)))$ and $x_n(\overline{\theta}_n(t)) \to x(t)$ for all $t \in [0,\tau[$. By Lemma 2.3, we conclude that $x(t) \in D(A(t))$ for all $t \in [0,\tau[$. Now, the weak convergence of $\dot{x}_n$ to $\dot{x}$ in $L^2([0,\tau], H)$ and the Mazur's Lemma entail

$$-\dot{x}(t) + g(t) \in \bigcap_n \overline{co}\Big\{-(\dot{x}_m(t) - g_m(t)) : m \;\geq\; n\Big\}, \text{ a.e. on } [0,\tau].$$

Next, for almost all $t \in [0,\tau]$ and all $y \in D(A(t))$, we have

$$\langle y, -\dot{x}(t) + g(t)\rangle \;\leq\; \inf_n \sup_{k\geq n}\langle y, -(\dot{x}_k(t) - g_k(t))\rangle.$$

By Lemma 2.4, there exists a sequence $(y_n)_n$ such that $y_n \in D(A(\overline{\theta}_n(t)))$, $y_n \to y$ and $A(\overline{\theta}_n(t))^0 y_n \to A(t)^0 y$. Since $A(\overline{\theta}_n(t))$ is monotone, we get

$$\Big\langle -(\dot{x}_n(t) - g_n(t)), y_n - x_n(\overline{\theta}_n(t))\Big\rangle \leq \Big\langle A(\overline{\theta}_n(t))^0 y_n, y_n - x_n(\overline{\theta}_n(t))\Big\rangle. \tag{15}$$

Consequently, by (11) and (15),

$$\begin{aligned}
&\langle -(\dot{x}_n(t) - g_n(t)), y - x(t)\rangle \\
= \;& \Big\langle -(\dot{x}_n(t) - g_n(t)), (y - y_n) - (x(t) - x_n(\overline{\theta}_n(t)))\Big\rangle \\
&\quad + \Big\langle -(\dot{x}_n(t) - g_n(t)), y_n - x_n(\overline{\theta}_n(t))\Big\rangle \\
\leq \;& L_1\Big(\|x_n(\overline{\theta}_n(t)) - x(t)\| + \|y_n - y\|\Big) + \langle A(\overline{\theta}_n(t))^0 y_n, y_n - x_n(\overline{\theta}_n(t))\rangle.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\langle -(\dot{x}_n(t) - g_n(t)), y - x(t)\rangle \le \langle A(t)^0 y, y - x(t)\rangle,$$

and, by Lemma 2.2, we get $-(\dot{x}(t) - g(t)) \in A(t)x(t)$. Finally, we conclude that $\dot{x}(t) \in -A(t)x(t) + G(t, T(t)x)$, for almost all $t \in [0, \tau]$. The proof is complete. □

4. Second result

This section concerns the second result of our paper. We will use the following assumptions.

(A1) Let $A(t) : D(A(t)) \subset H \longrightarrow 2^H$, for all $t \in [0, b]$, be a maximal monotone operator satisfying:

(a) there exists an absolutely continuous function $v : [0, b] \to \mathbb{R}$ such that

$$\operatorname{dis}(A(t), A(s)) \le |v(t) - v(s)|, \quad \forall t, s \in [0, b],$$

(b) there exists $c \ge 0$ such that

$$\|A(t)^0 x\| \le c(1 + \|x\|) \text{ for all } t \in [0, b] \text{ and } x \in D(A(t)),$$

(c) $\bigcup_{t\in[0,b]} D(A(t))$ is ball compact.

(A2) $G : [0, b] \times C_a \to 2^H$ is a set-valued map with nonempty closed values satisfying:

(i) for each $\psi \in C_a$, $t \mapsto G(t, \psi)$ is measurable,

(ii) there exists a function $m \in L^1([0, b], \mathbb{R}^+)$ such that for all $t \in [0, b]$ and for all $\psi_1, \psi_2 \in C_a$

$$d_H\Big(G(t, \psi_1), G(t, \psi_2)\Big) \le m(t)\|\psi_1 - \psi_2\|_\infty,$$

(iii) for all $\varphi \in C_a$, there exist $r > 0$ and functions $q, p \in L^1([0, b], \mathbb{R}^+)$ such that for all $t \in [0, b]$ and for all $\psi \in \overline{B}_a(\varphi, r)$

$$\|G(t, \psi)\| \le q(t) + p(t)\|\psi\|_\infty.$$

In the rest of this work, we prove the following theorem.

Theorem 4.1. *If assumptions (A1) and (A2) are satisfied, for all $\varphi \in C_a$ such that $\varphi(0) \in D(A(0))$, there exist $\tau > 0$ and a continuous function $x : [-a, \tau] \longrightarrow H$, that is absolutely continuous on $[0, \tau]$, such that x is solution of the evolution problem (1).*

Proof. Fix $\varphi \in C_a$ such that $x_0 := \varphi(0) \in D(A(0))$. There exist $r > 0$ and $q, p \in L^1([0, b], \mathbb{R}^+)$ such that

$$\|G(t, \psi)\| \le q(t) + p(t)\|\psi\|_\infty, \quad \forall (t, \psi) \in [0, b] \times \overline{B}_a(\varphi, r).$$

Let $\tau_1 \in]0, b]$ be such that

$$\int_0^{\tau_1} \Gamma(s) ds < \frac{r}{2},$$

with

$$\Gamma_1(t) = 2(q(t) + p(t)(r + \|\varphi\|_\infty)) + \frac{1}{2} + \frac{3}{2}c(1 + \|\varphi(0)\| + r) + \frac{3}{2}|\dot{v}(t)|$$

and

$$\Gamma(t) = \Gamma_1(t) + q(t) + p(t)(r + \|\varphi\|_\infty) \text{ for almost all } t \in [0, b].$$

For $\varepsilon > 0$, set

$$\eta_1(\varepsilon) \quad = \quad \sup\Big\{\gamma \in]0,\varepsilon] : \Big|\int_{t_1}^{t_2} \Gamma(s)ds\Big| < \varepsilon \text{ if } |t_1 - t_2| < \gamma\Big\},$$

and

$$\eta_2(\varepsilon) = \sup\Big\{\gamma \in]0,\varepsilon] : \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon \text{ if } |t_1 - t_2| < \gamma\Big\}.$$

Put $\eta(\varepsilon) = \min\{\eta_1(\varepsilon),\ \eta_2(\varepsilon)\}$ and $\tau = \min\{\frac{1}{2}\eta(\frac{r}{2}), \tau_1, a\}$. Set

$$\alpha(t) = \int_0^t \Gamma_1(s)ds, \quad \forall t \in [0,\tau].$$

We will use the following lemma to prove Theorem 4.1.

Lemma 4.2. *If assumptions (A1) and (A2) are satisfied, then for all $n \in \mathbb{N}^*$ and for all $y \in L^1([0,\tau], H)$, there exist a continuous mapping $x_n : [-a,\tau] \longrightarrow H$, step functions $\theta_n,\ \overline{\theta}_n : [0,\tau] \longrightarrow [0,\tau]$ and $g_n \in L^1([0,\tau], H)$ such that*

1. $x_n \equiv \varphi$ *on* $[-a,0],\ x_n(\overline{\theta}_n(t)) \in D(A(\overline{\theta}_n(t))),\ \forall t \in [0,\tau]$,
2. $x_n(t) \in \overline{B}(\varphi(0), r),\ T(\overline{\theta}_n(t))x_n \in \overline{B}_a(\varphi, r),\ \forall t \in [0,\tau]$,
3. $g_n(t) \in G(t, T(\theta_n(t))x_n),\ 0 \le \overline{\theta}_n(t) - \theta_n(t) \le \frac{\tau}{2^n},\ \theta_n(t) \le t \le \overline{\theta}_n(t), \forall t \in [0,\tau]$,
4. $\|g_n(t) - y(t)\| \le d(y(t), G(t, T(\theta_n(t))x_n)) + \frac{1}{n}$, *for almost all* $t \in [0,\tau]$,
5. $\|\dot{x}_n(t) - g_n(t)\| \le \Gamma_1(t),\ \|\dot{x}_n(t)\| \le \Gamma(t)$, *for almost all* $t \in [0,\tau]$,
6. $\frac{\alpha(\overline{\theta}_n(t)) - \alpha(\theta_n(t))}{\dot{\alpha}(t)(\overline{\theta}_n(t) - \theta_n(t))}\Big(\dot{x}_n(t) - g_n(t)\Big) \in -A(\overline{\theta}_n(t))x_n(\overline{\theta}_n(t))$, *a.e. on* $[0,\tau]$.

Proof. Fix $n \in \mathbb{N}^*$ and let $y : [0,\tau] \longrightarrow H$ be a measurable function. Consider a sequence $(P_n)_n$ of subdivisions of $[0,\tau]$:

$$P_n = \{t_0^n = 0 < t_1^n < \ldots < t_i^n < \ldots < t_{2^n}^n = \tau\},$$

where $t_i^n = i\frac{\tau}{2^n}$ for all $0 < i < 2^n$. Set $h_{i+1}^n = t_{i+1}^n - t_i^n$ for all $0 \le i \le 2^n - 1$. Let us define a sequence $(x_n)_n$ of approximate solutions as follows. Set $x_n(s) = \varphi(s)$ for all $s \in [-a,0]$. Put $x_0^n = \varphi(0) \in D(A(0))$. The set-valued map $t \longmapsto G(t, T(t_0^n)x_n)$ is measurable, in view of Lemma 2.6, there exists a function $g_0^n \in L^1([0,t_1^n], H)$ such that $g_0^n(t) \in G(t, T(t_0^n)x_n)$ for all $t \in [0,t_1^n]$ and

$$\|g_0^n(t) - y(t)\| \le d(y(t), G(t, T(t_0^n)x_n)) + \frac{1}{n}, \text{ for almost all } t \in [0,t_1^n].$$

Put

$$x_1^n = J_1^n\Big(x_0^n + \int_{t_0^n}^{t_1^n} g_0^n(s)\mathrm{d}s\Big),$$

where $J_1^n(x) := (I + h_1^n A(t_1^n))^{-1}(x)$, for all $x \in H$. Notice that, by construction, $x_1^n \in D(A(t_1^n))$ and

$$x_1^n - x_0^n - \int_{t_0^n}^{t_1^n} g_0^n(s)\mathrm{d}s \in -h_1^n A(t_1^n)x_1^n.$$

Hence, by (A1) and Lemma 2.1, we have

$$\begin{aligned}\|x_1^n - x_0^n\| &\le \left\| J_1^n\left(x_0^n + \int_{t_0^n}^{t_1^n} g_0^n(s)\mathrm{d}s\right) - J_1^n(x_0^n)\right\| + \|J_1^n(x_0^n) - x_0^n\| \\ &\le \int_{t_0^n}^{t_1^n} \|g_0^n(s)\|\mathrm{d}s + h_1^n\|A(t_0^n)^0 x_0^n\| + \mathrm{dis}(A(t_1^n), A(t_0^n)) \\ &\quad + \sqrt{h_1^n(1 + \|A(t_0^n)^0 x_0^n\|)\,\mathrm{dis}(A(t_1^n), A(t_0^n))} \\ &\le \int_{t_0^n}^{t_1^n} \|g_0^n(s)\|\mathrm{d}s + \int_{t_0^n}^{t_1^n} c(1 + \|x_0^n\|)ds + \int_{t_0^n}^{t_1^n} |\dot{v}(s)|ds \\ &\quad + \frac{1}{2}\int_{t_0^n}^{t_1^n} (1 + c(1 + \|x_0^n\|))ds + \frac{1}{2}\int_{t_0^n}^{t_1^n} |\dot{v}(s)|ds \\ &\le \int_{t_0^n}^{t_1^n} \left[q(s) + p(s)\|\varphi\|_\infty + \frac{1}{2} + \frac{3}{2}c(1 + \|x_0^n\|) + \frac{3}{2}|\dot{v}(s)|\right]ds.\end{aligned}$$

Then, using this last relation, we get

$$\left\| x_1^n - \left(x_0^n + \int_{t_0^n}^{t_1^n} g_0^n(s)ds\right)\right\| \le \int_{t_0^n}^{t_1^n} \Gamma_1(s)ds. \tag{16}$$

Now, set

$$x_n(t) = x_0^n + \frac{\alpha(t) - \alpha(t_0^n)}{\alpha(t_1^n) - \alpha(t_0^n)}\left(x_1^n - x_0^n - \int_{t_0^n}^{t_1^n} g_0^n(s)ds\right) + \int_{t_0^n}^{t} g_0^n(s)ds, \tag{17}$$

for all $t \in [t_0^n, t_1^n]$. From (A2), (16) and (17), we obtain

$$\begin{aligned}\|x_n(t) - \varphi(0)\| &\le \alpha(t) - \alpha(t_0^n) + \int_{t_0^n}^{t} (q(s) + p(s)\|\varphi\|_\infty)ds \\ &\le \int_{t_0^n}^{t} \Gamma_1(s)ds + \int_{t_0^n}^{t} (q(s) + p(s)\|\varphi\|_\infty)ds \\ &\le \int_{t_0^n}^{t} \Gamma(s)ds.\end{aligned}$$

Which is equivalent to $x_n(t) \in \overline{B}(\varphi(0), \frac{r}{2})$ for all $t \in [t_0^n, t_1^n]$. Now, we have to estimate $\|(T(t_1^n)x_n)(s) - \varphi(s)\|$ for each $s \in [-a, 0]$. If $-t_1^n \le s \le 0$, then $(t_1^n + s) \in [t_0^n, t_1^n]$. Thus, by the fact that $|s| \le t_1^n \le \tau < \eta(\frac{r}{2})$, we have

$$\begin{aligned}\|(T(t_1^n)x_n)(s) - \varphi(s)\| &= \|x_n(t_1^n + s) - \varphi(s)\| \\ &\le \|x_n(t_1^n + s) - \varphi(0)\| + \|\varphi(s) - \varphi(0)\| \\ &< r.\end{aligned}$$

Therefore, $T(t_1^n)x_n \in \overline{B}_a(\varphi, r)$. Next, we reiterate this process for constructing the sequences $(g_i^n)_i$ and $(x_i^n)_i$

and the function x_n satisfying, for all $0 \le i \le 2^n - 1$, the following assertions:

$$x_n(t) \in \overline{B}(\varphi(0), \tfrac{r}{2}),\ g_i^n(t) \in G(t, T(t_i^n)x_n), \quad \forall t \in [t_i^n, t_{i+1}^n]$$
$$\|x_n(t) - \varphi(0)\| \le \int_{t_0^n}^t \Gamma(s)ds, \quad \forall t \in [t_i^n, t_{i+1}^n]$$
$$x_0^n \in D(A(t_0^n)),\ x_{i+1}^n \in D(A(t_{i+1}^n)),\ T(t_i^n)x_n \in \overline{B}_a(\varphi, r),$$
$$\|g_i^n(t) - y(t)\| \le d(y(t), G(t, T(t_i^n)x_n)) + \tfrac{1}{n}, \text{ for almost all } t \in [t_i^n, t_{i+1}^n]$$
$$x_{i+1}^n = J_{i+1}^n\left(x_i^n + \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)\mathrm{d}s\right) \text{ with } J_{i+1}^n(x) := (I + h_{i+1}^n A(t_{i+1}^n))^{-1}(x),$$
$$\left\|x_{i+1}^n - \left(x_i^n + \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds\right)\right\| \le \int_{t_i^n}^{t_{i+1}^n} \Gamma_1(s)ds,$$
$$x_n(t) = x_i^n + \tfrac{\alpha(t)-\alpha(t_i^n)}{\alpha(t_{i+1}^n)-\alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds\right) + \int_{t_i^n}^t g_i^n(s)ds,\ \forall t \in [t_i^n, t_{i+1}^n].$$

We claim that x_n is absolutely continuous. Indeed, for all $0 \le i \le 2^n - 1$ and for all t and s in $[t_i^n, t_{i+1}^n],\ s < t$, one has

$$x_n(t) - x_n(s) = \frac{\alpha(t) - \alpha(s)}{\alpha(t_{i+1}^n) - \alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(\tau)d\tau\right) + \int_s^t g_i^n(\tau)d\tau.$$

Then, we get

$$\begin{aligned}\|x_n(t) - x_n(s)\| &\le \frac{\alpha(t) - \alpha(s)}{\alpha(t_{i+1}^n) - \alpha(t_i^n)}\left\|x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(\tau)d\tau\right\| + \int_s^t \|g_i^n(\tau)\|d\tau \\ &\le \int_s^t \Gamma(\tau)d\tau.\end{aligned}$$

By addition this last inequality holds for all $s,\ t \in [0, \tau]$ with $s < t$. Hence x_n is absolutely continuous. Remark that for all $0 \le i \le 2^n - 1$ and for almost all $t \in [t_i^n, t_{i+1}^n]$,

$$\dot{x}_n(t) = \frac{\dot{\alpha}(t)}{\alpha(t_{i+1}^n) - \alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds\right) + g_n(t). \tag{18}$$

Then, we obtain for almost all $t \in [t_i^n, t_{i+1}^n]$,

$$\|\dot{x}_n(t) - g_n(t)\| \le \Gamma_1(t) \text{ and } \|\dot{x}_n(t)\| \le \Gamma(t).$$

Now, we define the functions $\theta_n,\ \overline{\theta}_n : [0, \tau] \longrightarrow [0, \tau]$ and $g_n \in L^1([0, \tau], H)$ by setting for all $t \in [t_i^n, t_{i+1}^n[$,

$$\overline{\theta}_n(t) = t_{i+1}^n, \quad \theta_n(t) = t_i^n, \quad g_n(t) = g_i^n(t),$$
$$\overline{\theta}_n(\tau) = \tau, \quad \theta_n(\tau) = t_{2^n-1}^n \text{ and } g_n(\tau) = g_{2^n-1}^n(\tau).$$

Note that $0 \le \overline{\theta}_n(t) - \theta_n(t) \le \frac{\tau}{2^n}$ and $\theta_n(t) \le t \le \overline{\theta}_n(t)$ for all $t \in [0, \tau]$. Observe that, by construction, we have $g_n(t) \in G(t, T(\theta_n(t))x_n)$, for all $t \in [0, \tau]$ and

$$\|g_n(t) - y(t)\| \le d(y(t), G(t, T(\theta_n(t))x_n)) + \frac{1}{n}, \text{ for almost all } t \in [0, \tau].$$

Hence, from (18), we get

$$\gamma_n(t)\Big(\dot{x}_n(t) - g_n(t)\Big) \in -A(\overline{\theta}_n(t))x_n(\overline{\theta}_n(t)), \text{ for almost all } t \in [0, \tau],$$

where

$$\gamma_n(t) = \frac{\alpha(\overline{\theta}_n(t)) - \alpha(\theta_n(t))}{\dot{\alpha}(t)(\overline{\theta}_n(t) - \theta_n(t))}.$$

□

Now, we are ready to complete the proof of Theorem 4.1. By Lemma 4.2, we can define inductively sequences $(g_n)_{n\geq1}, (x_n)_{n\geq1} \subset C([-a,\tau], H)$ and $(\theta_n)_{n\geq1}, (\overline{\theta}_n)_{n\geq1} \subset S([0,\tau], [0,\tau])$, where $S([0,\tau], [0,\tau])$ denotes the space of step functions from $[0,\tau]$ into $[0,\tau]$, such that

$$x_n \equiv \varphi \text{ on } [-a, 0],\ x_n(\overline{\theta}_n(t)) \in D(A(\overline{\theta}_n(t))),\ \forall t \in [0,\tau], \tag{19}$$

$$x_n(t) \in \overline{B}(\varphi(0), r),\ T(\theta_n(t))x_n \in \overline{B}_a(\varphi, r),\ \forall t \in [0,\tau], \tag{20}$$

$$\|x_n(t) - x_n(s)\| \leq \int_s^t \Gamma(\tau)d\tau,\ \forall s,t \in [0,\tau],\ s < t, \tag{21}$$

$$g_n(t) \in G(t, T(\theta_n(t))x_n),\ 0 \leq \overline{\theta}_n(t) - \theta_n(t) \leq \tfrac{\tau}{2^n},\ \theta_n(t) \leq t \leq \overline{\theta}_n(t),\ \forall t \in [0,\tau],$$

$$\|g_{n+1}(t) - g_n(t)\| \leq d(g_n(t), G(t, T(\theta_{n+1}(t))x_{n+1})) + \tfrac{1}{n+1},\ \text{a.e. on } t \in [0,\tau], \tag{22}$$

$$\|\dot{x}_n(t) - g_n(t)\| \leq \Gamma_1(t),\ \|\dot{x}_n(t)\| \leq \Gamma(t),\ \text{for almost all } t \in [0,\tau], \tag{23}$$

$$\gamma_n(t)\Big(\dot{x}_n(t) - g_n(t)\Big) \in -A(\overline{\theta}_n(t))x_n(\overline{\theta}_n(t)),\ \text{for almost all } t \in [0,\tau].$$

Claim 4.3. *For all $t \in [0,\tau]$, the set $\{x_n(t), n \geq 1\}$ is relatively compact in H.*

Proof. Let $t \in [0,\tau]$ and $\varepsilon > 0$. There exists $n_0 \geq 1$ such that

$$\Big\| \int_t^{\bar{\theta}_n(t)} \Gamma(s)ds \Big\| \leq \varepsilon, \quad \forall n \geq n_0.$$

By (20),

$$\sup_{n\in\mathbb{N}^*} \|x_n\|_\infty \leq r + \|\varphi(0)\| := R_1,$$

which implies that $x_n(\overline{\theta}_n(t)) \in D(A([0,b])) \cap R_1\overline{B}$. Then, $\{x_n(\overline{\theta}_n(t)) : n \geq 1\}$ is relatively compact in H. Now, for all $t \in [0,\tau]$

$$\beta\{x_n(t) : n \geq n_0\} = \beta\{x_n(t) - x_n(\overline{\theta}_n(t)) + x_n(\overline{\theta}_n(t)) : n \geq n_0\}.$$

From Lemma 2.9 (iv), we get

$$\beta\{x_n(t) : n \geq n_0\} \leq \beta\{x_n(t) - x_n(\overline{\theta}_n(t)) : n \geq n_0\} + \beta\{x_n(\overline{\theta}_n(t)) : n \geq n_0\}.$$

Since the set $\{x_n(\bar{\theta}_n(t)) : n \geq n_0\}$ is relatively compact in H, by Lemma 2.9 (i), $\beta\{x_n(\overline{\theta}_n(t)) : n \geq n_0\} = 0$. Then

$$\beta\{x_n(t) : n \geq n_0\} \leq \beta\{x_n(t) - x_n(\overline{\theta}_n(t)) : n \geq n_0\} = \beta\Big\{ \int_{\bar{\theta}_n(t)}^t \dot{x}_n(s)ds : n \geq n_0\Big\}.$$

On the other hand, since

$$\Big\| \int_{\bar{\theta}_n(t)}^t \dot{x}_n(s)ds \Big\| \leq \int_t^{\bar{\theta}_n(t)} \Gamma(s)ds \leq \varepsilon,$$

for all $n \geq n_0$, we have

$$\Big\{ \int_{\bar{\theta}_n(t)}^t \dot{x}_n(s)ds : n \geq n_0\Big\} \subset \overline{B}(0,\varepsilon).$$

Then, by Lemma 2.9 (iii),(v), we obtain

$$\beta\{x_n(t) : n \geq n_0\} \leq \beta\{\overline{B}(0,\varepsilon)\} = 2\varepsilon.$$

Also, by Lemma 2.9 (vi), we get

$$\begin{aligned}\beta\{x_n(t):n\geq 1\} &= \beta\Big\{\{x_n(t):1\leq n\leq n_0\}\cup\{x_n(t):n\geq n_0\}\Big\}\\ &= \max\Big\{\beta\{x_n(t):1\leq n\leq n_0\},\beta\{x_n(t):n\geq n_0\}\Big\}\\ &= \beta\{x_n(t):n\geq n_0\}\end{aligned}$$

because $\beta\{x_n(t):1\leq n\leq n_0\}=0$. So

$$\beta\{x_n(t):n\geq 1\}\leq 2\varepsilon,$$

for all $\varepsilon>0$. We conclude that $\beta\{x_n(t):n\geq 1\}=0$ and the set $\{x_n(t):n\geq 1\}$ is relatively compact in H.
□

Now, by Ascoli-Arzelà Theorem, we can select a subsequence, again denoted by $(x_n)_n$ which converges uniformly to an absolutely continuous function x on $[0,\tau]$, moreover $\dot{x}_n$ converges weakly to $\dot{x}$ in $L^2([0,\tau],H)$. Also, since all functions x_n agree with φ on $[-a,0]$, we can obviously say that $(x_n)_n$ converges uniformly to x on $[-a,0]$, if we extend x in such a way that $x\equiv\varphi$ on $[-a,0]$. Additionally, observe that $(x_n(\overline{\theta}_n))_n$ converges uniformly to x on $[0,\tau]$. On the other hand, by (21), we have

$$\begin{aligned}\|x_n(\overline{\theta}_n(t))-x(t)\| &\leq \|x_n(\overline{\theta}_n(t))-x_n(t)\|+\|x_n(t)-x(t)\|\\ &\leq \int_t^{\overline{\theta}_n(t)}\Gamma(s)ds+\|x_n(t)-x(t)\|.\end{aligned}$$

The right term of the above inequality converges to 0. It follows that $(x_n(\overline{\theta}_n))_n$ converges uniformly to x on $[0,\tau]$. In addition, by the same technics of the previous section, we prove that, for all $t\in[0,\tau]$, $T(\theta_n(t))x_n$ and $T(\overline{\theta}_n(t))x_n$ converge to $T(t)x$ in C_a. Now, from (A2) and (22), for almost all $t\in[0,\tau]$, we have

$$\begin{aligned}\|g_{n+1}(t)-g_n(t)\| &\leq d_H(G(t,T(\theta_n(t))x_n),G(t,T(\theta_{n+1}(t))x_{n+1}))+\frac{1}{n+1}\\ &\leq m(t)\|T(\theta_n(t))x_n-T(\theta_{n+1}(t))x_{n+1}\|_\infty+\frac{1}{n+1}.\end{aligned}$$

Since the right term of the above relation converges to 0, for almost all $t\in[0,\tau]$, $(g_n(t))_{n\geq1}$ is a Cauchy sequence and $(g_n(t))_{n\geq1}$ converges to $g(t)$. Moreover, observe that

$$\begin{aligned}d(g(t),G(t,T(t)x)) &\leq \|g(t)-g_n(t)\|+d_H(G(t,T(\theta_n(t))x_n),G(t,T(t)x))\\ &\leq \|g(t)-g_n(t)\|+m(t)\|T(\theta_n(t))x_n-T(t)x\|_\infty.\end{aligned}$$

So, $g(t)\in G(t,T(t)x)$ for almost all $t\in[0,\tau]$. Let us prove that $x(t)\in D(A(t))$ on $[0,\tau]$. By (19), $x_n(\overline{\theta}_n(t))\in D(A(\overline{\theta}_n(t)))$ and $x_n(\overline{\theta}_n(t))\to x(t)$ for all $t\in[0,\tau]$. By Lemma 2.3, we conclude that $x(t)\in D(A(t))$ for all $t\in[0,\tau]$. Now, the weak convergence of $\dot{x}_n$ to $\dot{x}$ in $L^2([0,\tau],H)$ and the Mazur's Lemma entail

$$-\dot{x}(t)+g(t)\in\bigcap_n\overline{co}\Big\{-\gamma_m(t)(\dot{x}_m(t)-g_m(t)):m\ \geq\ n\Big\},\ \text{a.e. on }[0,\tau].$$

Note that by construction, $\lim_{n\to+\infty}\gamma_n(t)=1$. Next, for almost all $t\in[0,\tau]$ and all $y\in D(A(t))$, we have

$$\langle y,-\dot{x}(t)+g(t)\rangle\ \leq\ \inf_n\sup_{k\geq n}\langle y,-\gamma_k(t)(\dot{x}_k(t)-g_k(t))\rangle.$$

By Lemma 2.4, there exists a sequence $(y_n)_n$ such that,

$$y_n\in D(A(\overline{\theta}_n(t))),\ y_n\to y\text{ and }A(\overline{\theta}_n(t))^0y_n\to A(t)^0y.$$

Since $A(t)$ is monotone, we get

$$\Big\langle-\gamma_n(t)(\dot{x}_n(t)-g_n(t)),y_n-x_n(\overline{\theta}_n(t))\Big\rangle\leq\Big\langle A(\overline{\theta}_n(t))^0y_n,y_n-x_n(\overline{\theta}_n(t))\Big\rangle. \tag{24}$$

Consequently, by (23) and (24),

$$\begin{aligned}
&\Big\langle -\gamma_n(t)(\dot{x}_n(t)-g_n(t)), y-x(t)\Big\rangle \\
=\ &\Big\langle -\gamma_n(t)(\dot{x}_n(t)-g_n(t)), (y-y_n)-(x(t)-x_n(\overline{\theta}_n(t)))\Big\rangle \\
&\quad +\Big\langle -\gamma_n(t)(\dot{x}_n(t)-g_n(t)), y_n-x_n(\overline{\theta}_n(t))\Big\rangle \\
\leq\ &|\gamma_n(t)|\Gamma_1(t)\Big(\|x_n(\overline{\theta}_n(t))-x(t)\|+\|y_n-y\|\Big)+\langle A(\overline{\theta}_n(t))^0 y_n, y_n-x_n(\overline{\theta}_n(t))\rangle.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\Big\langle -\gamma_n(t)(\dot{x}_n(t)-g_n(t)), y-x(t)\Big\rangle \leq \langle A(t)^0 y, y-x(t)\rangle,$$

and by Lemma 2.2, we conclude that $-(\dot{x}(t)-g(t)) \in A(t)x(t)$. Finally, we obtain $\dot{x}(t) \in -A(t)x(t)+G(t,T(t)x)$, for almost all $t \in [0,\tau]$. Then the proof is complete. □

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