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Some remarks for the antisymmetrically connected spaces

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Abstract. In the previous studies, the notion of antisymmetrically connected T_0-quasi-metric space is described as a type of the connectivity in the framework of asymmetric topology. Actually, the theory of antisymmetric connectedness was established in terms of graph theory, as the natural counterpart of the connected complementary graph. In this paper, some significant properties of antisymmetrically connected T_0-quasi-metric spaces are presented.

Accordingly, we study some different aspects of the theory of antisymmetric connectedness in terms of asymmetric norms which associate the theory of quasi-metrics with functional analysis. In the light of this approach, antisymmetrically connected T_0-quasi-metric spaces are investigated and characterized for the first time in the theory of asymmetrically normed real vector spaces.

Besides these, many further observations about the antisymmetric connectedness are dealt with especially in the sense of their combinations such as products and unions through various theorems and examples in the context of T_0-quasi-metrics. Also, we examine the question of under what kind of quasi-metric mapping antisymmetric connectedness will be preserved.

1. Introduction and preliminaries

In [10], the theory of symmetric connectedness for a T_0-quasi-metric space was described and investigated in detail. In the same framework, the notion of *antisymmetrically connected* T_0*-quasi-metric* is introduced for the first time, as a kind of dual structure of symmetrically connected space. Actually, these types of T_0-quasi-metric spaces were especially discussed in the sense of graph theory [4, 9] as corresponding counterparts of the connectedness for both a graph and complementary graph.

When we consider this paper, firstly, it is natural to ask how antisymmetric connectedness behaves in the context of asymmetrically normed real vector spaces. Accordingly, many interesting properties and some characterizations of the antisymmetrically connected T_0-quasi-metric spaces will be presented in terms of asymmetric norms. Following that we observed some combinations of anstisymmetrically connected spaces such as products and unions in the context of T_0-quasi-metrics. With this viewpoint, it is also natural to inquire whether the images of antisymmetrically connected spaces under an isometric isomorphism have the same property or not. Hereby, we will also obtain some crucial and useful results within the framework of these questions.

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In the light of all these considerations, the content of paper is as follows:

Some necessary background material for the remaining of paper is presented in Section 1. After recalling the preliminary information, as one of the purposes of the paper, in Section 2 we studied some properties of the theory of antisymmetric connectedness in the framework of asymmetric norms peculiar to asymmetric topology. Indeed, it is known from [10] that the problem to determine the antisymmetry components of points in X turns out to be easier when it is formulated for a T_0-quasi-metric induced by the asymmetric norm of an asymmetrically normed real vector space which is introduced by Cobzaş (see [1]) in Functional Analysis. Specifically, some characterizations of antisymmetric connectedness are presented in the context of the T_0-quasi-metrics induced by the asymmetric norms.

As the last part of paper, Section 3 is devoted to discussing some observations about the products, unions,... of the antisymmetrically connected T_0-quasi-metric spaces and their preservation under the specific metric mappings. Additionally, we investigate a few future properties of these spaces besides the related (counter)examples.

The remainder of this section will present some background material on T_0-quasi-metrics and in particular, it consists of the required information about the theories of symmetric and antisymmetric connectedness.

Definition 1.1. Let (X, d) be a T_0-quasi-metric space. Then d is called a *T_0-quasi-metric* on X if

(a) $d(x, x) = 0$
(b) $d(x, y) = 0 = d(y, x) \Rightarrow x = y$
(c) $d(x, z) \leq d(x, y) + d(y, z)$

whenever $x, y, z \in X$. We will also say that (X, d) is *T_0-quasi-metric space*.

If d is a T_0-quasi-metric on X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a T_0-quasi-metric, called the *conjugate T_0-quasi-metric of d*. Obviously, a T_0-quasi-metric d on X satisfying $d = d^{-1}$ will become a *metric*.

For any T_0-quasi-metric d, note that

$$d^s = \sup\{d, d^{-1}\} = d \vee d^{-1}$$

is a metric and d^s is called the *symmetrization metric* of d.

We will use the notation τ_{d^s} to denote the topology induced by the (symmetrization) metric d^s.

The literature on the T_0-quasi-metric spaces is vast, so we only present here some basic facts which are relevant to our purpose. An adequate introduction to the theory of T_0-quasi-metrics and the motivation for their study may be obtained from the works [3, 5–8].

Example 1.2. On the set $\mathbb{R}$ of the reals take

$$u(x, y) = (x - y) \vee 0$$

whenever $x, y \in \mathbb{R}$. It is easy to verify that u satisfies the conditions of Definition 1.1, and so u is a T_0-quasi-metric, called the *standard* T_0-quasi-metric on $\mathbb{R}$.

Now, let us recall some crucial notions and examples related to the theories constructed in [10]:

Definition 1.3. Let us take a T_0-quasi-metric space (X, d).

i) A pair $(x, y) \in X \times X$ will be called *symmetric pair* if $d(x, y) = d(y, x)$.
ii) A finite sequence of points in X, starting at x and ending with y, is called a (finite) *symmetric path* $P_{x,y} = (x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$ (where $n \in \mathbb{N}$) from x to y provided that all the pairs (x_i, x_{i+1}) are symmetric where $i \in \{0, \ldots, n-1\}$.
iii) For any $x \in X$ the path $P_{x,x} = (x, x)$ or the pair (x, x) will be called a *loop*.

Note 1.4. Note that sometimes it is useful to assume that no point occurs twice in a path $P_{x,y}$, except that possibly $x = y$.

For a T_0-quasi-metric space (X, d), we take

$$Z_d = \{(x, y) \in X \times X : d(x, y) = d(y, x)\}$$

as the set of symmetric pairs in (X, d). It is clear that the relation Z_d is reflexive and symmetric.

Incidentally, note that $d^s(x, y) = d(x, y) = d^{-1}(x, y)$ for $(x, y) \in Z_d$.

Also,

$$Z_d(x) = \{y \in X \mid (x, y) \in Z_d\}$$

is called *symmetry set* of $x \in X$.

Definition 1.5. a) Let (X, d) be a T_0-quasi-metric space. We say that $x \in X$ is *symmetrically connected* to $y \in X$ if there is a symmetric path $P_{x,y}$, starting at the point x and ending at the point y.
By definition, it is easy to verify that "symmetric connectedness" is an equivalence relation on the set X.

b) The equivalence class of a point $x \in X$ with respect to the symmetric connectedness relation will be called the *symmetry component* of x.

More clearly, if C_d denotes the *symmetric connectedness relation* then the symmetry component of $x \in X$ is

$$C_d(x) = \{y \in X : \text{there is a symmetric path from } x \text{ to } y\}.$$

Therefore, we are now in a position to recall from [10] the following important notion via the above notations:

Definition 1.6. A T_0-quasi-metric space (X, d) such that all the C_d-equivalence classes of points in X agree with X, that is $C_d(x) = X$ for all $x \in X$, is called *symmetrically connected*.

Obviously, a T_0-quasi-metric space (X, d) is symmetrically connected if and only if for all $x, y \in X$, x and y are symmetrically connected (see Definition 1.5).

The next result is trivial from the above definition and it forms the fundamental motivation of the "symmetric connectedness theory" in the context of T_0-quasi-metric spaces.

Corollary 1.7. *Each metric space is symmetrically connected.*

Now we can recall from [10] a notion opposite to that of "metric".

Definition 1.8. We shall call a T_0-quasi-metric space (X, d) *antisymmetric* if $Z_d = \{(x, x) : x \in X\}$, that is, Z_d is equal to the *diagonal* Δ_X of $X \times X$.

Equivalently, it is clear that each symmetry component of (X, d) is a singleton if and only if (X, d) is antisymmetric space.

Example 1.9. Observe that $(\mathbb{R}, u)$ in Example 1.2 is an antisymmetric T_0-quasi-metric space by the fact that $Z_u = \triangle_{\mathbb{R}}$. Note that in $(\mathbb{R}, u)$, the only symmetric pairs are *trivial*, that is the pairs (x, x) for $x \in X$. Also, the T_0-quasi-metric space $(\mathbb{R}, u)$ is not symmetrically connected by the fact that $C_u(x) = \{x\} \neq \mathbb{R}$ for all $x \in \mathbb{R}$.

At this stage, we can turn our attention to the dual counterparts of some notions described above.

Definition 1.10. Let (X, d) be a T_0-quasi-metric space. A pair $(x, y) \in X \times X$ is called *antisymmetric* if it satisfies the condition $d(x, y) \neq d(y, x)$.

For a T_0-quasi-metric space (X, d) we will also study the complement set of Z_d in $X \times X$, that is, the set

$$R_d = \{(x, y) \in X \times X : d(x, y) \neq d(y, x)\}$$

of antisymmetric pairs of the space (X, d). It is clear that the relation R_d is symmetric, but neither reflexive nor transitive. Also, the set

$$R_d(x) = \{y \in X \mid \ (x, y) \in R_d\}$$

is called *antisymmetry set of* x.

Definition 1.11. A finite sequence of points in X, starting at x and ending with y, is called a (finite) *antisymmetric path* $P_{x,y} = (x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$ (where $n \in \mathbb{N}$) from x to y provided that all the pairs (x_i, x_{i+1}) are antisymmetric where $i \in \{0, \ldots, n-1\}$.

We are now in a position to recall the dual notion to symmetric connectedness as follows:

Definition 1.12. i) In a T_0-quasi-metric space (X, d), two points $x, y \in X$ will be called *antisymmetrically connected* if there is an antisymmetric path $P_{x,y} = (x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$, or $x = y$.
Now, if we take the relation

$$T_d := \{(x, y) \in X \times X : x \text{ and } y \text{ are antisymmetrically connected in } (X, d)\}$$

then T_d is an equivalence relation on X, obviously. In addition, note here that $R_d \subseteq T_d$.

ii) The equivalence class of a point $x \in X$ with respect to T_d will be called the *antisymmetry component* and it is denoted by

$$T_d(x) = \{y \in X : \text{there is an antisymmetric path from } x \text{ to } y\}.$$

iii) If $T_d = X \times X$, or $T_d(x) = X$ for each $x \in X$, then the T_0-quasi-metric space (X, d) will be called *antisymmetrically connected*.

Hence, (X, d) is antisymmetrically connected if and only if for all $x, y \in X$, x and y are antisymmetrically connected (see Definition 1.12).

Because of Definition 1.12 iii), it is trivial that the singleton sets and loops are antisymmetrically connected.

Incidentally, the proofs of the following observations are straightforward.

Proposition 1.13. *a)* $R_d \cup Z_d = X \times X$, $R_d \cap Z_d = \emptyset$ *and so,* $R_d(x) \cup Z_d(x) = X$, $R_d(x) \cap Z_d(x) = \emptyset$ *for each* $x \in X$.

b) The relation $R_d(Z_d)$ *is* $\tau_{d^s} \times \tau_{d^s}$*-open (closed) in* $X \times X$.

c) The antisymmetry set $R_d(x)$ *(the symmetry set* $Z_d(x)$*) of* $x \in X$ *is* τ_{d^s}*-open (closed) in* X.

d) $Z_d \cup T_d = X \times X$, $C_d \cap T_d \neq \emptyset$ *and* $C_d \cup R_d = X \times X$.

Example 1.14. The T_0-quasi-metric space $(\mathbb{R}, u)$ given in Example 1.2 is antisymmetrically connected but not symmetrically connected, by Definition 1.12 iii) and Definition 1.6.

After presenting the preliminary information, now we can start to the main ideas of this study.

2. Antisymmetric connectedness in asymmetrically normed real vector spaces

Asymmetrically normed real vector spaces in the sense of [1] are also investigated in [10] as a new approach to the theory of asymmetry measurement for T_0-quasi-metrics. Now, after recalling the notion of *asymmetric norm*, we will present some new considerations peculiar to this framework.

Definition 2.1. ([1]) Let X be a real vector space equipped with a given map $\|\cdot| : X \to [0,\infty)$ satisfying the conditions:

(a) $\|x| = \|-x| = 0$ if and only if $x = \mathbf{0}$.
(b) $\|\lambda x| = \lambda\|x|$ whenever $\lambda \geq 0$ and $x \in X$.
(c) $\|x + y| \leq \|x| + \|y|$ whenever $x, y \in X$.

Then $\|\cdot|$ is called an *asymmetric norm* and $(X, \|\cdot|)$ an *asymmetrically normed real vector space*. In (a), $\mathbf{0}$ denotes the zero vector of the vector space X.

Obviously, an asymmetric norm induces a T_0-quasi-metric on X with the equality $d_{\|\cdot|}(x,y) = \|x - y|$ for each $x, y \in X$, where $(X, \|\cdot|)$ is an asymmetrically normed real vector space. But, naturally some T_0-quasi-metrics may not be induced by an asymmetric norm:

Example 2.2. Consider the function s on $\mathbb{R}$ as follows:

$$s(x,y) = \begin{cases} \min\{x - y, 1\} \;;\; x \geq y \\ \quad 1 \qquad\quad ;\; x < y \end{cases}$$

for each $x, y \in \mathbb{R}$. It is easy to show that s is a T_0-quasi-metric, but it cannot be induced by an asymmetric norm.

Incidentally, the notation $d_{\|\cdot|}$ will be used for the T_0-quasi-metric induced by the asymmetric norm $\|\cdot|$.
Moreover, the function

$$\|\cdot|^s = \|\cdot| \vee \|\cdot|^{-1} = \|\cdot\|$$

defines the standard (symmetrization) norm on X, where $\|a|^{-1} = \|-a|$ for $a \in X$, and so

$$d^s_{\|\cdot|} = d_{\|\cdot|^s} = d_{\|\cdot\|}$$

Also, clearly we have that $Z_{d_{\|\cdot|}}(\mathbf{0}) = \{x \in X : d(\mathbf{0}, x) = d(x, \mathbf{0})\} = \{x \in X : \|-x| = \|x|\}$ and

$$R_{d_{\|\cdot|}}(\mathbf{0}) = \{x \in X : d_{\|\cdot|}(\mathbf{0}, x) \neq d_{\|\cdot|}(x, \mathbf{0})\} = \{x \in X : \|x| \neq \|-x|\}$$

If $(X, \|\cdot|)$ is an asymmetrically normed real vector space, then $C_{\|\cdot|} = \{x \in X : \|-x| = 0\}$ is known to be a proper cone [2] in X. Accordingly,

Proposition 2.3. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. In this case,*

$$R_{d_{\|\cdot|}}(\mathbf{0}) \cup X \setminus C_{\|\cdot|} = X \setminus \{\mathbf{0}\}.$$

Proof. Let $x \in R_{d_{\|\cdot|}}(\mathbf{0}) \cup X \setminus C_{\|\cdot|}$. Then $x \in R_{d_{\|\cdot|}}(\mathbf{0})$ or $x \in X \setminus C_{\|\cdot|}$. So, $\|x| \neq \|-x|$ that is $x \neq \mathbf{0}$ or $\|-x| \neq 0$ that is $x \neq \mathbf{0}$. Thus, $x \in X \setminus \{\mathbf{0}\}$.

On the other hand, let $a \in X \setminus \{\mathbf{0}\}$ then $a \neq \mathbf{0}$. Now, if $a \notin R_{d_{\|\cdot|}}(\mathbf{0}) \cup X \setminus C_{\|\cdot|}$ then $a \notin R_{d_{\|\cdot|}}(\mathbf{0})$ and $a \notin X \setminus C_{\|\cdot|}$. Thus $\|a| = \|-a|$, and $\|-a| = \mathbf{0}$ since $a \in C_{\|\cdot|}$. Finally, a must be $\mathbf{0}$ because of the definition of asymmetric norm. This gives a contradiction. □

Lemma 2.4. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. Then for $x \in X$,*

1. $R_{d_{\|\cdot|}}(x) = R_{d_{\|\cdot|}}(\mathbf{0}) + x$
2. $T_{d_{\|\cdot|}}(x) = T_{d_{\|\cdot|}}(\mathbf{0}) + x$

Proof. First of all, let us take $d = d_{\|\cdot|}$ for the simplicity in the proof.

(1) Let $y \in R_d(\mathbf{0})$. Then $\|y| \neq \|-y|$. Thus $d(x+y, x) = \|y|$ and $d(x, x+y) = \|-y|$ are not equal, so $R_d(\mathbf{0}) + x \subseteq R_d(x)$.

For the converse inclusion, let $y \in R_d(x)$. Then $d(y,x) \neq d(x,y)$. Thus $\|y-x| \neq \|x-y|$ and so, $y - x \in R_d(\mathbf{0})$. Therefore $y = (y-x) + x \in R_d(\mathbf{0}) + x$. Finally, we have the equality $R_d(x) = R_d(\mathbf{0}) + x$.

(2) Assume that $y \in T_d(x)$. Then there exists an antisymmetric path $P_{x,y} = (x_0, x_1, \ldots, x_n)$ from x to y, where $x_0 = x$, $x_n = y$. Define the path $P_{\mathbf{0},y-x}$ as $(x_0 - x, x_1 - x, \ldots, x_n - x)$. Then $P_{\mathbf{0},y-x}$ is an antisymmetric path from $\mathbf{0}$ to $y - x$. Thus $y - x \in T_d(\mathbf{0})$. Therefore $y \in T_d(\mathbf{0}) + x$ and $T_d(x) \subseteq T_d(\mathbf{0}) + x$.

Conversly, let $y \in T_d(\mathbf{0}) + x$. Then there exists $t \in T_d(\mathbf{0})$ with $y = t + x$. Furthermore, there is an antisymmetric path $P_{\mathbf{0},t} = (\mathbf{0}, x_1 \ldots, t)$ from $\mathbf{0}$ to t. Then define $P_{x,x+t}$ as the path $(x, x+x_1, \ldots, x+t)$. Obviously $P_{x,x+t}$ is an antisymmetric path from x to $x+t$. Therefore $y = t + x \in T_d(x)$ and we have established that $T_d(\mathbf{0}) + x \subseteq T_d(x)$. $\square$

Specifically, we can present the next characterization in the context of asymmetrically normed real vector spaces.

Theorem 2.5. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. Then, $(X, d_{\|\cdot|})$ is antisymmetrically connected if and only if $T_{d_{\|\cdot|}}(\mathbf{0}) = X$.*

Proof. ($\Longrightarrow$) Since (X, d) is antisymmetrically connected, we have $T_{d_{\|\cdot|}}(x) = X$ for all $x \in X$. So, $T_{d_{\|\cdot|}}(\mathbf{0}) = X$, in particular.

($\Longleftarrow$) We claim that (X, d) is antisymmetrically connected, that is for each $x \in X$ the equality $T_{d_{\|\cdot|}}(x) = X$ must be proved. Firstly, $T_{d_{\|\cdot|}}(x) \subseteq X$ is clear. For the other inclusion, let $y \in X$. Since X is a vector space, $y - x \in X$ and so $y - x \in T_{d_{\|\cdot|}}(\mathbf{0})$ by the hypothesis. Then we have an antisymmetric path $(\mathbf{0}, x_1, ..., x_n = y - x)$ from $\mathbf{0}$ to $y - x$. In this case, we get a new antisymmetric path $(x, x_1 + x ..., x_n + x = y))$ from x to y. Hence $y \in T_{d_{\|\cdot|}}(x)$, and we obtain the fact $T_{d_{\|\cdot|}}(x) = X$. $\square$

Example 2.6. Let $\mathbb{R}^2$ be equipped with its usual real vector space structure and the asymmetric norm $\|x| = x_1 \vee x_2 \vee 0$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Here $x_1 = -x_2$ if and only if $\|x| = \|-x|$ (see Example 32 in [10]).

First of all, according to definition of the asymmetric norm $\|.|$ on $\mathbb{R}^2$, the T_0-quasi-metric generated by $\|.|$ is

$$d_{\|.|}((x,y),(a,b)) = \|(x-a, y-b)| = (x-a) \vee (y-b) \vee 0$$

whenever $(x,y), (a,b) \in \mathbb{R}^2$.

In this case,

$$Z_{d_{\|.|}}(\mathbf{0}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = -x_2\}.$$

Also, we obtain the symmetry set

$$Z_{d_{\|.|}}((h_1, h_2)) = \{(x_1 + h_1, -x_1 + h_2) \mid \ x_1 \in \mathbb{R}\} = C_{d_{\|.|}}((h_1, h_2))$$

and the antisymmetry set

$$R_{d_{\|.|}}((h_1, h_2)) = \{(x_1 + h_1, x_2 + h_2) \mid \ x_1 \neq -x_2\}$$

of $h = (h_1, h_2) \in \mathbb{R}^2$.

We are now in a position to show that $T_{d_{\|.|}}(\mathbf{0}) = T_{d_{\|.|}}((0,0)) = \mathbb{R}^2$. The inclusion $T_{d_{\|.|}}((0,0)) \subseteq \mathbb{R}^2$ is clear. For the other side, let $(u,v) \in \mathbb{R}^2$ such that $u \neq -v$. Thus $(u,v) \in R_{d_{\|.|}}((0,0))$. So $(u,v) \in T_{d_{\|.|}}((0,0))$ by the fact that $R_d \subseteq T_d$ given in Definition 1.12 i).

Now for the assumption $u = -v$ we have two cases:

If $u = -v = 0$ then $(u,v) = (0,0) \in T_{d_{\|.|}}((0,0))$, trivially.

If $u = -v \neq 0$ then the pair $((u,v),(0,0))$ is a symmetric pair and moreover, the path $((u,v), (\frac{-u}{2}, \frac{v}{2}), (0,0))$ will be an antisymmetric path from (u,v) to $(0,0)$. That is, $(u,v) \in T_{d_{\|.|}}((0,0))$.

Finally, $T_{d_{\|.|}}((0,0)) = \mathbb{R}^2$. Thus, the T_0-quasi-metric space $(\mathbb{R}^2, d_{\|.|})$ will be antisymmetrically connected by Theorem 2.5.

Due to this example, it is obvious that any symmetric pair (here, $((u,-u),(0,0))$) can be antisymmetrically connected as well, from Definition 1.12. $\square$

By virtue of Theorem 2.5 we have the following also:

Corollary 2.7. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. Then $(X, d_{\|\cdot|})$ is antisymmetrically connected if and only if $T_{d_{\|\cdot|}}(\mathbf{0})$ is τ_{d^s}-open.*

Proof. ($\Longrightarrow$) Straightforward.

($\Longleftarrow$) Firstly, let us prove that $T_{d_{\|\cdot|}}(x)$ is τ_{d^s}-open for all $x \in X$. If $z \in T_{d_{\|\cdot|}}(x)$ then $z - x \in T_{d_{\|\cdot|}}(\mathbf{0})$ by Lemma 2.4(2). In addition, since $T_{d_{\|\cdot|}}(\mathbf{0})$ is τ_{d^s}-open, there exists $\epsilon > 0$ such that $B_{d^s_{\|\cdot|}}(z - x, \epsilon) \subseteq T_{d_{\|\cdot|}}(\mathbf{0})$. Therefore, in a similar manner it is easy to verify that $B_{d^s_{\|\cdot|}}(z, \epsilon) \subseteq T_{d_{\|\cdot|}}(x)$. Indeed, if $a \in B_{d^s_{\|\cdot|}}(z, \epsilon)$ then $d^s(z, a) < \epsilon$. On the other hand, $d^s(z-a, x-a) = \|z-x-a+x\| = \|z-a\| = d^s(z, a) < \epsilon$, and so $a-x \in B_{d^s_{\|\cdot|}}(z-x, \epsilon)$. By the fact that $B_{d^s_{\|\cdot|}}(z - x, \epsilon) \subseteq T_{d_{\|\cdot|}}(\mathbf{0})$ we have $a - x \in T_{d_{\|\cdot|}}(\mathbf{0})$, so $a \in T_{d_{\|\cdot|}}(\mathbf{0}) + x$ that is $a \in T_{d_{\|\cdot|}}(x)$. It completes the proof of $B_{d^s_{\|\cdot|}}(z, \epsilon) \subseteq T_{d_{\|\cdot|}}(x)$. Finally, $T_{d_{\|\cdot|}}(x)$ is τ_{d^s}-open for each $x \in X$. In this case, it is easy to show that $T_{d_{\|\cdot|}}(x)$ is τ_{d^s}-closed, so these sets are clopen. Also, it is well-known that the normed topological spaces are path-connected, so connected. Therefore, in the topological space $\tau^s_{d_{\|\cdot|}} = \tau_{\|\cdot|^s} = \tau_{d_{\|\cdot\|}}$ clopen sets must be $\emptyset$ and X. Moreover, clearly $T_{d_{\|\cdot|}}(x) \neq \emptyset$ since $x \in T_{d_{\|\cdot|}}(x)$, so the sets $T_{d_{\|\cdot|}}(x)$ must be X, that is X will be antisymmetrically connected by Theorem 2.5. □

Lemma 2.8. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. Suppose that $x \in R_{d_{\|\cdot|}}(\mathbf{0})$. Then $\lambda x \in R_{d_{\|\cdot|}}(\mathbf{0})$ whenever $\lambda \in \mathbb{R}$.*

Proof. This is obvious for $\lambda \geq 0$, since $\|x| \neq \|-x|$ implies that $\|\lambda x| \neq \|-\lambda x|$, by property (b) of an asymmetric norm, so $\lambda x \in R_{d_{\|\cdot|}}(\mathbf{0})$. It holds also also for $\lambda < 0$ if we take $\lambda' = -\lambda > 0$, which yields the conclusion. □

With the similar argument of Lemma 2.8, we can establish the following more general result, the proof of which is left to the reader.

Corollary 2.9. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space and $\lambda \in \mathbb{R} \setminus \{0\}$.[1)] If $(x = x_0, x_1, \ldots, x_n = y)$ is an antisymmetric path from x to y in X, then $(\lambda x, \lambda x_1, \ldots, \lambda y)$ is an antisymmetric path from λx to λy in X.*

Proposition 2.10. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space and $x, y \in X$. Take $n \in \mathbb{N}$. In this case, $y \in R^n_{d_{\|\cdot|}}(x)$ implies that $y \in R_{d_{\|\cdot|}}(\mathbf{0}) + \ldots + R_{d_{\|\cdot|}}(\mathbf{0}) + R_{d_{\|\cdot|}}(x)$, where the latter sum has n summands. (Here, $R^1_d = R_d$ and for each $n \in \mathbb{N}$ we have $R^{n+1}_d = R_d \circ R^n_d$, where $d = d_{\|\cdot|}$.)*

Proof. Note that $y \in R^n_{d_{\|\cdot|}}(x)$ implies that there are $x_1, \ldots, x_{n-1} \in X$ such that $(x, x_1), \ldots, (x_{n-1}, y) \in R_{d_{\|\cdot|}}$, which means that there exist $x_1, \ldots, x_{n-1} \in X$ such that $x_1 \in R_{d_{\|\cdot|}}(x), x_2 \in R_{d_{\|\cdot|}}(x_1), \ldots, y \in R_{d_{\|\cdot|}}(x_{n-1})$. By Lemma 2.4(1), the condition implies that there are $x_1, \ldots, x_{n-1} \in X$ such that $x_1 - x \in R_{d_{\|\cdot|}}(\mathbf{0}), x_2 - x_1 \in R_{d_{\|\cdot|}}(\mathbf{0}), \ldots, y - x_{n-1} \in R_{d_{\|\cdot|}}(\mathbf{0})$, which implies that $y-x \in R_{d_{\|\cdot|}}(\mathbf{0}) + \cdots + R_{d_{\|\cdot|}}(\mathbf{0})$, which finally entails that $y \in R_{d_{\|\cdot|}}(\mathbf{0}) + \cdots + R_{d_{\|\cdot|}}(\mathbf{0}) + R_{d_{\|\cdot|}}(x)$ (with n summands). □

Proposition 2.11. *For a T_0-quasi-metric space (X, d), we have*

$$T_d = (\bigcup_{n \in \mathbb{N}} R^n_d) \cup \Delta_X$$

That is, T_d is the transitive hull of $R_d \cup \Delta_X$.

Proof. Trivial from the definitions. □

Consequently, by virtue of Proposition 2.11, the next result will be obvious in the context of asymmetrically normed real vector spaces.

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1) It looks reasonable to exclude the case $\lambda = 0$ in order to avoid trivialities.

Corollary 2.12. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. In this case,*

$$T_{d_{\|\cdot|}}(\mathbf{0}) = \bigcup_{n\in\mathbb{N}} R^n_{d_{\|\cdot|}}(\mathbf{0}) \cup \{\mathbf{0}\}$$

Corollary 2.13. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. Then $T_{d_{\|\cdot|}}(\mathbf{0})$ is equal to the smallest linear subspace containing $R_{d_{\|\cdot|}}(\mathbf{0})$ (which we shall denote as its linear hull lin $R_{d_{\|\cdot|}}(\mathbf{0})$ of $R_{d_{\|\cdot|}}(\mathbf{0})$).*

Proof. Take $x \in T_{d_{\|\cdot|}}(\mathbf{0})$. Then by Corollary 2.12 there exists some $n \in \mathbb{N}$ such that $x \in R^n_{d_{\|\cdot|}}(\mathbf{0})$. So by Proposition 2.10 we see that $x = \sum_{i=1}^n t_i$ where $t_i \in R(\mathbf{0})$ for each $i = 1, \dots, n$. Therefore x obviously belongs to the linear hull of $R_{d_{\|\cdot|}}(\mathbf{0})$.

Now suppose that x is in the linear hull lin $R_{d_{\|\cdot|}}(\mathbf{0})$ of $R_{d_{\|\cdot|}}(\mathbf{0})$. Then $x = \sum_{i=1}^n a_i b_i$ where for each $i = 1, \dots, n$, a_i is a real number and $b_i \in R_{d_{\|\cdot|}}(\mathbf{0})$. However $t_i := a_i b_i \in R_{d_{\|\cdot|}}(\mathbf{0})$ by Lemma 2.8. Thus $x \in R_{d_{\|\cdot|}}(\mathbf{0}) + \dots + R_{d_{\|\cdot|}}(\mathbf{0})$ (n summands). At this stage, if take $h_j = \sum_{i=1}^j t_i$ for each $j \in \{1, \dots, n\}$ then

$$(\mathbf{0}, h_1), (h_1, h_2), \dots, (h_{n-1}, h_n) \in R_{d_{\|\cdot|}}$$

by the definition of R. Therefore $x = h_n \in R^n_{d_{\|\cdot|}}(\mathbf{0})$, that is $x \in T_{d_{\|\cdot|}}(\mathbf{0})$. □

For the next theorem, the following notion will be required:

Definition 2.14. Let (X, d) be an antisymmetric T_0-quasi-metric space and $A \subseteq X$. Then the space (A, d_A) is called *antisymmetric subspace* if d_A is antisymmetric T_0-quasi-metric on A, where

$$d_A(x, y) = d(x, y)$$

for all $x, y \in A$.

Theorem 2.15. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. Suppose that $R_{d_{\|\cdot|}}(\mathbf{0}) + R_{d_{\|\cdot|}}(\mathbf{0}) = R_{d_{\|\cdot|}}(\mathbf{0})$. Then $T_{d_{\|\cdot|}}(v) = R_{d_{\|\cdot|}}(v)$ and $T_{d_{\|\cdot|}}(v)$ is an antisymmetric subspace of X whenever $v \in X$.*

Proof. We have $T_{d_{\|\cdot|}}(\mathbf{0}) = \bigcup_{n\in\mathbb{N}} R^n_{d_{\|\cdot|}}(\mathbf{0}) \cup \{\mathbf{0}\}$ by Corollary 2.12 and therefore by Proposition 2.10 the fact that $R_{d_{\|\cdot|}}(\mathbf{0}) = T_{d_{\|\cdot|}}(\mathbf{0})$ follows from iterations of the hypothesis $R_{d_{\|\cdot|}}(\mathbf{0}) + R_{d_{\|\cdot|}}(\mathbf{0}) = R_{d_{\|\cdot|}}(\mathbf{0})$. Hence for each $v \in X$, $T_{d_{\|\cdot|}}(v) = R_{d_{\|\cdot|}}(v)$ by Proposition 2.10 and the equalities given in Lemma 2.4.

Let $x, y \in R_{d_{\|\cdot|}}(v)$. Then $x - v \in R_{d_{\|\cdot|}}(\mathbf{0})$ and $y - v \in R_{d_{\|\cdot|}}(\mathbf{0})$, hence $v - y \in R_{d_{\|\cdot|}}(\mathbf{0})$ by Lemma 2.8. Thus $x - y = x - v + v - y \in R_{d_{\|\cdot|}}(\mathbf{0})$ by our assumption. Consequently $d(x, y) = \|x - y| \neq \|-(x - y)| = d(y, x)$. Therefore d is antisymmetric T_0-quasi-metric on the subset $R_{d_{\|\cdot|}}(v)$ of X and so, $T_{d_{\|\cdot|}}(v)$ will be an antisymmetric subspace of X, clearly. □

3. Further properties of antisymmetrically connected T_0-quasi-metric spaces

First of all, let us describe a function on the product of two T_0-quasi-metric spaces as follows:

Definition 3.1. For the T_0-quasi-metric spaces (X, d), (Y, q), the product T_0-quasi-metric space $(X \times Y, D)$ is defined as

$$D : (X \times Y) \times (X \times Y) \longrightarrow [0, \infty)$$

$$D((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) \vee q(y_1, y_2)$$

It is easy to verify that D is a T_0-quasi-metric on $X \times Y$ since d, q are T_0-quasi-metrics. Accordingly, we have:

Theorem 3.2. *If (X, d) and (Y, q) are antisymmetrically connected T_0-quasi-metric spaces then the product space $(X \times Y, D)$ is antisymmetrically connected.*

Proof. Let us choose $(x, y) \in X\times Y$ and show that $T_D((x, y)) = X\times Y$: For $(a, b) \in X\times Y$ it is clear that $a, x \in X$ and $b, y \in Y$. Since (X, d) and (Y, q) are antisymmetrically connected, there is a path $P_{a,x} = (a = x_0, x_1, ..., x_n = x)$ $(n \in \mathbb{N})$, on X such that $d(x_i, x_{i+1}) \neq d(x_{i+1}, x_i)$ for $i = 0, 1, ..., n-1$ and there is a path $P_{b,y} = (b = y_0, y_1, ..., y_m = y)$ $(m \in \mathbb{N})$, on Y such that $q(y_j, y_{j+1}) \neq q(y_{j+1}, y_j)$ for $j = 0, 1, ..., m-1$.

In this case, $P_{((a,b),(x,y))} = ((x_0, y_0) = (a, b), (x_1, b), ..., (x, b), (x, y_1), (x, y_2), ..., (x, y))$ is an antisymmetric path on $X \times Y$ from (a, b) to (x, y).

Then there is an antisymmetric path from (a, b) to (x, y) on $X \times Y$, and $(a, b) \in T_D((x, y))$ and so, the product space $(X \times Y, D)$ is antisymmetrically connected. □

By virtue of Theorem 3.2, we have the following result with the help of induction:

Corollary 3.3. *The finite product of antisymmetrically connected T_0-quasi-metric spaces is antisymmetrically connected.* □

Example 3.4. Consider the sets $X_1 = \{0, 1\}$ with the usual metric $u^s = |x-y|$ and $X_2 = \{0, 1\}$ with the T_0-quasi-metric u, where $u(x, y) = (x - y) \vee 0$. Then the space (X_1, u^s) is not antisymmetrically connected because of the fact that $T_{u^s}(0) = \{0\}$. Also the space (X_2, u) is antisymmetric and so antisymmetrically connected since $T_u(0) = X_2$. Hence, if take the product T_0-quasi-metric space $(X_1 \times X_2, u^s \vee u)$ then it is not antisymmetrically connected by the fact that $T_{u^s\vee u}(0, 0) = \{(0, 0), (0, 1)\} \neq X_1 \times X_2$.

On the other hand, note that the equality $T_{u^s}(0) \times T_u(0) = T_{u^s\vee u}(0, 0)$ in this example.

Particularly, the next theorem can be presented.

Theorem 3.5. *Let $I = \{1, \ldots, n\}$. For each $i \in \{1, \ldots, n\}$ let (X_i, d_i) be an antisymmetrically connected T_0-quasi-metric space. Then $(\Pi_{i\in I}X_i, d)$ is an antisymmetrically connected T_0-quasi-metric space. Here on $\Pi_{i\in I}X_i$ we have defined the T_0-quasi-metric d by setting*

$$d((x_i)_{i\in I}, (y_i)_{i\in I}) = \sum_{i\in I} d_i(x_i, y_i)$$

whenever $(x_i)_{i\in I}, (y_i)_{i\in I} \in \Pi_{i\in I}X_i$.

Proof. Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \Pi_{i\in I}X_i$. Since (X_1, d_1) is antisymmetrically connected, then there is an antisymmetric path from x_1 to y_1 in (X_1, d_1). Keeping all the remaining $n-1$ coordinates fixed, we obtain an antisymmetric path with finitely many steps from $(x_1, x_2, \ldots, x_n)$ to $(y_1, x_2, \ldots, x_n)$ in $(\Pi_{i\in I}X_i, d)$. Also we can find an antisymmetric path with finitely many steps from x_2 to y_2 in (X_2, d_2). Thus, similarly we can obtain an antisymmetric path with finitely many steps from $(y_1, x_2, x_3, \ldots, x_n)$ to $(y_1, y_2, x_3, \ldots, x_n)$ in $(\Pi_{i\in I}X_i, d)$.

By continuing this process inductively over the remaining $n - 2$ coordinates and concatenating the n antisymmetric paths in $(\Pi_{i\in I}X_i, d)$, we have constructed an antisymmetric path in $(\Pi_{i\in I}X_i, d)$ from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_n)$.

Finally, we reached the fact that the product space $(\Pi_{i\in I}X_i, d)$ is antisymmetrically connected. □

Because of Definitions 1.10 and 1.12, the fact that $R_d \subseteq T_d$ is trivial for a T_0-quasi-metric space (X, d). Additionally, it is easy to prove the following observations via Definitions 1.8 and 1.12:

Lemma 3.6. *(a) A T_0-quasi-metric space (X, d) is antisymmetric if and only if $R_d = X \times X \setminus \Delta_X$ and $T_d = X \times X$.*

(b) If R_d is transitive then $R_d \cup \Delta_X = T_d$.

(c) For $x \in X$, the subspace $(R_d(x), d_{R_d(x)})$ of T_0-quasi-metric space (X, d) is antisymmetrically connected.

Incidentally, it is clear from the definitions, there is no space with more than one point that is both antisymmetrically connected and metric.

Example 3.7. Let us define a T_0-quasi-metric on the set $X = \{1, 2, 3\}$ by the matrix

$$\mathbf{F} = \begin{pmatrix} 0 & 9 & 8 \\ 7 & 0 & 1 \\ 6 & 1 & 0 \end{pmatrix}$$

That is, $F = (f_{ij})$ where $f(i, j) = f_{ij}$ for $i, j \in X$. It is easy to prove that f is a T_0-quasi-metric on X. Here note that $f(2, 3) = f(3, 2)$, so the T_0-quasi-metric space (X, f) is not antisymmetric, that is $R_f \setminus \Delta_X \neq T_f \setminus \Delta_X$ by Lemma 3.6 (a).

On the other hand, we can construct an antisymmetric path between 2 and 3 such as $(2, 1, 3)$ since $(2, 1)$ and $(1, 3)$ are antisymmetric pairs. Hence, (X, f) is antisymmetrically connected and moreover f is not a metric by the fact that $f(1, 2) \neq f(2, 1)$.

At this stage, let us turn our attention to the question of whether the antisymmetric connectedness is preserved under an isometric isomorphism.

Theorem 3.8. *Let (X, d), (Y, e) be T_0-quasi-metric-spaces and $f : (X, d) \longrightarrow (Y, e)$ an isometric isomorphism. Then, (X, d) is antisymmetrically connected $\Longleftrightarrow$ (Y, e) is antisymmetrically connected.*

Proof. The proof is obvious as expected, and left to the interested reader. □

For a T_0-quasi-metric d, the specialization order $\leq_d$ is described as

$$x \leq_d y \iff d(x, y) = 0.$$

Accordingly, we have the next fact:

Proposition 3.9. *Let (X, d) be a T_0-quasi-metric space with at least two points. If (X, d) has a top element with respect to the specialization order $\leq_d$ of d then (X, d) is antisymmetrically connected.*

Proof. Take $x, y \in X$ and the top element $z \in X$. Firstly, let us consider the case $x \neq z$, $y = z$. Then $x \leq_d z$, so $d(x, z) = 0$ and $d(z, x) \neq 0$. That is, $(x, y) = (x, z)$ is an antisymmetric pair, thus x and y are antisymmetrically connected. The case $y \neq z$, $x = z$ is similar.

Now assume that $x \neq z$ and $y \neq z$. Thus $x \leq_d z$ and $y \leq_d z$ since z is top element in X. That is, $d(x, z) = 0$ and $d(y, z) = 0$. In this case, if $d(z, x) = 0$ then $d(z, x) = d(x, z) = 0$, so we have $z = x$ which is a contradiction. Hence, $d(z, x) \neq d(x, z)$. Similarly, with the assumption $d(z, y) = 0$ we have another contradiction. That is, $d(z, y) \neq d(y, z)$. Finally, the path (x, z, y) will be an antisymmetric path from x to y in (X, d), and it means that (X, d) is antisymmetrically connected space. □

Lemma 3.10. *Let (X, d) be a T_0-quasi-metric space. Then (X, d) is antisymmetrically connected if and only if (X, d^{-1}) is antisymmetrically connected.*

Proof. ($\Longrightarrow$) Since (X, d) is antisymmetrically connected, there is a path $P_{x,y} = (x = x_0, x_1, ..., y = x_n)$ from x to y in (X, d) such that $d(x_i, x_{i+1}) \neq d(x_{i+1}, x_i)$ $(i = 0, ..., n-1)$ whenever $x, y \in X$. In this case, $d^{-1}(x_{i+1}, x_i) = d(x_i, x_{i+1}) \neq d(x_{i+1}, x_i) = d^{-1}(x_i, x_{i+1})$ $(i = 0, ..., n-1)$ which implies the path $(x = x_0, x_1, ..., y = x_n)$ $(i = 0, ..., n-1)$ will be an antisymmetric path from x to y in the space (X, d^{-1}).

($\Longleftarrow$) It can be seen easily with a similar method. □

Example 3.11. Consider the (unbounded) Sorgenfrey T_0-quasi-metric on $\mathbb{R}$ as follows:

$$d(x, y) = \begin{cases} x - y\,; & x \geq y \\ 1\,; & x < y \end{cases}$$

Because of the equality $d(x, x+1) = 1 = d(x+1, x)$, we have

$$Z_d(x) = \{y \in \mathbb{R} \mid \ y = x \mp 1\} = \{x - 1, x + 1\}$$

and

$$Z_d = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid \quad y = x \mp 1\}$$

Thus, $Z_d \neq \Delta_{\mathbb{R}}$, so $(\mathbb{R}, d)$ is neither an antisymmetric space nor a metric space.

However, the T_0-quasi-metric space $(\mathbb{R}, d)$ is antisymmetrically connected:

For every $x, y \in \mathbb{R}$, $x < y$, it is easy to verify that the path $P_{x,y}(x, x-2, y+2, y)$ is an antisymmetric path from x to y. Similarly, if $y < x$ then the path $P_{x,y}(y, y-2, x+2, x)$ is antisymmetric path from y to x. Hence, $(\mathbb{R}, d)$ is antisymmetrically connected.

Besides this, the dual T_0-quasi-metric space $(\mathbb{R}, d^{-1})$ is also antisymmetrically connected by Lemma 3.10.

Now we are in a position to discuss the unions of antisymmetyrically connected spaces. In this context, the following will be required for the next theorem.

Remark 3.12. Let (X, d) be a T_0-quasi-metric space and $A \subseteq X$. Thus with the notation d_A described in Definition 2.14, we have the inclusion

$$T_{d_A}(a) \subseteq T_d(a) \cap A$$

for all $a \in A$.

Incidentally, we can give an example for the fact $T_{d_A}(a) \neq T_d(a) \cap A$, as follows:

Example 3.13. Let $X = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ and define the function $e' : X \to [0, \infty)$ as

$$e'(x,y) = \begin{cases} |x-y| & ;\ x < y \text{ and } (x,y) \neq (2^{-(n+1)}, 2^{-n}),\ \forall n \in \mathbb{N} \\ 2|x-y| & ;\ \text{otherwise} \end{cases}$$

for $x, y \in X$.

It is proved in [5, Example 2.11] that e' is a T_0-quasi-metric on X. Also it is easy to show that $T_{e'}(\frac{1}{2}) = X$ and $T_{e'_{X\setminus\{0\}}}(\frac{1}{2}) = \{\frac{1}{2}\}$ for $\frac{1}{2} \in X \setminus \{0\} = A$. Thus, $T_{e'}(\frac{1}{2}) \cap A \nsubseteq T_{e'_A}(\frac{1}{2})$.

Additionally, we proved the following fact as in [5, Lemma 3.14]:

If (X, d) is a T_0-quasi-metric space and $A \subseteq X$ is τ_{d^s}-dense, then

$$T_{d_A}(x) = T_d(x) \cap A$$

for $x \in A$.

Theorem 3.14. *Let (X, d) be a T_0-quasi-metric space and $A_i \subseteq X$ for all $i \in I$. In this case, if the subspaces (A_i, d_{A_i}) are antisymmetrically connected for all $i \in I$ and $A_j \cap A_i \neq \emptyset$ for $i \neq j$, then $(\bigcup_{i\in I} A_i, d_{\bigcup_{i\in I} A_i})$ is antisymmetrically connected.*

Proof. The proof mimics that of the corresponding result for path connected spaces, because of Remark 3.12. □

Consequently, we have the following result via Theorem 3.14:

Corollary 3.15. *Let (X, d) be a T_0-quasi-metric space and $\bigcap_{i\in I} A_i \neq \emptyset$ whenever $A_i \subseteq X$ for all $i \in I$. If the subspaces (A_i, d_{A_i}) are antisymmetrically connected for all $i \in I$, then the T_0-quasi-metric space $(\bigcup_{i\in I} A_i, d_{\bigcup_{i\in I} A_i})$ is antisymmetrically connected.* □

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