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Canonical F-planar mappings of spaces with affine connection onto 3-symmetric spaces

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Abstract. We consider canonical F-planar mappings of spaces with affine connection onto 3-symmetric spaces. The main equations for the mappings have been obtained as a closed system of PDEs of Cauchy type in covariant derivatives. We have found the maximum numbers of essential parameters which the general solution of the system depends on.

1. Introduction.

The paper is devoted to further study of the theory of almost geodesic mappings of affinely connected spaces. The theory goes back to the paper by T. Levi-Civita [20], in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note a remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems.

The theory of geodesic mappings has been developed by T. Thomas, H. Weyl, P.A. Shirokov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, and others, see [23, 31].

Issues arisen by the exploration were studied by V.F. Kagan, G. Vrançeanu, Ya.L. Shapiro, D.V. Vedenyapin. The authors discover special classes of $(n-2)$-spaces. In [24] A.Z. Petrov introduced the notion of quasi-geodesic mappings. In particular, holomorphically projective mappings of Kählerian spaces are special quasi-geodesic mappings; they were examined by T. Otsuki, Y. Tashiro, M. Prvanović, J. Mikeš, and others, see [23, 31].

A natural generalization of these classes of mappings is the class of almost geodesic mappings introduced by Sinyukov (see [30–32]). He also specified three types of almost geodesic mappings π_1, π_2, π_3. The theory of almost geodesic mappings was developed by V.S. Sobchuk [33, 34], N.Y. Yablonskaya [40], V.E. Berezovski, J. Mikeš [2–15, 39], Lj.S. Velimirović, N. Vesić, M.S. Stanković, [25, 27, 28, 35–38] et al.

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F-planar mappings were introduced by N.S. Sinyukov and J. Mikeš (see [21]) as a further generalization of almost geodesic and holomorphically projective ones. In the same manner, F-planar mappings were studied between manifolds endowed with non-symmetric linear connection [26].

Definition and theorems of the F-planar mappings theory which are given below are presented in monographs [22, 23] and in an expository article [19].

Our study is restricted to the local theory, and we assume that all geometric objects we consider not only continuous but also sufficiently smooth.

2. Basic definitions of the *F*-planar mappings theory of spaces with affine connection.

Let us consider an n-dimensional space A_n with an affine torsion-free connection ∇ and affinor structure F ($\neq Id$). The space is referred to a local coordinate system $x = (x^1, x^2, \dots, x^n)$, $\Gamma^h_{ij}(x)$ and $F^h_i(x)$ ($\neq a \cdot \delta^h_i$) are components of ∇ and F, where δ^h_i is the Kronecker delta and a is some function.

A curve ℓ given by the equation $x^h = x^h(t)$ is called *F-planar* if its tangent vector $\lambda^h(t) \equiv dx^h(t)/dt$ subjected by the parallel translation along the curve belongs to the distribution generated by the vector fields λ and $F\lambda$.

According to this definition, a curve ℓ is F-planar if and only if the following condition holds

$$\lambda^h_{,\alpha}\lambda^\alpha = \rho_1(x)\lambda^h + \rho_2(x)\lambda^\alpha F^h_\alpha,$$

where $\rho_1(t)$ and $\rho_2(t)$ are some functions of the parameter t, we denote by comma "," the covariant derivative with respect to the connection ∇ of the space A_n.

The set of F-planar curves in a space A_n is a very wide one. It includes geodesics, planar curves, quasi-geodesic curves, and others.

Suppose two affinely connected spaces A_n and $\bar{A}_n$ are given and there are defined structural affinors F and $\bar{F}$ respectively. J. Mikeš and N.S. Sinyukov [21] introduced the following terms:

A mapping $\pi : A_n \to \bar{A}_n$ is an *F-planar mapping* if any F-planar curve of A_n is mapped under f onto an $\bar{F}$-planar curve in $\bar{A}_n$.

Let us consider the spaces A_n and $\bar{A}_n$ which are referred to a common coordinate system $x^1, x^2, \dots, x^n$ with respect to the F-planar mapping. The tensor

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{1}$$

is called a *deformation tensor* [1, 31]. Here $\Gamma^h_{ij}(x)$ and $\bar{\Gamma}^h_{ij}(x)$ are components of the affine connection in A_n and $\bar{A}_n$.

From [19] it follows that a mapping $\pi : A_n \to \bar{A}_n$ ($n > 2$) is F-planar if and only if the deformation tensor $P^h_{ij}(x)$ of the mapping π in a common coordinate system $x^1, x^2, \dots, x^n$ satisfies the condition

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)},$$

where $\psi_i(x)$ and $\varphi_i(x)$ are some covectors, we denote by the round parentheses an operation called *symmetrization* without division with respect to the indices i and j.

An F-planar mapping π for which $\psi_i \equiv 0$ is called *canonical*. It is known that any F-planar mapping can be written as the composition of a canonical F-planar mapping and a geodesic mapping. The latter may be referred to as a trivial F-planar mapping.

Hence, canonical F-planar mappings in a common coordinate system $x^1, x^2, \dots, x^n$ with respect to the mapping are determined by the equations

$$P^h_{ij} = F^h_{(i}\varphi_{j)}. \tag{2}$$

Suppose that the affinor F^h_i defined in the space A_n satisfies the condition

$$F^h_\alpha F^\alpha_i = e\delta^h_i, \tag{3}$$

where $e = \mp 1$. Such F-planar mappings we denote by $\pi(e)$, where $e = \mp 1$.

3. Canonical F-planar mappings $\pi(e)$ ($e = \mp 1$) of spaces with affine connections onto 3-symmetric spaces.

A space $\bar{A}_n$ with an affine connection is called *(locally) symmetric* if its Riemann tensor field is parallel. Symmetric spaces were introduced by P.A. Shirokov [29] and É. Cartan [17], see also S. Helgason [18]. Precisely, symmetric spaces satisfy the conditions $\bar{R}^h_{ijk;m} = 0$, where $\bar{R}^h_{ijk}$ is the Riemann tensor of the space $\bar{A}_n$, the symbol ";" denotes a covariant derivative with respect to the connection $\bar{\nabla}$ of the space $\bar{A}_n$.

A space $\bar{A}_n$ with an affine connection is called 2-*symmetric* and 3-*symmetric* if its Riemann tensor $\bar{R}^h_{ijk}$ satisfies the conditions $\bar{R}^h_{ijk;m_1m_2} = 0$ and $\bar{R}^h_{ijk;m_1m_2m_3} = 0$.

Canonical F-planar mappings $\pi(e)$ ($e = \mp 1$) of spaces with affine connection onto 2-symmetric spaces were studied in [16].

Obviously, symmetric spaces are 2-symmetric spaces, and 2-symmetric spaces are 3-symmetric ones.

Let us consider canonical F-planar mappings $\pi(e)$ ($e = \mp 1$) of spaces with affine connection onto 3-symmetric spaces $\bar{A}_n$. The mapping are characterized by the equations (2). Assume that the spaces A_n and $\bar{A}_n$ are referred to a common coordinate system $x^1, x^2, \dots, x^n$ and the affinor F^h_i defined in the space A_n satisfies the condition (3).

Since $\bar{R}^h_{ijk;m} = \dfrac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha}$, then taking account of (1) we can obtain

$$\bar{R}^h_{ijk;m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{4}$$

According to the definition of covariant derivative

$$\left(\bar{R}^h_{ijk;m}\right)_{,\rho_1} = \frac{\partial \bar{R}^h_{ijk;m}}{\partial x^{\rho_1}} + \Gamma^h_{\alpha\rho_1}\bar{R}^\alpha_{ijk;m} - \Gamma^\alpha_{i\rho_1}\bar{R}^h_{\alpha jk;m} - \Gamma^\alpha_{j\rho_1}\bar{R}^h_{i\alpha k;m} - \Gamma^\alpha_{k\rho_1}\bar{R}^h_{ij\alpha;m} - \Gamma^\alpha_{m\rho_1}\bar{R}^h_{ijk;\alpha},$$

and taking account of (1), we have

$$\left(\bar{R}^h_{ijk;m}\right)_{,\rho_1} = \bar{R}^h_{ijk;m\rho_1} - P^h_{\alpha\rho_1}\bar{R}^\alpha_{ijk;m} + P^\alpha_{i\rho_1}\bar{R}^h_{\alpha jk;m} + P^\alpha_{j\rho_1}\bar{R}^h_{i\alpha k;m} + P^\alpha_{k\rho_1}\bar{R}^h_{ij\alpha;m} + P^\alpha_{m\rho_1}\bar{R}^h_{ijk;\alpha}. \tag{5}$$

Differentiating (4) with respect to x^{ρ_1} in the space A_n, we get

$$\begin{aligned}\left(\bar{R}^h_{ijk;m}\right)_{,\rho_1} = \bar{R}^h_{ijk,m\rho_1} &+ P^h_{m\alpha,\rho_1}\bar{R}^\alpha_{ijk} + P^h_{m\alpha}\bar{R}^\alpha_{ijk,\rho_1} - P^\alpha_{mi,\rho_1}\bar{R}^h_{\alpha jk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk,\rho_1}\\ &-P^\alpha_{mj,\rho_1}\bar{R}^h_{i\alpha k} - P^\alpha_{mj}\bar{R}^h_{i\alpha k,\rho_1} - P^\alpha_{mk,\rho_1}\bar{R}^h_{ij\alpha} - P^\alpha_{mk}\bar{R}^h_{ij\alpha,\rho_1}.\end{aligned} \tag{6}$$

Substituting in (5) from (6), we have

$$\begin{aligned}\bar{R}^h_{ijk,m\rho_1} = \bar{R}^h_{ijk;m\rho_1} &- P^h_{\alpha\rho_1}\bar{R}^\alpha_{ijk;m} + P^\alpha_{i\rho_1}\bar{R}^h_{\alpha jk;m} + P^\alpha_{j\rho_1}\bar{R}^h_{i\alpha k;m} + P^\alpha_{k\rho_1}\bar{R}^h_{ij\alpha;m} + P^\alpha_{m\rho_1}\bar{R}^h_{ijk;\alpha} - P^h_{m\alpha,\rho_1}\bar{R}^\alpha_{ijk}\\ -P^h_{m\alpha}\bar{R}^\alpha_{ijk,\rho_1} &+ P^\alpha_{mi,\rho_1}\bar{R}^h_{\alpha jk} + P^\alpha_{mi}\bar{R}^h_{\alpha jk,\rho_1} + P^\alpha_{mj,\rho_1}\bar{R}^h_{i\alpha k} + P^\alpha_{mj}\bar{R}^h_{i\alpha k,\rho_1} + P^\alpha_{mk,\rho_1}\bar{R}^h_{ij\alpha} + P^\alpha_{mk}\bar{R}^h_{ij\alpha,\rho_1}.\end{aligned} \tag{7}$$

Taking account of (2) and (4) the equations (7) can be written as

$$\bar{R}^h_{ijk,m\rho_1} = \bar{R}^h_{ijk;m\rho_1} + \Theta^h_{ijkm\rho_1}, \tag{8}$$

where

$$\begin{aligned}\Theta^h_{ijkm\rho_1} = &- F^h_{(\alpha}\varphi_{\rho_1)}\left(\bar{R}^\alpha_{ijk,m} + \Theta^\alpha_{ijkm}\right) + F^\alpha_{(i}\varphi_{\rho_1)}\left(\bar{R}^h_{\alpha jk,m} + \Theta^h_{\alpha jkm}\right) + F^\alpha_{(j}\varphi_{\rho_1)}\left(\bar{R}^h_{i\alpha k,m} + \Theta^h_{i\alpha km}\right) + F^\alpha_{(k}\varphi_{\rho_1)}\left(\bar{R}^h_{ij\alpha,m} + \Theta^h_{ij\alpha m}\right)\\ &+F^\alpha_{(m}\varphi_{\rho_1)}\left(\bar{R}^h_{ijk,\alpha} + \Theta^h_{ijk\alpha}\right) - F^h_{(m}\varphi_{\alpha),\rho_1}\bar{R}^\alpha_{ijk} - F^h_{(m}\varphi_{\alpha)}\bar{R}^\alpha_{ijk,\rho_1} + F^\alpha_{(m}\varphi_{i),\rho_1}\bar{R}^h_{\alpha jk} + F^\alpha_{(m}\varphi_{i)}\bar{R}^h_{\alpha jk,\rho_1} + F^\alpha_{(m}\varphi_{j),\rho_1}\bar{R}^h_{i\alpha k}\\ &+F^\alpha_{(m}\varphi_{j)}\bar{R}^h_{i\alpha k,\rho_1} + F^\alpha_{(m}\varphi_{k),\rho_1}\bar{R}^h_{ij\alpha} + F^\alpha_{(m}\varphi_{k)}\bar{R}^h_{ij\alpha,\rho_1},\end{aligned} \tag{9}$$

$$\Theta^h_{ijkm} = F^h_{(m}\varphi_{\alpha)}\bar{R}^\alpha_{ijk} - F^\alpha_{(m}\varphi_{i)}\bar{R}^h_{\alpha jk} - F^\alpha_{(m}\varphi_{j)}\bar{R}^h_{i\alpha k} - F^\alpha_{(m}\varphi_{k)}\bar{R}^h_{ij\alpha}. \tag{10}$$

Taking account of structure of the tensor Θ^h_{ijkm} defined by the formulas (10), one could see that the tensor $\Theta^h_{ijkm\rho_1}$ defined by the formulas (9), depends on the tensors F^h_k, $\bar{R}^h_{ijk}$, φ_k and their covariant derivatives with respect to the connection of the space A_n. Also, we assume that the tensor F^h_k called affinor is given, and the tensor satisfies the conditions (3).

According to the definition of covariant derivative

$$\left(\bar{R}^h_{ijk;m\rho_1}\right)_{,\rho_2} = \frac{\partial \bar{R}^h_{ijk;m\rho_1}}{\partial x^{\rho_2}} + \Gamma^h_{\alpha\rho_2}\bar{R}^\alpha_{ijk;m\rho_1} - \Gamma^\alpha_{i\rho_2}\bar{R}^h_{\alpha jk;m\rho_1} - \Gamma^\alpha_{j\rho_2}\bar{R}^h_{i\alpha k;m\rho_1} - \Gamma^\alpha_{k\rho_2}\bar{R}^h_{ij\alpha;m\rho_1} - \Gamma^\alpha_{m\rho_2}\bar{R}^h_{ijk;\alpha\rho_1} - \Gamma^\alpha_{\rho_1\rho_2}\bar{R}^h_{ijk;m\alpha},$$

and taking account of (1), we get

$$\left(\bar{R}^h_{ijk;m\rho_1}\right)_{,\rho_2} = \bar{R}^h_{ijk;m\rho_1\rho_2} - P^h_{\alpha\rho_2}\bar{R}^\alpha_{ijk;m\rho_1} + P^\alpha_{i\rho_2}\bar{R}^h_{\alpha jk;m\rho_1} + P^\alpha_{j\rho_2}\bar{R}^h_{i\alpha k;m\rho_1} + P^\alpha_{k\rho_2}\bar{R}^h_{ij\alpha;m\rho_1} + P^\alpha_{m\rho_2}\bar{R}^h_{ijk;\alpha\rho_1} + P^\alpha_{\rho_1\rho_2}\bar{R}^h_{ijk;m\alpha}. \quad (11)$$

Because of (2) and (8) it follows from (11) that

$$\left(\bar{R}^h_{ijk;m\rho_1}\right)_{,\rho_2} = \bar{R}^h_{ijk;m\rho_1\rho_2} - F^h_{(\alpha}\varphi_{\rho_2)}\left(\bar{R}^\alpha_{ijk,m\rho_1} - \Theta^\alpha_{ijkm\rho_1}\right) + F^\alpha_{(i}\varphi_{\rho_2)}\left(\bar{R}^h_{\alpha jk,m\rho_1} - \Theta^h_{\alpha jkm\rho_1}\right) \quad (12)$$

$$+F^\alpha_{(j}\varphi_{\rho_2)}\left(\bar{R}^h_{i\alpha k,m\rho_1} - \Theta^h_{i\alpha km\rho_1}\right) + F^\alpha_{(k}\varphi_{\rho_2)}\left(\bar{R}^h_{ij\alpha,m\rho_1} - \Theta^h_{ij\alpha m\rho_1}\right) + F^\alpha_{(m}\varphi_{\rho_2)}\left(\bar{R}^h_{ijk,\alpha\rho_1} - \Theta^h_{ijk\alpha\rho_1}\right) + F^\alpha_{(\rho_1}\varphi_{\rho_2)}\left(\bar{R}^h_{ijk,m\alpha} - \Theta^h_{ijkm\alpha}\right).$$

Let us differentiate the equations (8) covariantly with respect to x^{ρ_2} in the space A_n. Taking account of (12), we get

$$\begin{aligned} \bar{R}^h_{ijk,m\rho_1\rho_2} = \bar{R}^h_{ijk;m\rho_1\rho_2} &- F^h_{(\alpha}\varphi_{\rho_2)}\left(\bar{R}^\alpha_{ijk,m\rho_1} - \Theta^\alpha_{ijkm\rho_1}\right) + F^\alpha_{(i}\varphi_{\rho_2)}\left(\bar{R}^h_{\alpha jk,m\rho_1} - \Theta^h_{\alpha jkm\rho_1}\right) \\ &+F^\alpha_{(j}\varphi_{\rho_2)}\left(\bar{R}^h_{i\alpha k,m\rho_1} - \Theta^h_{i\alpha km\rho_1}\right) + F^\alpha_{(k}\varphi_{\rho_2)}\left(\bar{R}^h_{ij\alpha,m\rho_1} - \Theta^h_{ij\alpha m\rho_1}\right) \\ &+F^\alpha_{(m}\varphi_{\rho_2)}\left(\bar{R}^h_{ijk,\alpha\rho_1} - \Theta^h_{ijk\alpha\rho_1}\right) + F^\alpha_{(\rho_1}\varphi_{\rho_2)}\left(\bar{R}^h_{ijk,m\alpha} - \Theta^h_{ijkm\alpha}\right) + \Theta^h_{ijkm\rho_1,\rho_2}. \end{aligned} \quad (13)$$

Suppose that the space with affine connection $\bar{A}_n$ is a 3-symmetric space, i.e. $\bar{R}^h_{ijk;m_1m_2m_3} = 0$, and from (13) we get

$$\begin{aligned} \bar{R}^h_{ijk,m\rho_1\rho_2} = &-F^h_{(\alpha}\varphi_{\rho_2)}\left(\bar{R}^\alpha_{ijk,m\rho_1} - \Theta^\alpha_{ijkm\rho_1}\right) + F^\alpha_{(i}\varphi_{\rho_2)}\left(\bar{R}^h_{\alpha jk,m\rho_1} - \Theta^h_{\alpha jkm\rho_1}\right) \\ &+F^\alpha_{(j}\varphi_{\rho_2)}\left(\bar{R}^h_{i\alpha k,m\rho_1} - \Theta^h_{i\alpha km\rho_1}\right) + F^\alpha_{(k}\varphi_{\rho_2)}\left(\bar{R}^h_{ij\alpha,m\rho_1} - \Theta^h_{ij\alpha m\rho_1}\right) \\ &+F^\alpha_{(m}\varphi_{\rho_2)}\left(\bar{R}^h_{ijk,\alpha\rho_1} - \Theta^h_{ijk\alpha\rho_1}\right) + F^\alpha_{(\rho_1}\varphi_{\rho_2)}\left(\bar{R}^h_{ijk,m\alpha} - \Theta^h_{ijkm\alpha}\right) + \Theta^h_{ijkm\rho_1,\rho_2}, \end{aligned} \quad (14)$$

where the tensor $\Theta^h_{ijkm\rho_1}$ is defined by formulas (9) and (10).

It is easy to see, that because of (9) and (10) the right hand side of the equations (14) depends on tensors F^h_k, $\bar{R}^h_{ijk}$, φ_k and on their first and second covariant derivatives with respect to the connection of the space A_n. Also, we assume, that the affinor F^h_i and its covariant derivatives $F^h_{i,j}$, $F^h_{i,jk}$ are known.

It is known [31] that the Riemann tensors of the spaces with affine connection A_n and $\bar{A}_n$ are related to each other by the equations

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{\alpha j} - P^\alpha_{ij}P^h_{\alpha k}. \quad (15)$$

Taking account of structure of the deformation tensor (2), transforming the equations (15), we get

$$\varphi_{i,j}F^h_k + \varphi_{k,j}F^h_i - \varphi_{i,k}F^h_j - \varphi_{j,k}F^h_i = B^h_{ijk}, \quad (16)$$

where

$$B^h_{ijk} = \bar{R}^h_{ijk} - R^h_{ijk} - \varphi_i\left(F^h_{k,j} - F^h_{j,k} + e\delta^h_k\varphi_j + \varphi_\alpha F^\alpha_k F^h_j - e\delta^h_j\varphi_k - \varphi_\alpha F^\alpha_j F^h_k\right) - \varphi_k\left(F^h_{i,j} + \varphi_\alpha F^\alpha_i F^h_j\right) + \varphi_j\left(F^h_{i,k} + \varphi_\alpha F^\alpha_i F^h_k\right).$$

Contracting (16) with the affinor F^m_ρ for indices ρ and h, we have

$$\delta^m_k \varphi_{i,j} + \delta^m_i \varphi_{k,j} - \delta^m_j \varphi_{i,k} - \delta^m_i \varphi_{j,k} = eB^\alpha_{ijk} F^m_\alpha. \tag{17}$$

Contracting (17) for m indices k, we get

$$\varphi_{k,j} - \varphi_{j,k} = \frac{e}{n+1} B^\alpha_{\beta jk} F^\beta_\alpha. \tag{18}$$

Finaly, let us contract (17) with respect to m and k. We obtain

$$n\varphi_{i,j} - \varphi_{j,i} = eB^\alpha_{ij\beta} F^\beta_\alpha. \tag{19}$$

Taking account of (18), the equations (19) could be written in the form

$$\varphi_{i,j} = \frac{e}{n-1}\Big(B^\alpha_{ij\beta} - \frac{1}{n+1} B^\alpha_{\beta ji}\Big) F^\beta_\alpha. \tag{20}$$

It should be remarked that the formulas (20) are obtained for the general case of canonical F-planar mappings $\pi(e)$ ($e = \mp 1$) of spaces with affine connection.

Let us introduce the tensors $\bar{R}^h_{ijkm}$ and $\bar{R}^h_{ijkm\rho_1}$ defined by

$$\bar{R}^h_{ijk,m} = \bar{R}^h_{ijkm}, \tag{21}$$

$$\bar{R}^h_{ijkm,\rho_1} = \bar{R}^h_{ijkm\rho_1}. \tag{22}$$

Taking account of (21) and (22), we could write the equations (14) in the form

$$\begin{aligned} \bar{R}^h_{ijkm\rho_1,\rho_2} = &-F^h_{(\alpha}\varphi_{\rho_2)}\Big(\bar{R}^\alpha_{ijkm\rho_1} - \Theta^\alpha_{ijkm\rho_1}\Big) + F^\alpha_{(i}\varphi_{\rho_2)}\Big(\bar{R}^h_{\alpha jkm\rho_1} - \Theta^h_{\alpha jkm\rho_1}\Big) + F^\alpha_{(j}\varphi_{\rho_2)}\Big(\bar{R}^h_{i\alpha km\rho_1} - \Theta^h_{i\alpha km\rho_1}\Big) \\ &+F^\alpha_{(k}\varphi_{\rho_2)}\Big(\bar{R}^h_{ij\alpha m\rho_1} - \Theta^h_{ij\alpha m\rho_1}\Big) + F^\alpha_{(m}\varphi_{\rho_2)}\Big(\bar{R}^h_{ijk\alpha\rho_1} - \Theta^h_{ijk\alpha\rho_1}\Big) + F^\alpha_{(\rho_1}\varphi_{\rho_2)}\Big(\bar{R}^h_{ijkm\alpha} - \Theta^h_{ijkm\alpha}\Big) + \Theta^h_{ijkm\rho_1,\rho_2}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \Theta^h_{ijkm\rho_1} = &-F^h_{(\alpha}\varphi_{\rho_1)}\Big(\bar{R}^\alpha_{ijkm} + \Theta^\alpha_{ijkm}\Big) + F^\alpha_{(i}\varphi_{\rho_1)}\Big(\bar{R}^h_{\alpha jkm} + \Theta^h_{\alpha jkm}\Big) + F^\alpha_{(j}\varphi_{\rho_1)}\Big(\bar{R}^h_{i\alpha km} + \Theta^h_{i\alpha km}\Big) + F^\alpha_{(k}\varphi_{\rho_1)}\Big(\bar{R}^h_{ij\alpha m} + \Theta^h_{ij\alpha m}\Big) \\ &+F^\alpha_{(m}\varphi_{\rho_1)}\Big(\bar{R}^h_{ijk\alpha} + \Theta^h_{ijk\alpha}\Big) - F^h_{(m}\varphi_{\alpha),\rho_1}\bar{R}^\alpha_{ijk} - F^h_{(m}\varphi_{\alpha)}\bar{R}^\alpha_{ijk\rho_1} + F^\alpha_{(m}\varphi_{i),\rho_1}\bar{R}^h_{\alpha jk} + F^\alpha_{(m}\varphi_{i)}\bar{R}^h_{\alpha jk\rho_1} + F^\alpha_{(m}\varphi_{j),\rho_1}\bar{R}^h_{i\alpha k} \\ &+F^\alpha_{(m}\varphi_{j)}\bar{R}^h_{i\alpha k\rho_1} + F^\alpha_{(m}\varphi_{k),\rho_1}\bar{R}^h_{ij\alpha} + F^\alpha_{(m}\varphi_{k)}\bar{R}^h_{ij\alpha\rho_1}, \end{aligned} \tag{24}$$

the tensor Θ^h_{ijkm} is defined by the formulas (10).

In the following we have assumed that in (24) covariant derivatives of the vector φ_i are expressed according to (20); in the tensor $\Theta^h_{ijkm\rho_1,\rho_2}$ we could see at (23), covariant derivatives of the vector φ_i are also expressed according to (20), and covariant derivatives of the tensor $\bar{R}^h_{ijk}$ are expressed according to (21).

Obviously, in the space A_n the equations (20), (21), (22), and (23) form a system of PDEs of Cauchy type with respect to the functions $\varphi_i(x)$, $\bar{R}^h_{ijk}(x)$, $\bar{R}^h_{ijkm}(x)$, $\bar{R}^h_{ijkm\rho_1}(x)$.

The functions $\bar{R}^h_{ijk}(x)$, and also $\bar{R}^h_{ijkl}(x)$, $\bar{R}^h_{ijklm}(x)$, must satisfy the algebraic conditions

$$\bar{R}^h_{i(jk)} = 0, \quad \bar{R}^h_{(ijk)} = 0, \qquad \bar{R}^h_{i(jk)l} = 0, \quad \bar{R}^h_{(ijk)l} = 0, \qquad \bar{R}^h_{i(jk)lm} = 0, \quad \bar{R}^h_{(ijk)lm} = 0. \tag{25}$$

Hence we have proved the theorem.

Theorem 3.1. *In order that a space A_n with an affine connection admit a canonical F-planar mapping $\pi(e)$ ($e = \mp 1$) onto a 3-symmetric space $\bar{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives* (20), (21), (22), (23), *and* (25) *have a solution with respect to the unknown functions $\varphi_i(x)$, $\bar{R}^h_{ijk}(x)$, $\bar{R}^h_{ijkl}(x)$, $\bar{R}^h_{ijklm}(x)$.*

Obviously, the general solution of the closed mixed system of Cauchy type (20), (21), (22), (23), and (25) depends on no more than

$$1/3\, n^2\,(n^2-1)\,(1+n+n^2) + n$$

essential parameters.

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