Filomat 37:20 (2023), 6843–6853
https://doi.org/10.2298/FIL2320843A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hyperbolic Ricci soliton on warped product manifolds

Shahroud Azami[a], Ghodratallah Fasihi-Ramandi[a]

[a]*Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran*

Abstract. In this paper, we investigate hyperbolic Ricci soliton as the special solution of hyperbolic geometric flow on warped product manifolds. Then, especially, we study these manifolds admitting either a conformal vector field or a concurrent vector field. Also, the question that:" whether or not a hyperbolic soliton reduces to an Einstein manifold?" is considered and answered. Finally, we obtain some necessary conditions for generalized Robertson-Walker space-time to be a hyperbolic Ricci soliton.

1. Introduction

The concept of warped product metrics was first introduced by Bishop and O'Nill [4] to construct examples of Riemannian manifolds with negative curvature. In pseudo-Riemannian geometry, using of warped product manifolds and their generic forms, many new examples with interesting curvature properties have been constructed. For instance, Einstein spaces [3, 22] and symmetric spaces [2].

On the other hand, geometric flows are important topic in differential geometry, because by these flows we can find canonical metrics on their underlying Riemannian manifolds. A geometric flow is an evolution of a geometric structure under a differential equation with a functional on a manifold.
One of these geometric flows is hyperbolic geometric flow which is a system of nonlinear evolution partial differential equations of second order, it is very similar to wave equation flow metrics, and defines as follows

$$\frac{\partial^2}{\partial t^2}g = -2Ric, \qquad g(0) = g_0, \qquad \frac{\partial g}{\partial t}(0) = k_0, \tag{1}$$

where k_0 is a symmetric 2-tensor field on M. Also, one can see that this flow is similar to Einstein equation

$$\frac{\partial^2}{\partial t^2}g_{ij} = -2R_{ij} - \frac{1}{2}g^{pq}\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{pq}}{\partial t} + g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t}.$$

The existences and uniqueness of (1) studied in [11] on closed Riemannian manifolds. Also, Lu in [25] studied the Ricci flow and hyperbolic geometric flow on warped product manifolds.
Suppose that $(M^n, g(t))$ is a solution of the hyperbolic geometric flow on a time interval (a, b) containing 0, and set $g_0 = g(0)$. We say that $g(t)$ is a self-similar solution of the hyperbolic geometric flow if there exist

2020 *Mathematics Subject Classification*. Primary 53E40; Secondary 53C25.
Keywords. Warped product manifolds; Hyperbolic Ricci soliton; Concurrent vector fields; Robertson-Walker space-time.
Received: 02 February 2023; Accepted: 05 March 2023
Communicated by Mića Stanković
Email addresses: azami@sci.ikiu.ac.ir (Shahroud Azami), fasihi@sci.ikiu.ac.ir (Ghodratallah Fasihi-Ramandi)

a scalar functions $\sigma(t)$ and a diffeomorphism ϕ_t on M^n such that $g(t) = \sigma(t)\phi_t^*(g_0)$ for all $t \in (a, b)$. We may assume without loss of the generality that $\sigma(0) = 2$, $\sigma'(0) = \lambda$, $\sigma''(0) = -2\mu$, and $\phi_0 = id$. Then we have

$$Ric(g_0) + \lambda\mathcal{L}_X g_0 + (\mathcal{L}_X \circ \mathcal{L}_X)g_0 = \mu g_0 \tag{2}$$

where $X = Y(0)$ and $Y(t)$ is the family of vector fields generating the diffeomorphisms ϕ_t. In this case, we say g_0 is a hyperbolic Ricci soliton and, we show it by $(M, g_0, X, \lambda, \mu)$. If X vanishes identically, a hyperbolic Ricci soliton is an Einstein metric. If $\lambda = \frac{1}{2}$ and the vector field X is a 2-Killing vector filed, i.e., $(\mathcal{L}_X \circ \mathcal{L}_X)g_0 = 0$ then a hyperbolic Ricci soliton is a Ricci soliton. When the vector field $X = \nabla f$ for some smooth functions $f : M \to \mathbb{R}$, we say that $(M, g_0, \nabla f, \lambda, \mu)$ is a gradient hyperbolic Ricci soliton. 2-Killing vector fields were firstly introduced by Németh in [28] and, Cruz Neto et al. in [10] showed the importance of 2-Killing vector fields on Lorentzian geometry. However all Killing vector fields are 2-Killing vector fields, but there are also examples of 2-Killing vector fields that are not Killing vector fields (see [28]).
The concept of Ricci solitons was introduced by Hamilton [18], which are natural generalizations of Einstein metrics. Since then, Ricci solitons have been extensively studied for different reasons and in different spaces [5, 7, 8, 15, 26, 27, 30, 31].
In [21], the authors obtain a criteria that the Riemannian manifold M is Einstein or a gradient Ricci soliton using of the second derivative of warping function f in the warped and Lorentzian warped product spaces of the form $\mathbb{R} \times_f M$ with gradient Ricci solitons. Also, in [1, 6, 13, 14, 19, 20, 23, 24, 36], have been studied the Ricci solitons and gradient Ricci solitons on warped product manifolds.

Let M_i, $i = 1, 2$ be two smooth pseudo-Riemannian manifolds with pseudo Riemannian metrics g_i for $i = 1, 2$. Let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the natural projections on M_1 and M_2. Also, let $f : M_1 \to (0, \infty)$ be a smooth positive function. The warped product manifold $M = M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ equipped with the metric $g = g_1 \oplus f^2 g_2$ defined by $g = \pi_1^*(g_1) \oplus (f \circ \pi_1)^2 \pi_2^* g_2$, where $*$ denotes the pull-back operator on tensors. The function f is called the warping function of the warped product manifold $M_1 \times_f M_2$. In following we assume that ∇^i, $i = 1, 2$, Ric^i and $\mathcal{L}^i$ are the Levi-Civita connections, Ricci tensors and Lie derivatives on M_i, respectively. Also, we denote the hessian of a smooth function f by H^f.

In this paper, we will consider the warped product metrics combining with hyperbolic Ricci solitons and we obtain some results about hyperbolic Ricci solitons on warped product manifolds.

2. Preliminaries

Now, we have the following two proposition from [4, 9, 29, 35].

Proposition 2.1. *Let $(M, g) = (M_1 \times_f M_2, g_1 \oplus f^2 g_2)$ be a warped product manifold with function $f > 0$ on M_1. Then*

1) $\nabla_{X_1} Y = \nabla^1_{X_1} Y_1$,

2) $\nabla_{X_1} Y_2 = \nabla_{Y_2} X_1 = \frac{X_1 f}{f} Y_2$,

3) $\nabla_{X_2} Y_2 = -f g_2(X_2, Y_2)\nabla f + \nabla^2_{X_2} Y_2$,

4) $(\mathcal{L}_Z g)(X, Y) = (\mathcal{L}^1_{Z_1} g_1)(X_1, Y_1) + (f^2 \mathcal{L}_{Z_2} g_2)(X_2, Y_2) + 2f(Z_1 f)g_2(X_2, Y_2)$,

for all vector fields $X = X_1 + X_2$, $Y = Y_1 + Y_2$ and $Z = Z_1 + Z_2$ on M where $X_i, Y_i, Z_i \in \mathcal{X}(M_i)$, $i = 1, 2$ and ∇f is the gradient of f.

Proposition 2.2. *Let $(M, g) = (M_1 \times_f M_2, g_1 \oplus f^2 g_2)$ be a warped product manifold with function $f > 0$ on M_1 and* $\dim(M_2) = n_2$. *Then*

1) $Ric(X_1, Y_1) = Ric^1(X_1, Y_1) - \frac{n_2}{f} H^f(X_1, Y_1)$, *2)* $Ric(X_1, Y_2) = 0$,

3) $Ric(X_2, Y_2) = Ric^2(X_2, Y_2) - f^\sharp g_2(X_2, Y_2)$,

for all vector fields $X_i, Y_i \in \mathcal{X}(M_i)$, $i = 1, 2$ *where* $f^\sharp = f\Delta f + (n_2 - 1)|\nabla f|^2$.

Corollary 2.3. *Let* $(M, g) = (M_1 \times_f M_2, g_1 \oplus f^2 g_2)$ *be a warped product manifold with function* $f > 0$ *on* M_1. *Then*

1) $(\mathcal{L}_Z\mathcal{L}_Z g)(X_1, Y_1) = (\mathcal{L}^1_{Z_1}\mathcal{L}^1_{Z_1} g_1)(X_1, Y_1)$,

2) $(\mathcal{L}_Z\mathcal{L}_Z g)(X_2, Y_2) = f^2(\mathcal{L}^2_{Z_2}\mathcal{L}^2_{Z_2} g_2)(X_2, Y_2) + 2Z_1(f^2)(\mathcal{L}^2_{Z_2} g_2)(X_2, Y_2)$
$+Z_1(Z_1(f^2))g_2(X_2, Y_2)$,

3) $(\mathcal{L}_Z\mathcal{L}_Z g)(X_1, Y_2) = -\frac{X_1 f}{f}\Big(f^2(\mathcal{L}^2_{Z_2} g_2)(Z_2, Y_2) + Z_1(f^2)g_2(Z_2, Y_2)\Big)$,

for all vector fields $Z = Z_1 + Z_2$ *on* M *and* $X_i, Y_i, Z_i \in \mathcal{X}(M_i)$, $i = 1, 2$.

Proof. From the Proposition 2.1 we have

$$\begin{aligned}
\mathcal{L}_Z X_1 &= \nabla_Z X_1 - \nabla_{X_1} Z = \nabla_{Z_1} X_1 + \nabla_{Z_2} X_1 - \nabla^1_{X_1} Z_1 \\
&= \nabla^1_{Z_1} X_1 + \frac{X_1 f}{f} Z_2 - \nabla^1_{X_1} Z_1 \\
&= \mathcal{L}^1_{Z_1} X_1 + \frac{X_1 f}{f} Z_2.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\mathcal{L}_Z\mathcal{L}_Z g)(X_1, Y_1) &= \mathcal{L}_Z(\mathcal{L}_Z g(X_1, Y_1)) - \mathcal{L}_Z g(\mathcal{L}_Z X_1, Y_1) - \mathcal{L}_Z g(X_1, \mathcal{L}_Z Y_1) \\
&= \mathcal{L}_Z(\mathcal{L}^1_{Z_1} g_1(X_1, Y_1)) - \mathcal{L}^1_{Z_1} g_1(\mathcal{L}_Z X_1, Y_1) - \mathcal{L}^1_{Z_1} g_1(X_1, \mathcal{L}_Z Y_1) \\
&= \mathcal{L}^1_Z(\mathcal{L}^1_{Z_1} g_1(X_1, Y_1)) - \mathcal{L}^1_{Z_1} g_1(\mathcal{L}^1_{Z_1} X_1 + \frac{X_1 f}{f} Z_2, Y_1) \\
&\quad -\mathcal{L}^1_{Z_1} g_1(X_1, \mathcal{L}^1_{Z_1} Y_1 + \frac{Y_1 f}{f} Z_2) \\
&= (\mathcal{L}^1_{Z_1}\mathcal{L}^1_{Z_1} g_1)(X_1, Y_1).
\end{aligned}$$

Using again the Proposition 2.1 we get

$$\begin{aligned}
\mathcal{L}_Z X_2 &= \nabla_Z X_2 - \nabla_{X_2} Z = \nabla_{Z_1} X_2 + \nabla_{Z_2} X_2 - \nabla_{X_2} Z_1 - \nabla_{X_2} Z_2 \\
&= \nabla^2_{Z_2} X_2 - \nabla^2_{X_2} Z_2 \\
&= \mathcal{L}^2_{Z_2} X_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\mathcal{L}_Z\mathcal{L}_Z g)(X_2, Y_2) &= \mathcal{L}_Z(\mathcal{L}_Z g(X_2, Y_2)) - \mathcal{L}_Z g(\mathcal{L}_Z X_2, Y_2) - \mathcal{L}_Z g(X_2, \mathcal{L}_Z Y_2) \\
&= \mathcal{L}_Z\left(f^2\mathcal{L}^2_{Z_1} g_2(X_2, Y_2) + Z_1(f^2)g_2(X_2, Y_2)\right) \\
&\quad -f^2\mathcal{L}^2_{Z_2} g_2(\mathcal{L}^2_{Z_2} X_2, Y_2) - Z_1(f^2)g_2(\mathcal{L}^2_{Z_2} X_2, Y_2) \\
&\quad -f^2\mathcal{L}^2_{Z_2} g_2(X_2, \mathcal{L}^2_{Z_2} Y_2) - Z_1(f^2)g_2(X_2, \mathcal{L}^2_{Z_2} Y_2) \\
&= f^2(\mathcal{L}^2_{Z_2}\mathcal{L}^2_{Z_2} g_2)(X_2, Y_2) + 2Z_1(f^2)(\mathcal{L}^2_{Z_2} g_2)(X_2, Y_2) \\
&\quad +Z_1(Z_1(f^2))g_2(X_2, Y_2).
\end{aligned}$$

Also, we have

$$
\begin{aligned}
(\mathcal{L}_Z\mathcal{L}_Zg)(X_1,Y_2) &= \mathcal{L}_Z(\mathcal{L}_Zg(X_1,Y_2)) - \mathcal{L}_Zg(\mathcal{L}_ZX_1,Y_2) - \mathcal{L}_Zg(X_1,\mathcal{L}_ZY_2)\\
&= -\mathcal{L}_Zg(\mathcal{L}^1_{Z_1}X_1 + \frac{X_1f}{f}Z_2,Y_2) - \mathcal{L}_Zg(X_1,\mathcal{L}^2_{Z_2}Y_2)\\
&= -\frac{X_1f}{f}\mathcal{L}_Zg(Z_2,Y_2)\\
&= -\frac{X_1f}{f}\Big(f^2(\mathcal{L}^2_{Z_2}g_2)(Z_2,Y_2) + Z_1(f^2)g_2(Z_2,Y_2)\Big).
\end{aligned}
$$

□

Theorem 2.4. *Let the connected warped product manifold $(M_1\times_f M_2, g_1\oplus f^2g_2, \xi=\xi_1+\xi_2,\lambda,\mu)$ be a hyperbolic Ricci soliton. Then either f is constant or the operator $T:\mathcal{X}(M_2)\to\mathbb{R}$ vanishes, where $T(X_2)=f^2(\mathcal{L}^2_{\xi_2}g_2)(X_2,\xi_2)+\xi_1(f^2)g_2(X_2,\xi_2)$.*

Proof. From the definition of hyperbolic Ricci soliton we get

$$Ric(X,Y)+\lambda\mathcal{L}_\xi g(X,Y)+(\mathcal{L}_\xi\circ\mathcal{L}_\xi)g(X,Y)=\mu g(X,Y), \tag{3}$$

for all vector fields X, Y on $M_1\times_f M_2$. If we assume that $X=X_1\in\mathcal{X}(M_1)$ and $Y=Y_2\in\mathcal{X}(M_2)$, then the part 4 of Proposition 2.1, the part 2 of Proposition 2.2, and the part 3 of Corollary 2.3, imply that

$$(X_1f)\Big(f^2(\mathcal{L}^2_{\xi_2}g_2)(\xi_2,Y_2)+\xi_1(f^2)g_2(\xi_2,Y_2)\Big)=0, \tag{4}$$

or equivalently $(X_1f)T(Y_2)=0$ for any vector fields $X_1\in\mathcal{X}(M_1)$ and $Y_2\in\mathcal{X}(M_2)$. This shows that f is constant or $T=0$. □

Theorem 2.5. *Let the warped product manifold $(M_1\times_f M_2, g_1\oplus f^2g_2, \xi=\xi_1+\xi_2,\lambda,\mu)$ be a hyperbolic Ricci soliton and $H^f=0$. Then the manifold $(M_1,g_1,\xi_1,\lambda,\mu)$ is a hyperbolic Ricci soliton.*

Proof. From the definition of hyperbolic Ricci soliton we get

$$Ric(X,Y)+\lambda\mathcal{L}_\xi g(X,Y)+(\mathcal{L}_\xi\circ\mathcal{L}_\xi)g(X,Y)=\mu g(X,Y), \tag{5}$$

for all vector fields X, Y on $M_1\times_f M_2$. If we assume that $X=X_1\in\mathcal{X}(M_1)$ and $Y=Y_1\in\mathcal{X}(M_1)$, then by $H^f=0$, the part 4 of Proposition 2.1, the part 1 of Proposition 2.2, and the part 1 of Corollary 2.3, we have

$$Ric^1(X_1,Y_1)+\lambda\mathcal{L}^1_{\xi_1}g(X_1,Y_1)+(\mathcal{L}^1_{\xi_1}\circ\mathcal{L}^1_{\xi})g_1(X_1,Y_1)=\mu g_1(X_1,Y_1), \tag{6}$$

that is $(M_1,g_1,\xi_1,\lambda,\mu)$ is a hyperbolic Ricci soliton. □

A pseudo Riemannian manifold (M,g) is an h-almost Ricci soliton if there exist a vector field $X\in\mathcal{X}(M)$, a smooth function $\gamma(x):M\to\mathbb{R}$, and a function $h:M\to\mathbb{R}$ such that

$$Ric+h\mathcal{L}_Xg=\gamma(x)g.$$

In this case we denote it by (M,g,X,h,γ). The h-almost Ricci solitons have been introduced by Pigola et al. [32] and Gomes et al. [17].

Theorem 2.6. *Let the warped product manifold $(M_1\times_f M_2, g_1\oplus f^2g_2, \xi_1+\xi_2,\lambda,\mu)$ be a hyperbolic Ricci soliton and ξ_2 be 2-Killing vector field. Then (M_2,g_2) is an h-almost Ricci soliton with parameters $h=\lambda f^2+2\xi_1(f^2)$ and $\gamma(x)=\mu f^2+f^\sharp-\lambda\xi_1(f^2)-\xi_1(\xi_1(f^2))$.*

Proof. From the definition of hyperbolic Ricci soliton we get

$$Ric(X,Y) + \lambda \mathcal{L}_X g(X,Y) + (\mathcal{L}_X \circ \mathcal{L}_X) g(X,Y) = \mu g(X,Y), \tag{7}$$

for all vector fields X, Y on $M_1 \times_f M_2$. If we assume that $X = X_2 \in \mathcal{X}(M_2)$ and $Y = Y_2 \in \mathcal{X}(M_2)$, then the part 4 of Proposition 2.1, the part 3 of Proposition 2.2, and the part 2 of Corollary 2.3, imply that

$$\begin{aligned} & Ric^2(X_2,Y_2) - f^\sharp g_2(X_2,Y_2) + \lambda f^2 \mathcal{L}^2_{\xi_2} g_2(X_2,Y_2) + \lambda \xi_1(f^2) g_2(X_2,Y_2) \\ & + f^2 \mathcal{L}^2_{\xi_2} \mathcal{L}^2_{\xi_2} g_2(X_2,Y_2) + 2\xi_1(f^2) \mathcal{L}^2_{\xi_2} g_2(X_2,Y_2) \\ & + \xi_1(\xi_1(f^2)) g_2(X_2,Y_2) = \mu f^2 g_2(X_2,Y_2). \end{aligned} \tag{8}$$

Since ξ_2 is a 2-Killing vector field then $\mathcal{L}^2_{\xi_2}\mathcal{L}^2_{\xi_2} g_2 = 0$ and we can write the equation (8) as

$$Ric^2 + (\lambda f^2 + 2\xi_1(f^2))\mathcal{L}^2_{\xi_2} g_2 = \Big(\mu f^2 + f^\sharp - \lambda \xi_1(f^2) - \xi_1(\xi_1(f^2))\Big) g_2.$$

This completes the proof of theorem. □

Definition 2.7. *A vector field ξ on a manifold (M,g) is called a conformal vector field if $\mathcal{L}_\xi g = \rho g$ for some smooth function $\rho : M \to \mathbb{R}$. If ρ is non-zero constant or zero, then ξ is called homothetic or Killing vector field, respectively.*

Theorem 2.8. *Let the warped product manifold $(M = M_1 \times_f M_2, g = g_1 \oplus f^2 g_2, \xi_1 + \xi_2, \lambda, \mu)$ be a hyperbolic Ricci soliton. Then g is an Einstein metric if*

i) *ξ_i is conformal vector field on M_i with factor ρ_i, $i = 1, 2$,*

ii) $\mu f^2 - \lambda \xi_1(f^2) - \xi_1(\xi_1(f^2) - \rho_2 \left(\lambda f^2 + 2\xi_1(f^2)\right) - f^2\Big(\xi_2(\rho_2) + \rho_2^2\Big)$

$= f^2\Big(\mu - \lambda\rho_1 - \xi_1(\rho_1) - \rho_1^2\Big).$

Proof. Since ξ_i is conformal vector field on M_i with factor ρ_i, $i = 1, 2$ we have $\mathcal{L}^1_{\xi_1} g_1 = \rho_1 g_1$ and $\mathcal{L}^2_{\xi_2} g_2 = \rho_2 g_2$. Therefore

$$\mathcal{L}^1_{\xi_1}\mathcal{L}^1_{\xi_1} g_1 = \Big(\xi_1(\rho_1) + \rho_1^2\Big) g_1, \qquad \mathcal{L}^2_{\xi_2}\mathcal{L}^2_{\xi_2} g_2 = \Big(\xi_2(\rho_2) + \rho_2^2\Big) g_2.$$

Since $(M_1 \times_f M_2, g_1 \oplus f^2 g_2, \xi_1 + \xi_2, \lambda, \mu)$ is a hyperbolic Ricci soliton we have

$$Ric(X_1,Y_1) + \lambda \mathcal{L}^1_{\xi_1} g_1(X_1,Y_1) + \mathcal{L}^1_{\xi_1}\mathcal{L}^1_{\xi_1} g_1(X_1,Y_1) = \mu g_1(X_1,Y_1)$$

then

$$Ric(X_1,Y_1) = \Big(\mu - \lambda\rho_1 - \xi_1(\rho_1) - \rho_1^2\Big) g_1(X_1,Y_1).$$

Similarly, as $(M_1 \times_f M_2, g_1 \oplus f^2 g_2, \xi_1 + \xi_2, \lambda, \mu)$ is a hyperbolic Ricci soliton we have

$$\begin{aligned} & Ric(X_2,Y_2) + \left(\lambda f^2 + 2\xi_1(f^2)\right) \mathcal{L}^2_{\xi_2} g_2(X_2,Y_2) + \left(\lambda \xi_1(f^2) + \xi_1(\xi_1(f^2))\right) g_2(X_2,Y_2) \\ & + f^2 \mathcal{L}^2_{\xi_2}\mathcal{L}^2_{\xi_2} g_2(X_2,Y_2) = \mu f^2 g_2(X_2,Y_2). \end{aligned}$$

Then

$$Ric(X_2,Y_2) = \Big(\mu f^2 - \lambda \xi_1(f^2) - \xi_1(\xi_1(f^2) - \rho_2\left(\lambda f^2 + 2\xi_1(f^2)\right) - f^2\Big(\xi_2(\rho_2) + \rho_2^2\Big)\Big) g_2(X_2,Y_2).$$

Therefore

$$Ric(X,Y) = \Big(\mu - \lambda\rho_1 - \xi_1(\rho_1) - \rho_1^2\Big) g(X,Y),$$

that is (M,g) is an Einstein manifold. □

Theorem 2.9. *Let manifold $(M_1, g_1, \xi_1, \lambda_1, \mu_1)$ be a hyperbolic Ricci soliton and (M_2, g_2) be an Einstein manifold with factor γ. Then the warped product manifold $(M = M_1 \times_f M_2, g = g_1 \oplus f^2 g_2, \xi = \xi_1 + \xi_2, \lambda_1, \mu_1)$ is a hyperbolic Ricci soliton if*

1) *ξ_2 is conformal vector field on M_2 with factor ρ,*
2) *f is a constant function or $T = 0$ and $H^f = 0$,*
3) *$\mu_1 f^2 = \gamma - (n_2 - 1)|\nabla f|^2 + \lambda_1\Big(f^2\rho + \xi_1(f^2)\Big) + f^2(\xi_2(\rho) + \rho^2) + 2\rho\xi_1(f^2) + \xi_1(\xi_1(f^2))$.*

Proof. We assume that $X_i \in \mathcal{X}(M_i)$, $i = 1, 2$. Since $H^f = 0$, we get $\Delta f = 0$. Since (M_2, g_2) is an Einstein manifold with factor γ and according to the Proposition 2.2, for $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ we have

$$\begin{aligned} Ric(X,Y) &= Ric(X_1,Y_1) + Ric(X_2,Y_2) \\ &= Ric^1(X_1,Y_1) - \frac{n_2}{f}H^f(X_1,Y_1) + Ric^2(X_2,Y_2) - f^\sharp g_2(X_2,Y_2) \\ &= Ric^1(X_1,Y_1) + (\gamma - (n_2-1)|\nabla f|^2)g_2(X_2,Y_2). \end{aligned} \tag{9}$$

Since $(M_1, g_1, \xi_1, \lambda_1, \mu_1)$ is a hyperbolic Ricci soliton we infer

$$Ric^1(X_1,Y_1) + \lambda_1 \mathcal{L}^1_{\xi_1} g_1(X_1,Y_1) + \mathcal{L}^1_{\xi_1}\mathcal{L}^1_{\xi_1} g_1(X_1,Y_1) = \mu_1 g_1(X_1,Y_1). \tag{10}$$

Now since ξ_2 is a conformal vector field with factor ρ and using part 4 of Proposition 2.1 we conclude that

$$\begin{aligned} (\mathcal{L}_\xi g)(X,Y) &= \mathcal{L}^1_{\xi_1} g_1(X_1,Y_1) + f^2\mathcal{L}^2_{\xi_2} g_2(X_2,Y_2) + \xi_1(f^2)g_2(X_2,Y_2) \\ &= \mathcal{L}^1_{\xi_1} g_1(X_1,Y_1) + \Big(f^2\rho + \xi_1(f^2)\Big)g_2(X_2,Y_2). \end{aligned} \tag{11}$$

Also, since f is a constant function or $T = 0$, then the Corollary 2.3 implies that

$$\begin{aligned} (\mathcal{L}_\xi\mathcal{L}_\xi g)(X,Y) &= (\mathcal{L}_\xi\mathcal{L}_\xi g)(X_1,Y_1) + (\mathcal{L}_\xi\mathcal{L}_\xi g)(X_2,Y_2) \\ &= (\mathcal{L}^1_{\xi_1}\mathcal{L}^1_{\xi_1} g_1)(X_1,Y_1) + f^2(\mathcal{L}^2_{\xi_2}\mathcal{L}^2_{\xi_2} g_2)(X_2,Y_2) \\ &\quad + 2\xi_1(f^2)(\mathcal{L}^2_{\xi_2} g_2)(X_2,Y_2) + \xi_1(\xi_1(f^2))g_2(X_2,Y_2) \\ &= (\mathcal{L}^1_{\xi_1}\mathcal{L}^1_{\xi_1} g_1)(X_1,Y_1) + \Big(f^2(\xi_2(\rho) + \rho^2) + 2\rho\xi_1(f^2) \\ &\quad + \xi_1(\xi_1(f^2))\Big)g_2(X_2,Y_2). \end{aligned} \tag{12}$$

By equations (9)-(12) we obtain

$$\begin{aligned} &Ric(X,Y) + \lambda_1(\mathcal{L}_\xi g)(X,Y) + (\mathcal{L}_\xi\mathcal{L}_\xi g)(X,Y) \\ &\quad = Ric^1(X_1,Y_1) + (\gamma - (n_2-1)|\nabla f|^2)g_2(X_2,Y_2) \\ &\qquad + \lambda_1\mathcal{L}^1_{\xi_1} g_1(X_1,Y_1) + \lambda_1\Big(f^2\rho + \xi_1(f^2)\Big)g_2(X_2,Y_2) \\ &\qquad + (\mathcal{L}^1_{\xi_1}\mathcal{L}^1_{\xi_1} g_1)(X_1,Y_1) + \Big(f^2(\xi_2(\rho) + \rho^2) + 2\rho\xi_1(f^2) + \xi_1(\xi_1(f^2))\Big)g_2(X_2,Y_2) \\ &\quad = \mu_1 g_1(X_1,Y_1) + \Big(\gamma - (n_2-1)|\nabla f|^2 + \lambda_1\Big(f^2\rho + \xi_1(f^2)\Big) + f^2(\xi_2(\rho) + \rho^2) \\ &\qquad + 2\rho\xi_1(f^2) + \xi_1(\xi_1(f^2))\Big)g_2(X_2,Y_2) \\ &\quad = \mu_1 g(X,Y). \end{aligned}$$

Therefore (M, g) is a hyperbolic Ricci soliton. □

Theorem 2.10. *Let the warped product manifold $(M = M_1 \times_f M_2, g = g_1 \oplus f^2 g_2, \xi = \xi_1 + \xi_2, \lambda, \mu)$ be a hyperbolic Ricci soliton. Then (M, g) is Einstein manifold if one of the following conditions holds.*

1) *$\xi = \xi_1$, ξ_1 is a Killing vector field on M_1 and $\lambda\xi_1(f^2) + \xi_1(\xi_1(f^2) = 0$.*

2) $\xi = \xi_2$, ξ_2 is a Killing vector field on M_2.

3) ξ_i is a Killing vector field on M_i, $i = 1, 2$ and $\lambda\xi_1(f^2) + \xi_1(\xi_1(f^2) = 0$.

Proof. If ξ_i is a Killing vector field on M_i, $i = 1, 2$ then

$$\mathcal{L}^i_{\xi_i} g_i = 0, \qquad \mathcal{L}^i_{\xi_i}\mathcal{L}^i_{\xi_i} g_i = 0, \quad i = 1, 2.$$

If $\xi = \xi_1$ and ξ_1 is a Killing vector field on M_1 and $\lambda\xi_1(f^2) + \xi_1(\xi_1(f^2) = 0$ then for any vector fields $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ where $X_i, Y_i \in \mathcal{X}(M_i)$, $i = 1, 2$ we have

$$(\mathcal{L}_{\xi_1} g)(X, Y) = \mathcal{L}^1_{\xi_1} g_1(X_1, Y_1) + \xi_1(f^2)g_2(X_2, Y_2) = \xi_1(f^2)g_2(X_2, Y_2),$$

and

$$\begin{aligned}(\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_1} g)(X, Y) &= (\mathcal{L}_{\xi_{1A}}\mathcal{L}_{\xi_1} g)(X_1, Y_1) + (\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_1} g)(X_2, Y_2)\\ &= (\mathcal{L}^1_{\xi_1}\mathcal{L}^1_{\xi_1} g_1)(X_1, Y_1) + +\xi_1(\xi_1(f^2))g_2(X_2, Y_2)\\ &= \xi_1(\xi_1(f^2))g_2(X_2, Y_2).\end{aligned}$$

Since $\lambda\xi_1(f^2) + \xi_1(\xi_1(f^2) = 0$, the hyperbolic Ricci soliton equation for (M, g) becomes

$$\begin{aligned}\mu g(X, Y) &= Ric(X, Y) + \lambda(\mathcal{L}_{\xi_1} g)(X, Y) + (\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_1} g)(X, Y)\\ &= Ric(X, Y) + \Big(\lambda\xi_1(f^2) + \xi_1(\xi_1(f^2))\Big)g_2(X_2, Y_2) = Ric(X, Y)\end{aligned}$$

that is (M, g) is an Einstein manifold.
If $\xi = \xi_2$ and ξ_2 is a Killing vector field on M_2 Then

$$(\mathcal{L}_{\xi_2} g)(X, Y) = f^2\mathcal{L}^2_{\xi_2} g_2(X_2, Y_2) = 0,$$

and

$$(\mathcal{L}_{\xi_2}\mathcal{L}_{\xi_2} g)(X, Y) = f^2(\mathcal{L}^2_{\xi_2}\mathcal{L}^2_{\xi_2} g_2)(X_2, Y_2) = 0.$$

Then the hyperbolic Ricci soliton equation for (M, g) becomes

$$\mu g(X, Y) = Ric(X, Y) + \lambda(\mathcal{L}_{\xi_2} g)(X, Y) + (\mathcal{L}_{\xi_2}\mathcal{L}_{\xi_2} g)(X, Y) = Ric(X, Y),$$

this shows that (M, g) is an Einstein manifold.
If ξ_i is a Killing vector field on M_i, $i = 1, 2$ then

$$\begin{aligned}(\mathcal{L}_{\xi} g)(X, Y) &= \mathcal{L}^1_{\xi_1} g_1(X_1, Y_1) + f^2\mathcal{L}^2_{\xi_2} g_2(X_2, Y_2) + \xi_1(f^2)g_2(X_2, Y_2)\\ &= \xi_1(f^2)g_2(X_2, Y_2),\end{aligned}$$

and

$$\begin{aligned}(\mathcal{L}_{\xi}\mathcal{L}_{\xi} g)(X, Y) &= (\mathcal{L}_{\xi}\mathcal{L}_{\xi} g)(X_1, Y_1) + (\mathcal{L}_{\xi}\mathcal{L}_{\xi} g)(X_2, Y_2)\\ &= (\mathcal{L}^1_{\xi_1}\mathcal{L}^1_{\xi_1} g_1)(X_1, Y_1) + f^2(\mathcal{L}^2_{\xi_2}\mathcal{L}^2_{\xi_2} g_2)(X_2, Y_2)\\ &\quad +2\xi_1(f^2)(\mathcal{L}^2_{\xi_2} g_2)(X_2, Y_2) + \xi_1(\xi_1(f^2))g_2(X_2, Y_2)\\ &= \xi_1(\xi_1(f^2))g_2(X_2, Y_2).\end{aligned}$$

Hence, since $\lambda\xi_1(f^2) + \xi_1(\xi_1(f^2) = 0$ the hyperbolic Ricci soliton equation for (M, g) gives

$$\begin{aligned}\mu g(X, Y) &= Ric(X, Y) + \lambda(\mathcal{L}_{\xi} g)(X, Y) + (\mathcal{L}_{\xi}\mathcal{L}_{\xi} g)(X, Y)\\ &= Ric(X, Y) + \Big(\lambda\xi_1(f^2) + \xi_1(\xi_1(f^2))\Big)g_2(X_2, Y_2)\\ &= Ric(X, Y).\end{aligned}$$

This completes the proof of theorem. □

A vector field Z on a pseudo Riemannian manifold M is said to be a concurent vector field if for any vector field $X \in \mathcal{X}(M)$,

$$\nabla_X Z = X.$$

Since for concurent vector field Z we have $(\mathcal{L}_Z g)(X,Y) = 2g(X,Y)$, then Z is a homothetic vector field. Also, if we assume $u = \frac{1}{2}g(Z,Z)$ then for any vector field X on M we get

$$g(X, \nabla u) = X(u) = g(\nabla_X Z, Z) = g(X, Z),$$

thus $Z = \nabla u$.

Theorem 2.11. *Let the connected warped product manifold $(M = M_1 \times_f M_2, g = g_1 \oplus f^2 g_2, \xi = \xi_1 + \xi_2, \lambda, \mu)$ be a hyperbolic Ricci soliton and ξ be a concurent vector field on M. If $\xi_2 \neq 0$, then M, M_1, and M_2 are Ricci flat, gradient hyperbolic Ricci solitons such that $\mu = 2\lambda + 4$.*

Proof. Since ξ is a concurent vector field on M we have $\mathcal{L}_\xi g = 2g$ and

$$\begin{aligned}
(\mathcal{L}_\xi \mathcal{L}_\xi g)(X,Y) &= \mathcal{L}_\xi(\mathcal{L}_\xi g(X,Y)) - \mathcal{L}_\xi g(\mathcal{L}_\xi X, Y) - \mathcal{L}_\xi g(X, \mathcal{L}_\xi Y) \\
&= 2\mathcal{L}_\xi(g(X,Y)) - 2g(\mathcal{L}_\xi X, Y) - 2g(X, \mathcal{L}_\xi Y) \\
&= 2(\mathcal{L}_\xi g)(X,Y) = 4g(X,Y)
\end{aligned}$$

for any vectors fields X, Y on M. Definition of hyperbolic Ricci soliton yields

$$Ric(X,Y) = \Big(\mu - 2\lambda - 4\Big) g(X,Y). \tag{13}$$

In (13) suppose that $X = X_2 \in \mathcal{X}(M_2)$ and $Y = Y_2 \in \mathcal{X}(M_2)$, then

$$Ric^2(X_2, Y_2) = \Big((\mu - 2\lambda - 4\Big) f^2 + f^\sharp) g_2(X_2, Y_2). \tag{14}$$

Since ξ is a concurrent vector field on M we get

$$\nabla_{X_1}\xi = X_1, \qquad \nabla_{X_2}\xi = X_2, \ \ \forall X_1 \in \mathcal{X}(M_1), \ X_2 \in \mathcal{X}(M_2). \tag{15}$$

On the other hand , the part 1 of Proposition 2.1 gives

$$\nabla_{X_1}\xi = \nabla^1_{X_1}\xi_1. \tag{16}$$

Thus equations (15) and (16) give $\nabla^1_{X_1}\xi_1 = X_1$, that is ξ_1 is a concurrent vector field on M_1. Using the Proposition (2.1) again we obtain

$$X_2 = \nabla_{X_2}\xi = \nabla_{X_2}\xi_1 + \nabla_{X_2}\xi_2 = \frac{\xi_1 f}{f} X_2 - f g_2(\xi_2, X_2)\nabla f + \nabla^2_{X_2}\xi_2. \tag{17}$$

Then $\nabla f = 0$, this shows that $f = c$ is constant. Therefore the equation (17) becomes $\nabla^2_{X_2}\xi_2 = X_2$, that is ξ_2 is a concurrent vector field on M_2. Also, since $f = c$ we have $f^\sharp = 0$ and we can write (14) as

$$Ric^2(X_2, Y_2) = c^2(\mu - 2\lambda - 4) g_2(X_2, Y_2). \tag{18}$$

If we assume that $X_2 = Y_2 = \xi_2$ then

$$Ric^2(\xi_2, \xi_2) = c^2(\mu - 2\lambda - 4)|\xi_2|^2_2. \tag{19}$$

Let $\{\xi_2, e_1, \cdots, e_{n_2-1}\}$ be orthogonal basis of $\mathcal{X}(M_2)$, then the curvature tensor R^2 of M_2 is given by

$$\begin{aligned}
R^2(\xi_2, e_i, \xi_2, e_i) &= g(R^2(\xi_2, e_i)\xi_2, e_i) \\
&= g_2(\nabla_{\xi_2}\nabla_{e_i}\xi_2 - \nabla_{e_i}\nabla_{\xi_2}\xi_2 - \nabla_{[\xi_2, e_i]}\xi_2, e_i) \\
&= g_2(\nabla_{\xi_2} e_i - \nabla_{e_i}\xi_2 - [\xi_2, e_i], e_i) = 0.
\end{aligned}$$

Hence, $Ric^2(\xi_2, \xi_2) = 0$. Replacing it in equation (19) we infer $\mu - 2\lambda - 4 = 0$ and so equations (13) and (18) imply that $Ric(X, Y) = Ric^2(X_2, Y_2) = 0$. Therefore M and M_2 are Ricci flat. If we consider $X = X_1 \in \mathcal{X}(M_1)$ and $Y = Y_1 \in \mathcal{X}(M_1)$ then

$$0 = Ric(X_1, Y_1) = Ric^1(X_1, Y_1) - \frac{n_2}{f} H^f(X_1, Y_1) = Ric^1(X_1, Y_1). \tag{20}$$

This shows that also M_1 is Ricci flat. Thus, the manifods M_1 and M_2 are gradient hyperbolic Ricci soliton with the same factors λ and μ such that $\mu = 2\lambda + 4$. Notice that ξ and ξ_i are gradient vector fields with potential functions $u = \frac{1}{2}g(\xi, \xi)$ and $u_i = \frac{1}{2}g(\xi_i, \xi_i)$, respectively, where $i = 1, 2$. □

In [16], the authors using two $(0, 2)$ tensor fields, have defined bi-conformal vector fields. Then De et al. in [13] defined Ricci bi-conformal vector fields by taking the metric tensor field g and the Ricci tensor field Ric as the two tensor fields as follows.

Definition 2.12. *A vector field X on a Riemannian manifold (M, g) is called Ricci bi-conformal vector field if it is satisfies the following equations*

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta Ric(Y, Z) \tag{21}$$

and

$$(\mathcal{L}_X Ric)(Y, Z) = \alpha Ric(Y, Z) + \beta g(Y, Z) \tag{22}$$

for some non-zero smooth functions α and β.

Theorem 2.13. *Let the warped product manifold $(M^n = M_1 \times_f M_2, g = g_1 \oplus f^2 g_2, \xi = \xi_1 + \xi_2, \lambda, \mu)$ be a hyperbolic Ricci soliton and admits a Ricci bi-conformal vector field ξ as (21) and (22). Then the manifold M is an Einstein manifold or*

$$1 + \lambda\beta + 2\alpha\beta + \xi(\beta) = 0, \qquad \lambda\alpha + \xi(\alpha) + \alpha^2 + \beta^2 - \mu = 0. \tag{23}$$

Proof. Using (21) and (22) we get

$$\mathcal{L}_\xi \mathcal{L}_\xi g = (\xi(\alpha) + \alpha^2 + \beta^2)g + (\xi(\beta) + 2\alpha\beta)Ric. \tag{24}$$

Substititing (21) and (24) into hyperbolic Ricci soliton equation, we conclude

$$(1 + \lambda\beta + 2\alpha\beta + \xi(\beta))\, Ric + \left(\lambda\alpha + \xi(\alpha) + \alpha^2 + \beta^2 - \mu\right) g = 0. \tag{25}$$

If $1 + \lambda\beta + 2\alpha\beta + \xi(\beta) = 0$ then $\lambda\alpha + \xi(\alpha) + \alpha^2 + \beta^2 - \mu = 0$. Otherwise, that is, if $1 + \lambda\beta + 2\alpha\beta + \xi(\beta) \neq 0$ then by taking trace of (25) we have

$$\lambda\alpha + \xi(\alpha) + \alpha^2 + \beta^2 - \mu = -(1 + \lambda\beta + 2\alpha\beta + \xi(\beta))\,\frac{R}{n}. \tag{26}$$

Replacing it in (25) yields $Ric = \frac{R}{n}g$. This shows that manifold M is Einstein. □

3. Hyperbolic Ricci soliton on generalized Robertson-Walker space-time

In this section we will consider hyperbolic Ricci solitons on generalized Robertson-Walker space-time and indicate some necassary conditions for this space-time to be hyperbolic Ricci soliton. Let (N, g_N) be an n-dimensional Riemannian manifold, $f : I \to (0, \infty)$ be a smooth function on open, connected subinterval I of $\mathbb{R}$ and dt^2 be the Euclidean metric tensor on I. Then $(n + 1)$-dimensional product manifold $I \times N$ with the metric $g = -dt^2 \oplus f^2 g_N$ is called a generalized Robertson-Walker space-time and is denoted by $M = I \times_f N$ (see [33, 34]).

Theorem 3.1. *Let the generalized Roberston-Walker space-time* $(M = I\times_f N, g = -dt^2 \oplus f^2 g_N, \nabla u, \lambda, \mu)$ *be a gradient hyperbolic Ricci soliton where* $u = \int_a^t f(r)dr$ *for some constant* $a \in I$. *Then* $Ric = (\mu - 2\lambda\dot{f} - 2f\ddot{f} - 4\dot{f}^2)g$.

Proof. Let $\xi = \nabla u$, then $\xi = f(t)\partial_t$ where $\partial_t = \frac{\partial}{\partial t} \in \mathcal{X}(I)$. Thus the vector field ξ is prependicular to M. Assume that $\{\partial_t, \partial_1, \partial_2, \cdots, \partial_n\}$ is an orthogonal basis for $\mathcal{X}(M)$. The Hessian tensor of function u is given by

$$H^u(X, Y) = g(\nabla_X \nabla u, Y) = (Xf)g(\partial_t, Y) + fg(\nabla_X \partial_t, Y), \quad \forall X, Y \in \mathcal{X}(M).$$

Now, since $\nabla_{\partial_t}\partial_t = 0$, $\nabla_{\partial_i}\partial_t = \frac{\dot{f}}{f}\partial_i$, we have

$$\begin{aligned} H^u(\partial_t, \partial_t) &= (\partial_t f)g(\partial_t, \partial_t) + fg(\nabla_{\partial_t}\partial_t, \partial_t) = \dot{f}g(\partial_t, \partial_t), \\ H^u(\partial_t, \partial_i) &= (\partial_t f)g(\partial_t, \partial_i) + fg(\nabla_{\partial_t}\partial_t, \partial_i) = \dot{f}g(\partial_t, \partial_i), \quad \forall i = 1, 2, ..., n, \\ H^u(\partial_i, \partial_j) &= (\partial_i f)g(\partial_t, \partial_j) + fg(\nabla_{\partial_i}\partial_t, \partial_j) = \dot{f}g(\partial_i, \partial_j), \quad \forall i, j = 1, 2, ..., n. \end{aligned}$$

Therefore $H^u(X, Y) = \dot{f}g(X, Y)$,

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \nabla u, Y) + g(X, \nabla_Y \nabla u) = 2H^u(X, Y) = 2\dot{f}g(X, Y),$$

and

$$\begin{aligned} (\mathcal{L}_\xi \mathcal{L}_\xi g)(X, Y) &= \mathcal{L}_\xi(\mathcal{L}_\xi g(X, Y)) - (\mathcal{L}_\xi g)(\mathcal{L}_\xi X, Y) - (\mathcal{L}_\xi g)(X, \mathcal{L}_\xi Y) \\ &= 2\mathcal{L}_\xi(\dot{f}g(X, Y)) - 2\dot{f}g(\mathcal{L}_\xi X, Y) - 2\dot{f}g(X, \mathcal{L}_\xi Y) \\ &= 2(\xi\dot{f})g(X, Y) + 2\dot{f}(\mathcal{L}_\xi g)(X, Y) \\ &= (2f\ddot{f} + 4\dot{f}^2)g(X, Y). \end{aligned}$$

Since $(M = I \times_f N, g = -dt^2 \oplus f^2 g_N, \nabla u, \lambda, \mu)$ is a gradient hyperbolic Ricci soliton we conclude

$$Ric = \mu g - \lambda \mathcal{L}_\xi g - \mathcal{L}_\xi \mathcal{L}_\xi g = (\mu - 2\lambda\dot{f} - 2f\ddot{f} - 4\dot{f}^2)g.$$

□

References

[1] M. Barari and A. Razavi, *Generalized Ricci solitons on twisted products*, Çankaya university Journal of science and Engineering, **15**(2) (2018), 76-87.
[2] M. Bertola and Goutheir, *Lie triple system and warped products*, Rend. Math. Appl., **7** (21) (2001), 275-293.
[3] A. L. Besse, *Einstein manifolds, Ergebnisse der mathematics used threr Grenzgeiete*, **3** (10), Springer-Verlag, Berlin, 1987.
[4] R. L. Bishop, B. O'Nill, *Manifolds of negative curvature*, Trans. Am. Math. Soc., **145** (1969), 1-49.
[5] S. Brendle, *Rotational symmetry of Ricci solitons in higher dimensions*, Journal of Differential Geometry, **97** (2014) 191-214.
[6] M. Brozos-Vázquez, E. Garcia-Rio and S. Gavino-Fernandez, *Locally conformally flat Lorentzian gradient Ricci solitons*, J. Geom. Anal., **23** (2013), 1196-1212.
[7] H., -D. Cao and D. Zhou, *On complete gradient shrinking Ricci solitons*, Journal of Differential Geometry, **85** (2010), 175-186.
[8] B.-Y. Chen, *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujevac Journal of Mathematics, **41**(2) (2017), 239-250.
[9] B. Y. Chen, *Differential geometry of warped product manifolds and submanifolds*, World Scientific, 2017.
[10] J. X. Cruz Neto, I. D. Melo and P. A. Sousa, *Non-existence of strictly monotone vector fields on certain Riemannian manifolds*, Acta Math. Hungar, **146** (2015), 240-246.
[11] W. R. Dai, D. X. Kong, K. Liu, *Hyperbolic gometric flow (I): short-time existence and nonlinear stability*, Pure and applied mathematics quarterly, **6** (2010), 331-359.
[12] U. C. De, C. A. Mantica, S. Shenawy and B. Ünal, *Ricci soliton on singly warped product manifolds and applications*, J. Geom. Phys., **166** (2021) 104257.
[13] U. C. De, A. Sardar, and A. Sarkar, *Some conformal vector fields and conformal Ricci solitons on N(k)-contact metric manifolds*, AUT J. Math. Com., **2** (1) (2021), 61-71.
[14] F. E. S. Feitosa, A. A. Freitas Filho and J. N. V. Gomes, *On the construction of gradient Ricci soliton warped product*, Nonlinear Analysis, **161** (2017), 30-43.
[15] M. Fernández-López and E. García-Río, *Rigidity of shrinking Ricci solitons*, Mathematische Zeitschrift, **269** (2011), 461-466.
[16] A. Garcia-Parrado and J. M. M. Senovilla, *Bi-conformal vector fields and their applications*, Classical and Quantum Gravity, **21** (8) (2004), 2153-2177.

[17] J. M. Gomes, Q. Wang, C. Xia, *On the h-almost Ricci soliton*, J. Geom. Phys., **114**(2017), 216-222.
[18] R.S. Hamilton, The Ricci flow on surfaces, *Contemporary mathematics*, **71**(1988), 237-261.
[19] F. Karaca and C. Ozgur, *Gradient Ricci solitons on multiply warped product manifolds*, Filomat, **32**(12) (2018), 4221-4228.
[20] J. Kim, *Some doubly-warped product gradient Ricci solitons*, Commun. Korean Math. Soc., **31**(3) (2016), 625-635.
[21] B. H. Kim, S. D. Lee, J. H. Choi, and Y. O. Lee, *On warped product spaces with a certain Ricci condition*, Bull. Korean Math. Soc., **50** (2013), 683-1691.
[22] D.-S. Kim and Y. H. Kim, *Compact Einstein warped product spaces with nonpositive scalar curvature*, Proc. Amer. Soc., **131** (2003), 2573-2576.
[23] S. D. Lee, B. H. Kim, and J. H. Choi, *On a classification of warped product spaces with gradient Ricci solitons*, Korean J. Math., **24** (4) (2016), 627-636.
[24] S. D. Lee, B. H. Kim, and J. H. Choi, *Warped product spaces with Ricci conditions*, Turk J. Math., **41** (2017), 1365-1375.
[25] W.-J. Lu, *Geometric flows on warped products manifold*, Taiwanese Journal of mathematics, **17**(5) (2013), 1791-1817.
[26] M. Manev, *Ricci-Like solitons with vertical potential on Sasaki-like almost contact B-metric manifolds*, Results Math. **75**, 136 (2020).
[27] O. Munteanu and N. Sesum, *On gradient Ricci solitons*, Journal of Geometric Analysis, **23** (2013), 539-561.
[28] S. Z. Németh, *Five kinds of monotone vector fields*, Pure Math. Appl., **9**(1998), 417-428.
[29] B. O'Nill, *Semi-Riemannian geometry with applications to relativity*, academic Press, Limited, London, 1983.
[30] P. Petersen and W. Wylie, *Rigidity of gradient Ricci solitons*, Pacific Journal of Mathematics, **241** (2009) 329-345.
[31] P. Petersen and W. Wylie, *On the classification of gradient Ricci solitons*, Geom. Topol., **14** (2010), 2277-2300.
[32] S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti, *Ricci almost solitons*, Ann. Scuola, Norm. Sup. Pisa Cl. Sci., **10**(2011), 757-799.
[33] M. Sánchez, *On the geometry of generalized Robertson-Walker spacetimes: geodesics*, Gen. Relativ. Gravit., **30** (1998), 915-932.
[34] M. Sánchez, *On the geometry of generalized Robertson-Walker spacetimes: curvature and Killing fields*, J. Geom. Phys., **31** (1999). 1-15.
[35] S. Shenawy, B. Ünal, *2-Killing vector fields on warped product manifolds*, Int. J. Math. **26** (2015), 1550065.
[36] M. L. Sousa, R. Pina, *Gradient Ricci solitons with structure of warped product*, Results Math **71**, (2017), 825-840.