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Existence of positive solutions for singular fractional boundary value problems with p-Laplacian

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Abstract. In this paper, we obtain the existence of positive solutions for the singular fractional boundary value problem with p-Laplacian. The existence of positive solutions is established using the Avery-Peterson fixed point theorem. In addition, we include an example for the demonstration of our main result.

1. Introduction

Singular fractional differential equations arise in many engineering and scientific disciplines as the modeling of problems in mathematics and physics such as gas dynamics, chemical reactions, nuclear physics, atomic calculations and the studies of atomic structures. Many authors study on singular boundary value problems using variational methods, fixed point theory, see, [1–6]. There has been noticeable development in the study of singular fractional differential equations in recent year, see, [7–14] and references therein. Recently, many important results related to the boundary value problem of fractional differantial equations with p-Laplacian operator have been obtained, see, [21–28].

In [9], the author established the existence of positive solutions for the singular fractional boundary value problem with p-Laplacian

$$\begin{aligned}&[\phi_p(D^{\alpha}_{0^+}u(t))]' + f(t,u(t)) = 0, \quad 0 < t < 1,\\ &u'(0) = 0, \quad u(1) - \gamma u(\eta) = 0,\end{aligned}$$

where $1 < \alpha \leq 2$, $D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f(t,u)$ is singular at $u = 0$.

In [7], using the Avery-Peterson's fixed point theorem, the authors got the positive solutions for the following singular fractional boundary value problems

$$\begin{aligned}&{}^cD^{\alpha}_{0^+}u(t) + f(t,u(t),u'(t)) = 0, \quad 0 < t < 1,\\ &u(0) = u''(0) = 0, \quad u'(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j),\end{aligned}$$

where $2 < \alpha \leq 3$, $\eta_j \geq 0$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{j-1} < \xi_j < \ldots < 1$ $(j = 1, 2, \ldots)$, $f(t,x,y)$ is singular at t=0.

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In [8], the authors established the existence of positive solutions for the singular fractional boundary value problem

$$D^{\alpha}_{0^+}u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0, 1),$$
$$u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \quad D^{p}_{0^+}u(t)|_{t=1} = \sum_{i=1}^{m} a_i D^{q}_{0^+}u(t)|_{t=\xi_i},$$

where $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, $n \geq 3$, $\xi_i \in \mathbb{R}$ for all $i = 1, \ldots, m$ ($m \in \mathbb{N}$), $0 < \xi_1 < \xi_2 < \ldots < \xi_m < 1$, f is singular at $t = 0$ or $t = 1$.

In [11], the authors investigated positive solutions for the singular fractional boundary value problem with parameters

$$D^{\alpha}_{0^+}u(t) + \lambda h(t) f(t, u(t)) = 0, \quad t \in (0, 1),$$
$$u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \quad D^{\beta_0}_{0^+}u(1) = \sum_{i=1}^{m} \int_0^1 D^{\beta_i}_{0^+}u(t)dH_i(t),$$

where $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, $n \geq 3$, $f(t, u)$ is singular at $u = 0$ and $h(t)$ singular at $t = 0$.

Motivated by the above papers, we investigate the existence of at least three positive solutions for the boundary value problem with p-Laplacian :

$$[\phi_p(D^{\alpha}_{0^+}u(t))]' + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = u'(0) = 0, \quad D^{\alpha-1}_{0^+}u(1) = \sum_{i=1}^{m-2} a_i D^{\alpha-1}_{0^+}u(\xi_i), \tag{2}$$

where $\alpha \in \mathbb{R}$, $2 < \alpha \leq 3$, $\xi_i \in \mathbb{R}$ for all $i = 1, 2, ..., m-2$ ($m \in \mathbb{N}$), $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $D^{\alpha}_{0^+}$ is the Riemann-Liouville derivative of order α, $f(t, x, y)$ may be singular at $t = 0$. In this paper, we will always suppose that the following conditions hold.

(H1) $a_i > 0$ and $\sum_{i=1}^{m-2} a_i < 1$ for all $i = 1, 2, ..., m-2$ ($m \in \mathbb{N}$),

(H2) $f(t, x, y) : (0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, and there exists a constant $0 < \sigma < 1$ such that $t^{\sigma} f(t, x, y)$ is continuous in $[0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$.

By using Avery-Peterson fixed point theorem in [12], we get the existence of positive solutions for the BVP (1)-(2). Thus, this results can be considered as a contribution to this field. The organization of this paper is as follows. In section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. In section 3, we give and prove our main result. Finally, we give an example to illustrate how the main result can be used in practice.

2. Preliminaries

In this section, we present some necessary definitions and lemmas, which can be found in [16–19].

Definition 2.1. *The integral*

$$I^{\alpha}_{0^+}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

where $\alpha > 0$, is called the Riemann-Liouville fractional integral of order α.

Definition 2.2. *For a function $y(t)$ given in the interval $[0, \infty)$, the expression*

$$D^{\alpha}_{0^+}y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha+1} y(s)ds,$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of number α, is called Riemann-Liouville fractional derivative of order α.

Remark 2.3. *From the definition of the Riemann-Liouville fractional derivative, we quote for* $\mu > -1$, *then*

$$D_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha}$$

In particular $D_{0^+}^{\alpha} t^{\alpha-m} = 0$ $(m = 1, 2, \ldots, N)$, N is the smallest integer greater than or equal to α.

Lemma 2.4. *([13]) Assume that* $u \in C(0,1) \bigcap L^1(0,1)$, *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \bigcap L^1(0,1)$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_N t^{\alpha-N},$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, *where* N *is the smallest integer greater than or equal to* α.

Lemma 2.5. *We consider the fractional differential equation*

$$[\phi_p(D_{0^+}^{\alpha} u(t))]' + h(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = u'(0) = 0, \quad D_{0^+}^{\alpha-1} u(1) = \sum_{i=1}^{m-2} a_i D_{0^+}^{\alpha-1} u(\xi_i), \tag{4}$$

with the boundary conditions (4), where $h \in C(0,1) \bigcap L^1(0,1)$. *We denote by* $\Delta = \Gamma(\alpha)\left(1 - \sum_{i=1}^{m-2} a_i\right)$.

Then the unique solution $u \in C[0,1]$ *of problem (3), (4) is given by*

$$u(t) = \frac{t^{\alpha-1}}{\Delta} \int_0^1 \omega(s) ds - \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds \tag{5}$$

where $\omega(s) = \phi_q\left(\int_0^s h(\tau) d\tau\right)$. $\phi_q(u)$ *is the inverse function of* $\phi_p(u)$, *i.e.* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Lemma 2.5, we deduce that the solution $u \in C(0,1) \bigcap L^1(0,1)$ of the fractional differential equation (5) is given by

$$\begin{aligned} u(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - I_{0^+}^{\alpha}\left(\phi_q \int_0^t h(s) ds\right) \\ &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\phi_q \int_0^s h(\tau) d\tau\right) ds \\ &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds \end{aligned}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, 3$. By using the conditions $u(0) = u'(0) = 0$, we obtain $c_2 = c_3 = 0$. Then we conclude

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds \tag{6}$$

For the obtained function (6), we find

$$D_{0^+}^{\alpha-1} u(1) = c_1 \Gamma(\alpha) - \int_0^1 \omega(s) ds$$

$$\sum_{i=1}^{m-2} a_i D_{0^+}^{\alpha-1} u(\xi_i) = c_1 \Gamma(\alpha) \sum_{i=1}^{m-2} a_i - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s) ds$$

Then the condition $D_{0^+}^{\alpha-1}u(1) = \sum_{i=1}^{m-2} a_i D_{0^+}^{\alpha-1}u(\xi_i)$ gives us

$$c_1 = \frac{1}{\Delta}\int_0^1 \omega(s)ds - \frac{1}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds$$

Therefore, the unique solution of the problem (3) is given by

$$u(t) = \frac{t^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds$$

□

Lemma 2.6. *Suppose that the condition (H1) and (H2) hold, then $u(t)$ is nonnegative and nondecreasing function*

Proof. It is obvious that $\omega(s) \geq 0$,

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds \\ &\geq \frac{t^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^1 \omega(s)ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 \omega(s)ds \\ &= \frac{t^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^1 \omega(s)ds - \frac{t^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds + \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^1 \omega(s)ds \\ &= 0 \end{aligned}$$

Therefore, we see that $u(t)$ is nonnegative.

It is similar to the proof of $u'(t) \geq 0$, we can obtain $u'(t) \geq 0$, so $u(t)$ is nondecreasing. The proof is complete. □

We consider the Banach space $B = C^1[0,1]$ with the norm

$$\|u\| = \max\left\{\max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)|\right\}$$

and we define the cone

$$P = \{u \in B : u(t) \geq 0, u'(t) \geq 0, t \in [0,1]\}$$

and operator $T : P \to B$ given by

$$Tu(t) = \frac{t^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds$$

Lemma 2.7. *$T : P \to P$ is completely continuous operator.*

Proof. For $u \in P$, in view of Lemma 2.6, we see that $Tu(t)$ is nonnegative and nondecreasing, consequently, we have $T : P \to P$. By (H2), we can easily get that $T : P \to P$ is continuous. Now, we will prove that T is compact in bounded subsets of its domain.

Let $\Omega \subset P$ be bounded. By (H2), we get that there exists a constant $L > 0$ such that $t^{\sigma} f(t, u(t), u'(t)) \leq L$, $t \in [0, 1]$, $u \in \Omega$. Thus, for $u \in \Omega$, $t \in [0, 1]$, we have

$$\begin{aligned}
\omega(s) &= \phi_q\left(\int_0^s t^{-\sigma} t^{\sigma} f(t, u(t), u'(t))dt\right) \\
&\leq \phi_q\left(\int_0^1 L t^{-\sigma} dt\right) \\
&= \frac{L^{q-1}}{(1-\sigma)^{q-1}}
\end{aligned}$$

So we get

$$\begin{aligned}
Tu(t) &= \frac{t^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds \\
&\leq \frac{t^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds \\
&\leq \frac{1}{\Delta}\int_0^1 \frac{L^{q-1}}{(1-\sigma)^{q-1}}ds \\
&\leq \frac{(\alpha-1)L^{q-1}}{\Delta(1-\sigma)^{q-1}}
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
(Tu)'(t) &= \frac{(\alpha-1)t^{\alpha-2}}{\Delta}\int_0^1 \omega(s)ds - \frac{(\alpha-1)t^{\alpha-2}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}\omega(s)ds \\
&\leq \frac{(\alpha-1)}{\Delta}\int_0^1 \frac{L^{q-1}}{(1-\sigma)^{q-1}}ds \\
&= \frac{(\alpha-1)L^{q-1}}{\Delta(1-\sigma)^{q-1}}
\end{aligned}$$

Consequently, $\|Tu\| \leq \frac{(\alpha-1)L^{q-1}}{\Delta(1-\sigma)^{q-1}}$. In the following we will proof that $T(\Omega)$ is equicontinuous.

For $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, $u \in \Omega$, we have

$$\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
&= \left| c_1 t_2^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1}\omega(s)ds - c_1 t_1^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\omega(s)ds \right| \\
&\leq c_1 \left| t_2^{\alpha-1} - t_1^{\alpha-1} \right| + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2-s)^{\alpha-1}\omega(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-1}\omega(s)ds \right| \\
&\leq \left(\frac{1}{\Delta}\int_0^1 \omega(s)ds\right) \left| t_2^{\alpha-1} - t_1^{\alpha-1} \right| + \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_2} (t_2-s)^{\alpha-1}\omega(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-1}\omega(s)ds\right) \\
&\leq \frac{L^{q-1}}{\Delta(1-\sigma)^{q-1}} \left| t_2^{\alpha-1} - t_1^{\alpha-1} \right| + \frac{L^{q-1}}{\Gamma(\alpha+1)(1-\sigma)^{q-1}} \left| t_2^{\alpha} - t_1^{\alpha} \right| \\
&\leq \frac{L^{q-1}}{\Delta(1-\sigma)^{q-1}} 2|t_2 - t_1| + \frac{L^{q-1}}{\Gamma(\alpha)(1-\sigma)^{q-1}} |t_2 - t_1|
\end{aligned}$$

Similarly, we get

$$
\begin{aligned}
&|(Tu)'(t_2) - (Tu)'(t_1)| \\
&= \left| c_1(\alpha-1)t_2^{\alpha-2} - \frac{1}{\Gamma(\alpha-1)}\int_0^{t_2}(t_2-s)^{\alpha-2}\omega(s)ds - c_1(\alpha-1)t_1^{\alpha-2} + \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}\omega(s)ds \right| \\
&\leq c_1(\alpha-1)\left|t_2^{\alpha-2} - t_1^{\alpha-2}\right| + \frac{1}{\Gamma(\alpha-1)}\left|\int_0^{t_2}(t_2-s)^{\alpha-2}\omega(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-2}\omega(s)ds\right| \\
&\leq \left(\frac{1}{\Delta}\int_0^1 \omega(s)ds\right)(\alpha-1)\left|t_2^{\alpha-2} - t_1^{\alpha-2}\right| + \frac{1}{\Gamma(\alpha-1)}\left(\int_0^{t_2}(t_2-s)^{\alpha-2}\omega(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-2}\omega(s)ds\right) \\
&\leq \frac{L^{q-1}}{\Delta(1-\sigma)^{q-1}}(\alpha-1)\left|t_2^{\alpha-2} - t_1^{\alpha-2}\right| + \frac{L^{q-1}}{\Gamma(\alpha)(1-\sigma)^{q-1}}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| \\
&\leq \frac{L^{q-1}}{\Delta(1-\sigma)^{q-1}}2\left|t_2 - t_1\right| + \frac{L^{q-1}}{\Gamma(\alpha)(1-\sigma)^{q-1}}2\left|t_2 - t_1\right|
\end{aligned}
$$

We have the right-hand side of the above inequalities tends to zero if $t_2 \to t_1$. Using Arzela–Ascoli Theorem, we have T is a completely continuous operator.

□

Let γ and θ be nonnegative, continuous and convex functional on P, Φ and ψ be a nonnegative continuous functional on P. Then, for positive numbers h, r, c and d, we define the following sets:

$$
\begin{aligned}
P(\gamma, d) &= \{x \in P : \gamma(x) < d\}, \\
P(\gamma, \Phi, r, d) &= \{x \in P : r \leq \Phi(x), \gamma(x) \leq d\}, \\
P(\gamma, \theta, \Phi, r, c, d) &= \{x \in P : r \leq \Phi(x), \theta(x) \leq c, \gamma(x) \leq d\}, \\
R(\gamma, \psi, h, d) &= \{x \in P : h \leq \psi(x), \gamma(x) < d\}.
\end{aligned}
$$

We will use the following fixed point theorem of Avery and Peterson to study the problem (1), (2).

Theorem 2.8. *([12]) Let P be a cone in a real Banach space E. Let γ and θ be nonnegative, continuous and convex functionals on P, Φ be a nonnegative, continuous and concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers d and M,*

$$
\Phi(x) \leq \psi(x) \quad \textit{and} \quad \|x\| \leq M\gamma(x)
$$

for all $x \in \overline{P(\gamma, d)}$. Suppose that

$$
T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}
$$

is completely continuous and there exist positive numbers h, r, c with $h < r$, such that the following conditions are satisfied:

(S1) $\{x \in P(\gamma, \theta, \Phi, r, c, d) : \Phi(x) > r\} \neq \emptyset$ and $\Phi(Tx) \geq r$ for $x \in P(\gamma, \theta, \Phi, r, c, d)$;
(S2) $\Phi(Tx) > r$ for $x \in P(\gamma, \Phi, r, d)$ with $\theta(Tx) > c$;
(S3) $0 \notin R(\gamma, \psi, h, d)$ and $\psi(Tx) < h$ for $x \in R(\gamma, \psi, h, d)$ with $\psi(x) = h$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$, such that

$$
\gamma(x_i) \leq d, \quad \textit{for } i = 1, 2, 3,
$$

and

$$
r < \Phi(x_1), \quad h < \psi(x_2), \quad \gamma(x_2) < r, \quad \psi(x_3) < h.
$$

3. Main result

To prove that (1), (2) has three positive solutions, the following three functionals are defined by $\Phi(u) = \min_{t\in[\xi_{m-2},1]} |u(t)|$ and convex functionals $\gamma(u) = \max_{t\in[0,1]} |u'(t)|$, $\psi(u) = \theta(u) = \max_{t\in[0,1]} |u(t)|$ on P.

Theorem 3.1. *Assume that there exist positive constants h, r, c, d with $h < r$, $c > max\left\{\frac{1}{\xi_{m-2}^{\alpha-1}}, e^{1-\frac{\xi_{m-2}^{\alpha-1}}{2}}\right\} r$, $d \geq c$, and f holds the following conditions:*

(H3) $t^{\sigma} f(t, u, u') \leq (dM_1)^{p-1}, \quad for\ (t, u, u') \in [0, 1] \times [0, d] \times [0, d];$
(H4) $f(t, u, u') > (rM_2)^{p-1}, \quad for\ (t, u, u') \in [0, 1] \times [r, c] \times [r, c];$
(H5) $t^{\sigma} f(t, u, u') < (hM_1)^{p-1}, \quad for\ (t, u, u') \in [0, 1] \times [0, h] \times [0, h].$

where $M_1 = \frac{\Delta(1-\sigma)^{q-1}}{(\alpha-1)}$ and $M_2 = \frac{\Delta q}{\xi_{m-2}^{\alpha-1} - \xi_{m-2}^{\alpha+q-1}}$. Then the problem (1),(2) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\gamma(u_i) \leq d, \quad for\ i = 1, 2, 3,$$

and

$$r < \Phi(u_1), \quad h < \psi(u_2), \quad \gamma(u_2) < r, \quad \psi(u_3) < h.$$

Proof. First of all, we prove $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

For $u \in \overline{P(\gamma, d)}$,By assumption (H3), we get

$$\begin{aligned}
\omega(s) &= \phi_q\left(\int_0^s t^{-\sigma} t^{\sigma} f(t, u(t), u'(t))dt\right) \\
&\leq \phi_q\left(\int_0^1 (dM_1)^{p-1} t^{-\sigma} dt\right) \\
&= \frac{dM_1}{(1-\sigma)^{q-1}}
\end{aligned}$$

then

$$\begin{aligned}
\gamma(Tu(t)) &= \max_{t\in[0,1]} |(Tu)'(t)| \\
&\leq \frac{(\alpha-1)}{\Delta} \int_0^1 \omega(s)ds \\
&\leq \frac{(\alpha-1)}{\Delta} \int_0^1 \frac{dM_1}{(1-\sigma)^{q-1}} ds \\
&= \frac{(\alpha-1)dM_1}{\Delta(1-\sigma)^{q-1}} \\
&= d
\end{aligned}$$

So we obtain $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

Take $u(t) = re^{t-0,5\xi_{m-2}^{\alpha-1}}$, $t \in [0, 1]$. By simple calculation, we can get that $u \in P$, $\gamma(u) < c$, $\psi(u) = \theta(u) < c$ and $\Phi(u) > r$.

$$\{u \in P(\gamma, \theta, \Phi, r, c, d) : \Phi(u) > r\} \neq \emptyset$$

For by (H4), we get

$$\begin{aligned}
\Phi(Tu(t)) &= \min_{t\in[\xi_{m-2},1]} |Tu(t)| = |Tu(\xi_{m-2})| \\
&= \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\xi_{m-2}} (\xi_{m-2}-s)^{\alpha-1}\omega(s)ds \\
&\geq \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_{m-2}} \omega(s)ds - \frac{\xi_{m-2}^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\xi_{m-2}} \omega(s)ds \\
&= \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\int_{\xi_{m-2}}^1 \omega(s)ds \\
&= \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\int_{\xi_{m-2}}^1 \phi_q\left(\int_0^s f(t,u(t),u'(t))dt\right)ds \\
&\geq \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\int_{\xi_{m-2}}^1 \phi_q\left(\int_0^s (M_2 r)^{p-1}dt\right)ds \\
&\geq \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\int_{\xi_{m-2}}^1 \phi_q\left((M_2 r)^{p-1}s\right)ds \\
&= \frac{M_2 r(\xi_{m-2}^{\alpha-1} - \xi_{m-2}^{\alpha+q-1})}{\Delta q} \\
&= r
\end{aligned}$$

So, the condition (S1) of Theorem 3.1 holds.

Take $u \in P(\gamma, \Phi, r, d)$ and $\theta(Tu(t)) > c$. Considering $Tu \in P$, we get

$$\begin{aligned}
\theta(Tu(t)) &= \max_{t\in[0,1]} |Tu(t)| = |Tu(1)| \\
&= \frac{1}{\Delta}\int_0^1 \omega(s)ds - \frac{1}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\omega(s)ds
\end{aligned}$$

and

$$\begin{aligned}
\Phi(Tu(t)) &= \min_{t\in[\xi_{m-2},1]} |Tu(t)| = |Tu(\xi_{m-2})| \\
&= \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\int_0^1 \omega(s)ds - \frac{\xi_{m-2}^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{\xi_{m-2}^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\xi_{m-2}} (1-\frac{s}{\xi_{m-2}})^{\alpha-1}\omega(s)ds \\
&\geq \xi_{m-2}^{\alpha-1}\left(\frac{1}{\Delta}\int_0^1 \omega(s)ds - \frac{1}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\omega(s)ds\right) \\
&= \xi_{m-2}^{\alpha-1}(\theta(Tu(t))) \\
&\geq \xi_{m-2}^{\alpha-1}c \\
&= r
\end{aligned}$$

This shows that the condition (S2) is satisfied.

In the following we will prove that the condition (S3) is satisfied.

Assume that $u \in R(\gamma, \psi, h, d)$ with $\psi(u) = h$. Then by (H5), we have

$$\begin{aligned}\omega(s) &= \phi_q\left(\int_0^s t^{-\sigma}t^{\sigma} f(t, u(t), u'(t))dt\right)\\ &\leq \phi_q\left(\int_0^1 (hM_1)^{p-1} t^{-\sigma} dt\right)\\ &= \frac{hM_1}{(1-\sigma)^{q-1}}\end{aligned}$$

So we get

$$\begin{aligned}\psi(Tu(t)) &= \max_{t\in[0,1]} |Tu(t)| = |Tu(1)|\\ &= \frac{1}{\Delta}\int_0^1 \omega(s)ds - \frac{1}{\Delta}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \omega(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{(\alpha-1)}\omega(s)ds\\ &\leq \frac{(\alpha-1)}{\Delta}\int_0^1 \frac{hM_1}{(1-\sigma)^{q-1}}\\ &= \frac{(\alpha-1)hM_1}{\Delta(1-\sigma)^{q-1}}\\ &= h\end{aligned}$$

Thus, the condition (S3) is satisfied. By Theorem 3.1, we can get that (1),(2) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\gamma(u_i) \leq d, \quad for\ i = 1, 2, 3,$$

and

$$r < \Phi(u_1), \quad h < \psi(u_2), \quad \gamma(u_2) < r, \quad \psi(u_3) < h.$$

□

Example 3.2. *Consider the following boundary value problem*

$$[\phi_p(D_{0^+}^{\alpha} u(t))]' + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1 \tag{7}$$

$$u(0) = u'(0) = 0, \quad D_{0^+}^{\alpha-1}u(1) = \sum_{i=1}^{m-2} a_i D_{0^+}^{\alpha-1} u(\xi_i) \tag{8}$$

where $\sqrt{t}f(t, u, v)$ *is continuous in* $[0, 1] \times \mathbb{R}^+, \times\mathbb{R}^+$, $\sqrt{t}f(t, u, v) \leq 1500$, *for* $(t, u, v) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$, *such that*

$$f(t,u,v) = \begin{cases} \frac{1}{\sqrt{t}}(e^u + e^v), & (t,u,v) \in (0,1] \times [0,3] \times [0,3]\\ \frac{50}{\sqrt{t}}(e^{\sqrt{u}} + e^{\sqrt{v}}), & (t,u,v) \in (0,1] \times [5,17] \times [5,17]\\ \frac{1499}{\sqrt{t}}, & (t,u,v) \in (0,1] \times [100,\infty) \times [100,\infty)\end{cases}$$

Corresponding to Theorem 3.1, we take $\alpha = \frac{7}{3}, p = 3, \sigma = \frac{1}{2}, m = 4, \xi_1 = \frac{1}{3}, \xi_2 = \frac{2}{3}, a_1 = \frac{1}{2}, a_2 = \frac{1}{4}$. *Let* $h = 3, r = 5, c = 17, d = 150$. *By simple calculations, we can get that the conditions of Theorem 3.1 are satisfied. So, the problem (7), (8) has at least three positive solutions* u_1, u_2, u_3 *satisfying*

$$\gamma(u_i) \leq 150, \quad for\ i = 1, 2, 3,$$

and

$$5 < \Phi(u_1), \quad 3 < \psi(u_2), \quad \gamma(u_2) < 5, \quad \psi(u_3) < 3.$$

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