Filomat 37:20 (2023), 6665–6675
https://doi.org/10.2298/FIL2320665X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some new midpoint and trapezoidal type inequalities in multiplicative calculus with applications

Jianqiang Xie[a], **Muhammad Aamir Ali**[b,*], **Thanin Sitthiwirattham**[c]

[a] *School of Mathematical Sciences, Anhui University, Hefei 230601, China*
[b] *School of Mathematical Sciences, Nanjing Normal University, China*
[c] *Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand*

Abstract. In this paper, we use multiplicative twice differentiable functions and establish two new multiplicative integral identities. Then, we use convexity for multiplicative twice differentiable functions and establish some new midpoint and trapezoidal type inequalities in the framework of multiplicative calculus. Finally, we give some applications to special means of real numbers to make these inequalities more interesting for the readers.

1. Introduction

In calculus and analysis, integration and differentiation are two fundamental operations. In actuality, these are the infinitesimal forms of the operations on numbers for subtraction and addition, respectively.

New definitions of differentiation and integration in which the roles of addition and subtraction move to multiplication and division and introduce a new calculus called multiplicative calculus. This is also called non-Newtonian calculus. Despite answering all the conditions expected from calculus, multiplicative calculus is not so much popular as the Newton and Leibnitz unfortunately.

Since the applications of multiplicative calculus are relatively limited than the calculus of Newton and Leibnitz. Therefore a well-developed tool with a wider scope has already been made, the question of whether it is fair to design a new tool with a limited scope arises. The solution is comparable to the question of why mathematicians use a polar coordinate system when a rectangular coordinate system that better describes points on a plane exists. We believe that the mathematical instrument of multiplicative calculus can be particularly helpful for the study of economics and finance.

Assume for motivation's sake that by depositing \$$\theta$, one will receive \$$\vartheta$ after a year. The original number then fluctuates ϑ/θ times. How frequently does it change each month? Assume that the change

2020 *Mathematics Subject Classification*. 26D10, 26A51, 26D15.
Keywords. Hermite–Hadamard inequality; Multiplicative calculus; Convex functions.
Received: 21 January 2023; Accepted: 22 February 2023
Communicated by Dragan S. Djordjević
This work is partially supported by National Natural Science Foundation of China (No. 11971241). This work is also supported by National Research Council of Thailand (NRCT) and Suan Dusit University (No. N42A650384). This work was supported by project of Quality Engineering of Anhui University (Grant no. 2022xjzlgc305) and the Natural Science Foundation of Anhui Province (Grant no. 2108085QA18).
* Corresponding author: Muhammad Aamir Ali
Email addresses: xiejq1025@163.com (Jianqiang Xie), mahr.muhammad.aamir@gmail.com (Muhammad Aamir Ali), thanin_sit@dusit.ac.th (Thanin Sitthiwirattham)

over a month is p times for this. The total then becomes $\vartheta = \theta p^{12}$ for a year. The formula for computing p is now $p = (\vartheta/\theta)^{\frac{1}{12}}$. Assuming that deposits fluctuate daily, hourly, minutely, secondarily, etc. and that the function Λ indicating its value at various time points, is the formula

$$\lim_{h\to 0}\left(\frac{\Lambda(x+h)}{h}\right)^{\frac{1}{h}}. \tag{1}$$

The above formula shows that how the value of $\Lambda(x)$ varies at moment x. For the compare of definition (1), the definition of derivative is

$$\Lambda'(x) = \lim_{h\to 0}\frac{\Lambda(x+h)-\Lambda(x)}{h}. \tag{2}$$

We observe that the difference in (2) is replaced by division and the division by h is replace by the raising to the reciprocal power $1/h$. The limit (1) is called multiplicative derivative.

Analysis has long been the dominant area of mathematics, and inequalities play a major role in analysis. The significance of inequalities has long been recognized in the field of mathematics. The mathematical roots of the theory of inequality were set by eminent mathematicians in the 18^{th} and 19^{th} centuries. The impact of inequalities was significant in the years that followed, and many well-known mathematicians were drawn to the subject. In the 20^{th} century, it was the ground breaking work of G. H. Hardy, J. E. Littlewoods, and G. Polya which was published in 1934 that sparked the development of the field as a subfield of modern mathematics.

The Hermite-Hadamard inequality, named after Charles Hermite and Jacques Hadamard and commonly known as Hadamard's inequality, says that if a function $\Lambda : [\theta, \vartheta] \to \mathbb{R}$ is convex, the following double inequality holds:

$$\Lambda\left(\frac{\theta+\vartheta}{2}\right) \leq \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Lambda(x)\,dx \leq \frac{\Lambda(\theta)+\Lambda(\vartheta)}{2}. \tag{3}$$

If Λ is a concave mapping, the above inequality holds in the opposite direction. The inequality (3) can be proved using the Jensen inequlity. There has been much research done in the direction of Hermite-Hadamard for different kinds of convexities. For example, in [1–4], the authors established some inequalities linked with midpoint and trapezoid formulas of numerical integration for convex functions.

In [5] and [6], the authors used twice differentiability and established the following formulas of finding the error bounds for midpoint and trapezoidal type inequalities:

Theorem 1.1. *Let $\Lambda : [\theta, \vartheta] \to \mathbb{R}$ be a twice differentiable and integrable functions. If Λ'' is convex, then the following inequalities holds:*

$$\begin{aligned} &\left|\frac{\Lambda(\theta)+\Lambda(\vartheta)}{2} - \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Lambda(x)\,dx\right| \\ \leq\ & \frac{(\vartheta-\theta)^2}{12}\left[\frac{|\Lambda''(\theta)|+\Lambda''(\vartheta)}{2}\right] \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Lambda(x)\,dx - \Lambda\left(\frac{\theta+\vartheta}{2}\right)\right| \\ \leq\ & \frac{(\vartheta-\theta)^2}{24}\left[\frac{|\Lambda''(\theta)|+\Lambda''(\vartheta)}{2}\right]. \end{aligned} \tag{5}$$

Very recently, Ali et al. [7] proved the Hermite–Hadamard type inequality in the framework of multiplicative calculus and stated as:

Theorem 1.2. *Let $\Lambda : [\theta, \vartheta] \to \mathbb{R}^+$ be a multiplicative convex function, then the following inequality holds:*

$$\Lambda\left(\frac{\theta+\vartheta}{2}\right) \leq \left(\int_\theta^\vartheta (\Lambda(x))^{dx}\right)^{\frac{1}{\vartheta-\theta}} \leq \sqrt{\Lambda(\theta)\Lambda(\vartheta)}. \tag{6}$$

After the work of Ali et al. [7], many researchers started work in this directions and proved different variants of integral inequalities in the setting of multiplicative calculus. For example, Khan and Budak [8] gave some estimates for the midpoint and trapezoidal inequalities in multiplicative calculus, the Hermite–Hadamard type inequalities for general multiplicative convex functions were proved in [9] and Özcan used the multiplicative preinvexity and established Hermite–Hadamard type inequalities in [10]. For multiplicative s-convex and multiplicative s-preinvex functions, the Hermite–Hadamard type inequalities were found in [11, 12] and for h-preinvex functions proved in [13]. Ali et al. [14] established some Ostrowski's and Simpson's type inequalities for multiplicative convex functions and give their applications. Budak and Özcelik [15] used multiplicative fractional integrals and established Hermite–Hadamard type inequalities. In [16], Fu et al. introduced multiplicative tempered fractional integrals and established some new fractional Hermite–Hadamard type inequalities for multiplicative convex functions. Ali et al. [17] introduced the notions of multiplicative interval-valued integral and established some new Hermite–Hadamard type inequalities for interval–valued multiplicative convex functions. Very recently, some new Simpson's and Newton's type inequalities in the setting of multiplicative calculus were established in [18].

Inspired by the ongoing studies, we establish some new trapezoidal and midpoint type inequalities linked with the left and right sides of the inequality (6) for multiplicative twice differentiable multiplicative convex functions. The newly established inequalities are the multiplicative versions of the inequalities (4) and (5), where all subtraction and addition operations behave as a multiplication and division. finally, we give some applications of newly established inequalities. Since multiplicative calculus is modern calculus with lot of applications in banking and finance, therefore the study about multiplicative calculus is valuable.

2. Preliminaries

In this section, we recall some concepts of multiplicative calculus and convexity.

Definition 2.1. *[19] Let $\Lambda : \mathbb{R} \to \mathbb{R}^+$ be a positive function. The multiplicative derivative of the function Λ is given by*

$$\frac{d^*\Lambda}{d\rho}(\rho) = \Lambda^*(\rho) = \lim_{h\to 0}\left(\frac{\Lambda(\rho+h)}{\Lambda(\rho)}\right)^{1/h}.$$

If Λ has positive values and is differentiable at ρ, then Λ^* exists and the relation between Λ^* and ordinary derivative Λ' is as follows:

$$\Lambda^*(\rho) = e^{\left[\log \Lambda(\rho)\right]'} = e^{\frac{\Lambda'(\rho)}{\Lambda(\rho)}}.$$

If, additionally, the second derivative of Λ at ρ exists, then by an easy substitution, we obtain

$$\Lambda^{**}(\rho) = e^{\left[\log \circ \Lambda^*(\rho)\right]'} = e^{\left[\log \Lambda(\rho)\right]''}.$$

Here $(\ln \Lambda)''(\rho)$ exists because $\Lambda''(\rho)$ exist. Repeating this procedure n times, we conclude that if Λ is a positive function and its nth derivative at ρ exists, then $\Lambda^{*(n)}(\rho)$ exists and

$$\Lambda^{*(n)}(\rho) = e^{\left(\log \Lambda\right)^{(n)}(\rho)}, \quad n = 1, 2, \cdots.$$

For more details and properties of multiplicative calculus, one can consult [19].

We also recall that the concept of the * integral called multiplicative integral is denoted by $\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}$ which introduced by Bashirov et al. in [19]. While the sum of the terms of product is used in the definition of a classical Riemann integral of Λ on $[\theta, \vartheta]$, the product of terms raised to power is used in the definition multiplicative integral of Λ on $[\theta, \vartheta]$.
There is the following relation between Riemann integral and multiplicative integral [19]:

Proposition 2.2. *[19] If Λ is Riemann integrable on $[\theta, \vartheta]$, then Λ is multiplicative integrable on $[\theta, \vartheta]$ and*

$$\int_{\theta}^{\vartheta}(\Lambda(x))^{dx} = e^{\int_{\theta}^{\vartheta}\log(\Lambda(x))dx}.$$

Moreover, Bashirov et al. [19] showed that multiplicative integrable has the following results and properties:

Proposition 2.3. *If Λ is positive and Riemann integrable on $[\theta, \vartheta]$, then Λ is multiplicative integrable on $[\theta, \vartheta]$ and*
(i) $\int_{\theta}^{\vartheta}((\Lambda(x))^{p})^{dx} = \left(\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}\right)^{p}$,
(ii) $\int_{\theta}^{\vartheta}(\Lambda(x)g(x))^{dx} = \int_{\theta}^{\vartheta}(\Lambda(x))^{dx}.\int_{\theta}^{\vartheta}(g(x))^{dx}$,
(iii) $\int_{\theta}^{\vartheta}\left(\frac{\Lambda(x)}{g(x)}\right)^{dx} = \frac{\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}}{\int_{\theta}^{\vartheta}(g(x))^{dx}}$,
(iv) $\int_{\theta}^{\vartheta}(\Lambda(x))^{dx} = \int_{\theta}^{c}(\Lambda(x))^{dx}.\int_{c}^{\vartheta}(\Lambda(x))^{dx}, \ \theta \leq c \leq \vartheta.$
(v) $\int_{\theta}^{\theta}(\Lambda(x))^{dx} = 1$ *and* $\int_{\theta}^{\vartheta}(\Lambda(x))^{dx} = \left(\int_{\vartheta}^{\theta}(\Lambda(x))^{dx}\right)^{-1}$.

Theorem 2.4. *[19] Let $\Lambda : [\theta, \vartheta] \to \mathbb{R}$ be multiplicative differentiable, let $g : [\theta, \vartheta] \to \mathbb{R}$ be differentiable so the function Λ^{g} is multiplicative integrable. Then*

$$\int_{\theta}^{\vartheta}(\Lambda^{*}(x)^{g(x)})^{dx} = \frac{\Lambda(\vartheta)^{g(\vartheta)}}{\Lambda(\theta)^{g(\theta)}}.\frac{1}{\int_{\theta}^{\vartheta}(\Lambda(x)^{g'(x)})^{dx}}.$$

Lemma 2.5. *[14] Let $\Lambda : [\theta, \vartheta] \to \mathbb{R}$ be multiplicative differentiable, let $g : [\theta, \vartheta] \to \mathbb{R}$ and $h : J \subset \mathbb{R} \to [\theta, \vartheta]$ be two differentiable functions. Then we have*

$$\int_{\theta}^{\vartheta}(\Lambda^{*}(h(x))^{g(x)h'(x)})^{dx} = \frac{\Lambda(h(\vartheta))^{g(\vartheta)}}{\Lambda(h(\theta))^{g(\theta)}}.\frac{1}{\int_{\theta}^{\vartheta}(\Lambda\left(h\left(x\right)\right)^{g'(x)})^{dx}}.$$

For the our main results we need to following definition.

Definition 2.6. *[20] A non-empty set K is said to be convex, if for every $\theta, \vartheta \in K$ we have*

$$\theta + \rho(\vartheta - \theta) \in K, \ \forall \rho \in [0, 1].$$

Definition 2.7. *[20] A function Λ is said to be convex function on set K, if*

$$\Lambda(\rho x + (1-\rho)y) \leq \rho\Lambda(x) + (1-\rho)\Lambda(y), \ \forall \rho \in [0,1].$$

Definition 2.8. *[20] A function Λ is said to be log or multiplicative convex function on set K, if*

$$\Lambda(\rho x + (1-\rho)y) \leq [\Lambda(x)]^{\rho} \cdot [\Lambda(y)]^{1-\rho}, \forall \rho \in [0,1].$$

3. Multiplicative Integral Identities

In this section, we establish two integral identity associated with the twice multiplicative differentiable function.

Lemma 3.1. *Let $\Lambda : [\theta, \vartheta] \to \mathbb{R}^+$ be a twice multiplicative differentiable function over (θ, ϑ). If Λ^{**} is integrable function, then following equality holds:*

$$\sqrt{\Lambda(\theta)\Lambda(\vartheta)}\left(\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}\right)^{\frac{1}{\theta-\vartheta}} = \left(\int_0^1\left([\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)]^{\rho(1-\rho)}\right)^{d\rho}\right)^{\frac{(\vartheta-\theta)^2}{2}}. \tag{7}$$

Proof. From the fundamental rules of multiplicative integration and changing the variables of integration, we have

$$\begin{aligned}
&\left(\int_0^1\left([\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)]^{\rho(1-\rho)}\right)^{d\rho}\right)^{\frac{(\vartheta-\theta)^2}{2}}\\
=\ & e^{\frac{(\vartheta-\theta)^2}{2}\int_0^1 \rho(1-\rho)(\ln\circ\Lambda)''(\rho\vartheta+(1-\rho)\theta)d\rho}\\
=\ & e^{\frac{\vartheta-\theta}{2}\rho(1-\rho)(\ln\circ\Lambda)'(\rho\vartheta+(1-\rho)\theta)\big|_0^1 - \frac{\vartheta-\theta}{2}\int_0^1(1-2\rho)(\ln\circ\Lambda)'(\rho\vartheta+(1-\rho)\theta)d\rho}\\
=\ & e^{-\frac{1}{2}\left[(1-2\rho)(\ln\circ\Lambda)(\rho\vartheta+(1-\rho)\theta)\big|_0^1 + 2\int_0^1(\ln\circ\Lambda)(\rho\vartheta+(1-\rho)\theta)d\rho\right]}\\
=\ & e^{\frac{1}{2}[\ln\Lambda(\vartheta)\Lambda(\theta)] - \int_0^1(\ln\circ\Lambda)(\rho\vartheta+(1-\rho)\theta)d\rho}\\
=\ & \frac{e^{\ln\sqrt{\Lambda(\theta)\Lambda(\vartheta)}}}{e^{\int_0^1(\ln\circ\Lambda)(\rho\vartheta+(1-\rho)\theta)d\rho}}\\
=\ & \sqrt{\Lambda(\theta)\Lambda(\vartheta)}\left(\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}\right)^{\frac{1}{\theta-\vartheta}}.
\end{aligned}$$

Thus, the proof is completed. □

Lemma 3.2. *Let $\Lambda : [\theta, \vartheta] \to \mathbb{R}^+$ be a twice multiplicative differentiable function over (θ, ϑ). If Λ^{**} is integrable function, then following equality holds:*

$$\begin{aligned}
\frac{\left(\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}\right)^{\frac{1}{\vartheta-\theta}}}{\Lambda\left(\frac{\theta+\vartheta}{2}\right)} =\ & \left[\left(\int_0^{\frac{1}{2}}\left([\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)]^{\rho^2}\right)^{d\rho}\right)\right]^{\frac{(\vartheta-\theta)^2}{2}} \\
& \times\left[\left(\int_{\frac{1}{2}}^1\left([\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)]^{(1-\rho)^2}\right)^{d\rho}\right)\right]^{\frac{(\vartheta-\theta)^2}{2}}.
\end{aligned} \tag{8}$$

Proof. From basic rules of multiplicative integration, we have

$$\begin{aligned} I_1 &= \left[\left(\int_0^{\frac{1}{2}} \left(\left[\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right]^{\rho^2}\right)^{d\rho}\right)\right]^{\frac{(\vartheta-\theta)^2}{2}} \\ &= e^{\frac{(\vartheta-\theta)^2}{2}\int_0^{\frac{1}{2}}\rho^2(\ln\circ\Lambda)''(\rho\vartheta+(1-\rho)\theta)d\rho} \\ &= e^{\frac{\vartheta-\theta}{8}(\ln\circ\Lambda)'\left(\frac{\theta+\vartheta}{2}\right)-\frac{1}{2}\ln\Lambda\left(\frac{\theta+\vartheta}{2}\right)+\int_0^{\frac{1}{2}}\ln\Lambda(\rho\vartheta+(1-\rho)\theta)d\rho} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \left[\left(\int_{\frac{1}{2}}^{1} \left(\left[\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right]^{(1-\rho)^2}\right)^{d\rho}\right)\right]^{\frac{(\vartheta-\theta)^2}{2}} \\ &= e^{-\frac{\vartheta-\theta}{8}(\ln\circ\Lambda)'\left(\frac{\theta+\vartheta}{2}\right)-\frac{1}{2}\Lambda\left(\frac{\theta+\vartheta}{2}\right)+\int_{\frac{1}{2}}^{1}\ln\Lambda(\rho\vartheta+(1-\rho)\theta)d\rho}. \end{aligned}$$

We get the following resultant equality from the above two equalities

$$\begin{aligned} I_1 \times I_2 &= e^{\int_0^1 \ln\Lambda(\rho\vartheta+(1-\rho)\theta)d\rho-\ln\Lambda\left(\frac{\theta+\vartheta}{2}\right)} \\ &= \frac{e^{\frac{1}{\vartheta-\theta}\int_\theta^\vartheta \ln\Lambda(x)dx}}{\Lambda\left(\frac{\theta+\vartheta}{2}\right)} \\ &= \frac{\left(\int_\theta^\vartheta (\Lambda(x))^{dx}\right)^{\frac{1}{\vartheta-\theta}}}{\Lambda\left(\frac{\theta+\vartheta}{2}\right)}. \end{aligned}$$

Thus, the proof is completed. □

4. Trapezoidal type Inequalities

Some new trapezoidal type inequalities in the setting of multiplicative calculus are established in this section.

Theorem 4.1. *Under the assumptions of Lemma 3.1. If Λ^{**} is multiplicative convex function, then the following inequality holds:*

$$\begin{aligned} &\left|\sqrt{\Lambda(\theta)\Lambda(\vartheta)}\left(\int_\theta^\vartheta (\Lambda(x))^{dx}\right)^{\frac{1}{\theta-\vartheta}}\right| \\ \leq\ & \left(\left[\Lambda^{**}(\theta)\right]\left[\Lambda^{**}(\vartheta)\right]\right)^{\frac{(\vartheta-\theta)^2}{24}}. \end{aligned} \tag{9}$$

Proof. From the equality (7) and the multiplicative convexity of Λ^{**}, we have

$$\begin{aligned} &\left|\sqrt{\Lambda(\theta)\Lambda(\vartheta)}\left(\int_\theta^\vartheta (\Lambda(x))^{dx}\right)^{\frac{1}{\theta-\vartheta}}\right| \\ \leq\ & \exp\left[\frac{(\vartheta-\theta)^2}{2}\int_0^1 \left|\ln\left(\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right)^{\rho(1-\rho)}\right| d\rho\right] \\ =\ & \exp\left[\frac{(\vartheta-\theta)^2}{2}\int_0^1 \left|\rho(1-\rho)\ln\left(\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right)\right| d\rho\right] \\ \leq\ & \exp\left[\frac{(\vartheta-\theta)^2}{2}\int_0^1 \rho(1-\rho)\left(\rho\ln\Lambda^{**}(\vartheta)+(1-\rho)\ln\Lambda^{**}(\theta)\right)d\rho\right] \end{aligned}$$

$$= \quad \left(\left[\Lambda^{**}(\theta) \right] \left[\Lambda^{**}(\vartheta) \right] \right)^{\frac{(\vartheta-\theta)^2}{24}}.$$

□

Theorem 4.2. *Under the assumptions of Lemma 3.1. If* $(\ln(\Lambda^{**}))^q$, $q > 1$ *is convex function, then the following inequality holds:*

$$\begin{aligned} &\left| \sqrt{\Lambda(\theta)\Lambda(\vartheta)} \left(\int_\theta^\vartheta (\Lambda(x))^{dx} \right)^{\frac{1}{\theta-\vartheta}} \right| \\ \leq\ & \left(\sqrt{(\Lambda^{**}(\vartheta))(\Lambda^{**}(\theta))} \right)^{\frac{(\vartheta-\theta)^2}{2}\left[\beta(p+1,p+1)\right]^{\frac{1}{p}}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 7 and Hölder inequality, we have

$$\begin{aligned} &\left| \sqrt{\Lambda(\theta)\Lambda(\vartheta)} \left(\int_\theta^\vartheta (\Lambda(x))^{dx} \right)^{\frac{1}{\theta-\vartheta}} \right| \\ \leq\ & \exp\left[\frac{(\vartheta-\theta)^2}{2} \int_0^1 \rho(1-\rho) \left| \ln(\Lambda^{**}(\rho\vartheta + (1-\rho)\theta)) \right| d\rho \right] \\ \leq\ & \exp\left[\frac{(\vartheta-\theta)^2}{2} \left(\int_0^1 (\rho(1-\rho))^p d\rho \right)^{\frac{1}{p}} \left(\int_0^1 (\ln(\Lambda^{**}(\rho\vartheta + (1-\rho)\theta)))^q d\rho \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Applying convexity of $(\ln(\Lambda^{**}))^q$, Using $\sum_{i=1}^{n} (a_i + b_i)^s \leq \sum_{i=1}^{n} a_i^s + b_i^s$ for all $s \in [0, 1)$ and $\left(\frac{1}{2}\right)^{\frac{1}{q}} \leq \frac{1}{2}$, we have

$$\begin{aligned} &\left| \sqrt{\Lambda(\theta)\Lambda(\vartheta)} \left(\int_\theta^\vartheta (\Lambda(x))^{dx} \right)^{\frac{1}{\theta-\vartheta}} \right| \\ \leq\ & \exp\left[\frac{(\vartheta-\theta)^2}{2} \left(\int_0^1 (\rho(1-\rho))^p d\rho \right)^{\frac{1}{p}} \left(\int_0^1 \left[\rho(\ln(\Lambda^{**}(\vartheta))^q + (1-\rho)(\ln\Lambda^{**}(\theta))^q) \right] d\rho \right)^{\frac{1}{q}} \right] \\ =\ & \exp\left[\frac{(\vartheta-\theta)^2}{2} \left[\beta(p+1, p+1)\right]^{\frac{1}{p}} \right. \\ & \left. \times \left(\frac{1}{2}\right)^{\frac{1}{q}} \ln(\Lambda^{**}(\vartheta)) + \left(\frac{1}{2}\right)^{\frac{1}{q}} \ln(\Lambda^{**}(\theta)) \right] \\ \leq\ & \exp\left[\frac{(\vartheta-\theta)^2}{2} \left[\beta(p+1, p+1)\right]^{\frac{1}{p}} \left(\ln \sqrt{(\Lambda^{**}(\vartheta))(\Lambda^{**}(\theta))} \right) \right] \\ =\ & \left(\sqrt{(\Lambda^{**}(\vartheta))(\Lambda^{**}(\theta))} \right)^{\frac{(\vartheta-\theta)^2}{2}\left[\beta(p+1,p+1)\right]^{\frac{1}{p}}}. \end{aligned}$$

Thus, the proof is completed. □

5. Midpoint Type Inequalities

Some new midpoint type inequalities in the setting of multiplicative calculus are established in this section.

Theorem 5.1. *Under the assumptions of Lemma 3.2. If Λ^{**} is multiplicative convex function, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\left(\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}\right)^{\frac{1}{\vartheta-\theta}}}{\Lambda\left(\frac{\theta+\vartheta}{2}\right)}\right|\\ \leq\ &\left([\Lambda^{**}(\theta)][\Lambda^{**}(\vartheta)]\right)^{\frac{(\vartheta-\theta)^2}{48}}.\end{aligned}$$

Proof. From the equality (8) and the multiplicative convexity of Λ^{**}, we have

$$\begin{aligned}&\left|\frac{\left(\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}\right)^{\frac{1}{\vartheta-\theta}}}{\Lambda\left(\frac{\theta+\vartheta}{2}\right)}\right|\\
\leq\ &\exp\left[\frac{(\vartheta-\theta)^2}{2}\left(\int_0^{\frac{1}{2}}\rho^2\left|\ln\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right|d\rho\right)\right.\\
&\left.+\frac{(\vartheta-\theta)^2}{2}\left(\int_{\frac{1}{2}}^{1}(1-\rho)^2\left|\ln\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right|d\rho\right)\right]\\
\leq\ &\exp\left[\frac{(\vartheta-\theta)^2}{2}\left(\int_0^{\frac{1}{2}}\rho^2\left(\rho\ln\Lambda^{**}(\vartheta)+(1-\rho)\ln\Lambda^{**}(\theta)\right)d\rho\right)\right.\\
&\left.+\int_{\frac{1}{2}}^{1}(1-\rho)^2\left(\rho\ln\Lambda^{**}(\vartheta)+(1-\rho)\Lambda^{**}(\theta)\right)d\rho\right]\\
=\ &\exp\left[\frac{(\vartheta-\theta)^2}{2}\left(\frac{\ln\Lambda^{**}(\vartheta)}{64}+\frac{5\ln\Lambda^{**}(\theta)}{192}\right)\right.\\
&\left.+\frac{(\vartheta-\theta)^2}{2}\left(\frac{5\ln\Lambda^{**}(\vartheta)}{192}+\frac{\ln\Lambda^{**}(\theta)}{64}\right)\right]\\
=\ &\left[\Lambda^{**}(\theta)\Lambda^{**}(\vartheta)\right]^{\frac{(\vartheta-\theta)^2}{48}}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 5.2. *Under the assumptions of Lemma 3.2. If $(ln\Lambda^{**})^q$, $q>1$ is a convex function, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\left(\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}\right)^{\frac{1}{\vartheta-\theta}}}{\Lambda\left(\frac{\theta+\vartheta}{2}\right)}\right|\\ \leq\ &\left(\sqrt{\Lambda^{**}(\theta)\Lambda^{**}(\vartheta)}\right)^{\frac{(\vartheta-\theta)^2}{2}\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{\frac{1}{p}}}.\end{aligned}$$

Proof. From the equality (8) and Hölder's inequality, we have

$$\left|\frac{\left(\int_{\theta}^{\vartheta}(\Lambda(x))^{dx}\right)^{\frac{1}{\vartheta-\theta}}}{\Lambda\left(\frac{\theta+\vartheta}{2}\right)}\right|$$

$$
\begin{aligned}
&\leq \exp\left[\frac{(\vartheta-\theta)^2}{2}\left(\int_0^{\frac{1}{2}}\rho^2\left|\ln\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right|d\rho\right)\right.\\
&\quad\left.+\frac{(\vartheta-\theta)^2}{2}\left(\int_{\frac{1}{2}}^1(1-\rho)^2\left|\ln\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right|d\rho\right)\right]\\
&\leq \exp\left[\frac{(\vartheta-\theta)^2}{2}\left(\int_0^{\frac{1}{2}}\rho^{2p}d\rho\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\ln\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right|^q d\rho\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\frac{(\vartheta-\theta)^2}{2}\left(\int_{\frac{1}{2}}^1(1-\rho)^{2p}\,d\rho\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\ln\Lambda^{**}(\rho\vartheta+(1-\rho)\theta)\right|^q d\rho\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Now applying convexity of $\ln\Lambda^{**}$, Using $\sum_{i=1}^{n}(a_i+b_i)^s \leq \sum_{i=1}^{n} a_i^s + b_i^s$ for all $s\in[0,1)$ and $\left(\frac{1}{8}\right)^{\frac{1}{q}}+\left(\frac{3}{8}\right)^{\frac{1}{q}}\leq\frac{1}{2}$, we have

$$
\begin{aligned}
&\left|\frac{\left(\int_\theta^\vartheta(\Lambda(x))^{dx}\right)^{\frac{1}{\vartheta-\theta}}}{\Lambda\left(\frac{\theta+\vartheta}{2}\right)}\right|\\
&\leq \exp\left[\frac{(\vartheta-\theta)^2}{2}\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{\frac{1}{p}}\left(\left[\int_0^{\frac{1}{2}}\rho(\ln\Lambda^{**}(\vartheta))^q+(1-\rho)(\ln\Lambda^{**}(\theta))^q\right]d\rho\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\frac{(\vartheta-\theta)^2}{2}\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{\frac{1}{p}}\left(\left[\int_{\frac{1}{2}}^1\rho(\ln\Lambda^{**}(\vartheta))^q+(1-\rho)(\ln\Lambda^{**}(\theta))^q\right]d\rho\right)^{\frac{1}{q}}\right]\\
&= \exp\left(\frac{(\vartheta-\theta)^2}{2}\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{(\ln\Lambda^{**}(\vartheta))^q}{8}+\frac{3(\ln\Lambda^{**}(\theta))^q}{8}\right)^{\frac{1}{q}}\right.\right.\\
&\quad\left.\left.+\left(\frac{3(\ln\Lambda^{**}(\vartheta))^q}{8}+\frac{(\ln\Lambda^{**}(\theta))^q}{8}\right)^{\frac{1}{q}}\right]\right)\\
&\leq \exp\left(\frac{(\vartheta-\theta)^2}{2}\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{\frac{1}{p}}\left[\left(\ln\Lambda^{**}(\vartheta)^{\left(\frac{1}{8}\right)^{\frac{1}{q}}}\Lambda^{**}(\theta)^{\left(\frac{3}{8}\right)^{\frac{1}{q}}}\right)+\left(\ln\Lambda^{**}(\vartheta)^{\left(\frac{3}{8}\right)^{\frac{1}{q}}}\Lambda^{**}(\theta)^{\left(\frac{1}{8}\right)^{\frac{1}{q}}}\right)\right]\right)\\
&\leq \left(\sqrt{\Lambda^{**}(\theta)\Lambda^{**}(\vartheta)}\right)^{\frac{(\vartheta-\theta)^2}{2}\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{\frac{1}{p}}}.
\end{aligned}
$$

Thus, the proof is completed. □

6. Applications

In this section, we give applications to special special means of real number. For $\theta,\vartheta\in\mathbb{R}$ with $\theta,\vartheta>0$ and $\theta\neq\vartheta$, we have

(1) Arithmetic mean

$$A(\theta,\vartheta)=\frac{\theta+\vartheta}{2};$$

(2) Geometric mean

$$G(\theta,\vartheta)=\sqrt{\theta\vartheta};$$

(3) Logarithmic mean

$$L(\theta,\vartheta)=\frac{\vartheta-\theta}{\ln|\vartheta|-\ln|\theta|};$$

(4) Harmonic mean

$$H(\theta,\vartheta)=\frac{2\theta\vartheta}{\theta+\vartheta}$$

(5) Generalized logarithmic mean

$$L_p(\theta,\vartheta)=\left(\frac{\vartheta^{p+1}-\theta^{p+1}}{(p+1)(\vartheta-\theta)}\right)^{\frac{1}{p}}.$$

Proposition 6.1. *For $\theta,\vartheta\in\mathbb{R}$ with $\theta,\vartheta>0$ and $\theta\neq\vartheta$, the following inequality holds:*

$$\left|\frac{e^{A\left(\theta^2,\vartheta^2\right)}}{e^{L_2^2(\theta,\vartheta)}}\right|\leq e^{\frac{(\vartheta-\theta)^2}{6}}.$$

Proof. This inequality can be derived from Theorem 4.1 for the function $\Lambda(x)=e^{x^2}$. □

Proposition 6.2. *For $\theta,\vartheta\in\mathbb{R}$ with $\theta,\vartheta>0$ and $\theta\neq\vartheta$, the following inequality holds:*

$$\left|\frac{e^{A\left(\theta^2,\vartheta^2\right)}}{e^{L_2^2(\theta,\vartheta)}}\right|\leq e^{(\vartheta-\theta)^2\left[\beta\left(p+1,p+1\right)\right]^{\frac{1}{p}}}.$$

Proof. This inequality can be derived from Theorem 4.2 for the function $\Lambda(x)=e^{x^2}$. □

Proposition 6.3. *For $\theta,\vartheta\in\mathbb{R}$ with $\theta,\vartheta>0$ and $\theta\neq\vartheta$, the following inequality holds:*

$$\left|\frac{e^{H^{-1}(\theta,\vartheta)}}{e^{L^{-1}(\theta,\vartheta)}}\right|\leq e^{\frac{(\vartheta-\theta)^2}{6}H^{-1}(\theta,\vartheta)}.$$

Proof. This inequality can be derived from Theorem 4.1 for the function $\Lambda(x)=e^{\frac{1}{x}}$. □

Proposition 6.4. *For $\theta,\vartheta\in\mathbb{R}$ with $\theta,\vartheta>0$ and $\theta\neq\vartheta$, the following inequality holds:*

$$\left|\frac{e^{H^{-1}(\theta,\vartheta)}}{e^{L^{-1}(\theta,\vartheta)}}\right|\leq e^{2\left[\beta\left(p+1,p+1\right)\right]^{\frac{1}{p}}(\vartheta-\theta)^2H^{-1}(\theta,\vartheta)}.$$

Proof. This inequality can be derived from Theorem 4.2 for the function $\Lambda(x)=e^{\frac{1}{x}}$. □

Proposition 6.5. *For $\theta,\vartheta\in\mathbb{R}$ with $\theta,\vartheta>0$ and $\theta\neq\vartheta$, the following inequality holds:*

$$\left|\frac{e^{L_2^2(\theta,\vartheta)}}{e^{A^2(\theta,\vartheta)}}\right|\leq e^{\frac{(\vartheta-\theta)^2}{12}}.$$

Proof. This inequality can be derived from Theorem 5.1 for the function $\Lambda(x)=e^{x^2}$. □

Proposition 6.6. *For $\theta,\vartheta\in\mathbb{R}$ with $\theta,\vartheta>0$ and $\theta\neq\vartheta$, the following inequality holds:*

$$\left|\frac{e^{L_2^2(\theta,\vartheta)}}{e^{A^2(\theta,\vartheta)}}\right|\leq e^{(\vartheta-\theta)^2\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{\frac{1}{p}}}.$$

Proof. This inequality can be derived from Theorem 5.2 for the function $\Lambda(x) = e^{x^2}$. $\square$

Proposition 6.7. *For $\theta, \vartheta \in \mathbb{R}$ with $\theta, \vartheta > 0$ and $\theta \neq \vartheta$, the following inequality holds:*

$$\left| \frac{e^{L^{-1}(\theta,\vartheta)}}{e^{\frac{1}{2A(\theta,\vartheta)}}} \right| \leq e^{\frac{(\vartheta-\theta)^2}{12} H^{-1}(\theta,\vartheta)}.$$

Proof. This inequality can be derived from Theorem 5.1 for the function $\Lambda(x) = e^{\frac{1}{x}}$. $\square$

Proposition 6.8. *For $\theta, \vartheta \in \mathbb{R}$ with $\theta, \vartheta > 0$ and $\theta \neq \vartheta$, the following inequality holds:*

$$\left| \frac{e^{L^{-1}(\theta,\vartheta)}}{e^{\frac{1}{2A(\theta,\vartheta)}}} \right| \leq e^{2\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{\frac{1}{p}} (\vartheta-\theta)^2 H^{-1}(\theta,\vartheta)}.$$

Proof. This inequality can be derived from Theorem 5.2 for the function $\Lambda(x) = e^{\frac{1}{x}}$. $\square$

7. Conclusion

In this work, we establish some new inequalities of trapezoidal and midpoint type for multiplicative twice differentiable convex functions using the notions of multiplicative calculus. The newly established inequalities can be helpful in finding the error bounds for trapezoidal formula in multiplicative numerical integrations. It is a good and new problem that the upcoming researchers can obtain similar inequalities for other convexities and for the functions of two variables in the setting of multiplicative calculus.

References

[1] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Appl. Math. Comput.*, **147** (2004), 137-146.

[2] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.*, **11** (1998), 91-95.

[3] N. Alp, M. Z. Sarikaya, M. Kunt and İ. İşcan, q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ. Sci.*, **30** (2018), 193–203.

[4] S. Bermudo, P. Kórus and J. N. Valdés, On q-Hermite–Hadamard inequalities for general convex functions. *Acta Math. Hung.*, **162** (2020), 364-374.

[5] M. Z. Sarikaya and N. Aktan, On the generalization of some integral inequalities and their applications. *Mathematical and Computer Modelling*, ***54*** (2011), 2175-2182.

[6] M. Z. Sarikaya, A. Saglam and H. Yildirim, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex. *Int. J. Open Problems Comput. Math.*, **5** (2012).

[7] M. A. Ali, M. Abbas, Z. Zhang, I. B. Sial and R. Arif, On Integral Inequalities for Product and Quotient of Two Multiplicatively Convex Functions. *Asian Res. J. Math.*, **12 (**2019**)**, 1-11.

[8] S. Khan and H. Budak, On midpoint and trapezoidal type inequalities for multiplicative integral. *Mathematica*, **59** (2017), 124-133.

[9] M. A. Ali, M. Abbas and A. A. Zafar, On some Hermite-Hadamard integral inequalities in multiplicative calculus. *J. Inequal. Spec. Funct.*, **10** (2019), 111-122.

[10] S. Özcan, Some integral inequalities of Hermite-Hadamard type for multiplicatively preinvex functions. *AIMS Math.*, **5** (2020), 1505-1518.

[11] S. Özcan, Hermite-Hadamard type ınequalities for multiplicatively s-convex functions. *Cumhuriyet Sci. J.*, **41** (2020), 245-259.

[12] S. Özcan, Some Integral Inequalities of Hermite-Hadamard Type for Multiplicatively s-Preinvex Functions. *Internat. J. Math. Model. Comput.*, **9** (2019), 253-266.

[13] S. Özcan, Hermite-Hadamard Type Inequalities for Multiplicatively h-Preinvex Functions. *Turkish J. Math. Anal. Number Theory*, **9** (2021), 65-70.

[14] M. A. Ali, H. Budak, M.Z. Sarikaya and Z. Zhang, Ostrowski and Simpson type inequalities for multiplicative integrals. *Proyecciones*, **40** (2021), 743-763.

[15] H. Budak and K. Özçelik, On Hermite-Hadamard type inequalities for multiplicative fractional integrals. *Miskolc Math. Notes*, **21** (2020), 91-99.

[16] H. Fu, Y. Peng and T. Du, Some inequalities for multiplicative tempered fractional integrals involving the λ-incomplete gamma functions. *AIMS Math.*, **6** (2021), 7456-7478.

[17] M. A. Ali, Z. Zhang, H. Budak and M. Z. Sarikaya, On Hermite-Hadamard type inequalities for interval-valued multiplicative integrals. *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.*, **69** (2020), 1428-1448.

[18] S. Chasreechai, M. A. Ali, S. Naowarat, T. Sitthiwirattham and K. Nonlaopon, On Some Simpson's and Newton's Type Inequalities in Multiplicative Calculus with Applications. *AIMS Math.*, 2022, in press.

[19] A. E. Bashirov, E. M Kurpınar and A. Özyapıcı, Multiplicative calculus and its applications. *J. Math. Anal. Appl.*, **337 (**2008), 36-48.

[20] C. Niculescu and L. E. Persson, Convex functions and their applications. New York: Springer; 2006.