Filomat 37:20 (2023), 6877–6890
https://doi.org/10.2298/FIL2320877T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nonlocal semilinear Φ-Caputo fractional evolution equation with a measure of noncompactness in Banach space

Abdellah Taqbibt[a], M'hamed Elomari[a], Said Melliani[a]

[a]*Laboratory of Applied Mathematics and Scientific Calculus Sultan Moulay Slimane University, PO Box 523, FST, Beni Mellal, 23000, Morocco*

Abstract. The aim of this work is to study the existence of solutions for nonlocal fractional differential equations inclusions involving Φ-Caputo fractional dirivative in Banach space. The proofs are based on the noncompactness measure method. As application, we give an example is given to illustrate the theoretical results.

1. Introduction

Let $(\Sigma, \|\ .\ \|)$ be a Banach space and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \Sigma$ be a densely defined linear operator. The aim of this paper is to study the following semilinear fractional evolution problem with nonlocal conditions

$$\begin{cases} \left({}^{c}D_{0^+}^{\lambda,\Phi}\mu\right)(\tau) = \mathcal{A}\mu(\tau) + \varrho_\mu(\tau), \qquad \tau \in \Upsilon^* =]0, \Lambda]; \\ \varrho_\mu \in \Psi\Big(\tau, \mu(\tau)\Big), \\ \left(I_{0^+}^{1-\lambda,\Phi}\mu\right)(\tau) = \rho\Big(\mu(.)\Big)(:= \mu_0). \end{cases} \tag{1}$$

where ${}^{c}D_{0^+}^{\lambda,\Phi}$ is the Φ-Caputo fractional derivative of order $\lambda \in]0, 1[$, $\Lambda > 0$, Ψ is the multivalued function and $I_{0^+}^{1-\lambda,\Phi}$ is the Φ-Riemann-Liouville integral at order $1 - \lambda$.
Due to its huge applications in different fields such as physics, chemistry, engineering, finance and other sciences, fractional calculus has become an indispensable branch of mathematics. As an extension of the traditional integer calculus, which has the properties of an infinity memory and is hereditary. The fractional calculus plays a crucial role to give a real modeling for many real-world phenomena, which pushes researchers to study its qualitative aspects, in order to show the exact results. The study of this theory of fractional calculus has developed considerably during the 19th and 20th centuries. To present a common expression for various approaches of the fractional derivative, Almida in [1] tried to introduce a function in the definition of the approach of Caputo and he succeeded in unifying the approaches of Caputo and that of Hadamard, this type of fractional derivative is called Φ-Caputo fractional derivative. In other hand, the mathematical analysis is considered one of the fundamental mathematical domains, it has been developing in recent years by an influx of results and theorems, especially the fixed point theory, which

2020 *Mathematics Subject Classification*. 47J35.
Keywords. Φ-Caputo fractional evolution equation; Measure of noncompactness
Received: 02 February 2023; Accepted: 19 February 2023
Communicated by Maria Alessandra Ragusa
Email addresses: abdellah.taqbibt@usms.ma (Abdellah Taqbibt), m.elomari@usms.ma (M'hamed Elomari), s.melliani@usms.ma (Said Melliani)

is considered as the beating heart in the search for solutions in the case of linear or nonlinear problems, for more details see [14–16, 19, 20]. We follow the same strategy as in the case of the ordinary differential equation of the first order, then we seek an equivalent integral equation and we define an operator of a Banach space in itself, then we seek to show the existence of a fixed point under certain conditions see for example[11, 12]. Morever there are several fixed point theorems, we seek to apply the one that is suitable for each problem. In 1930, Schauder improved and generalized the existence and uniqueness problem of Brower and insists on the continuous map on a convex and compact set unlike Brower who studies only compactness because of their study which is performed on the topology. In recent years, many articles have been devoted to the notion of noncompactness measure. This has been defined in several ways. Where noncompactness measures have also proven very useful in metric fixed point theory, giving existence or stability results for non-expansive and uniformly Lipschitz maps based on certain coefficients defined in terms of such measures. This concept was initiated by the important work of Kuratowski [8]. In 1955, G. Darbo, used the concept of noncompactness measure, to prove a theorem guaranteeing the existence of fixed points of condensing operators [7]. This theorem has found an abundance of applications to prove the existence of solutions for a wide class of differential and integral equations. It is worth mentioning that Darbo's theorem extends both the classical principle of Banach's contraction and Schauder's fixed point theorem extended to noncompact operators [5].
The structure of this document is given in this order. In Section 2, we give some preliminaries, definitions and results that we will need to show our main results. In Section 3, we show the existence of solutions for the semilinear fractional evolution problem with nonlocal conditions **(1)** by introducing the noncompactness measure. After that, we give a concrete example to illustrate our main results.

2. Preliminaries

This section deals with some preliminaries, definitions and properties of the Φ-caputo fractional derivatives and the measure of noncompactness. For more details, we refer the reader to [2, 9].
Let $\Upsilon = [0, \Lambda]$, $\Lambda > 0$. We denote by Θ the set of all smoth functions $\Phi : \Upsilon \to \mathbb{R}$ satisfying

$$\Phi \in C^n(\Upsilon, \mathbb{R}) \quad \text{and} \quad \Phi'(t) > 0 \quad \text{for all } t \in \Upsilon.$$

Definition 2.1. *[2] Let $\Phi \in \Theta$, $\lambda > 0$ and $\varrho \in L^1([\Upsilon, \mathbb{R})$. The Φ-Riemann-Liouville fractional integral at order λ of ϱ is defined by*

$$\left(I_{0^+}^{\lambda,\Phi}\varrho\right)(\tau) = \frac{1}{\Gamma(\lambda)}\int_0^\tau \Phi'(s)(\Phi(\tau) - \Phi(s))^{\lambda-1}\varrho(s)ds. \tag{2}$$

Definition 2.2. *[2] Let $\Phi \in \Theta$, $\lambda > 0$ and $\varrho \in C^{n-1}(\Upsilon, \mathbb{R})$. The Φ-Caputo fractional derivative at order λ of ϱ is defined by*

$$\left({}^C D_{0^+}^{\lambda,\Phi}\varrho\right)(\tau) = \frac{1}{\Gamma(n-\lambda)}\int_0^\tau \Phi'(s)(\Phi(\tau) - \Phi(s))^{n-\lambda-1}\varrho_\Phi^{[n]}(s)ds, \tag{3}$$

where

$$\varrho_\Phi^{[n]}(s) = \left(\frac{1}{\Phi'(s)}\frac{d}{ds}\right)^n \varrho(s) \quad \text{and} \quad n = E(\lambda) + 1,$$

and $E(\lambda)$ denotes the integer part of λ.

Remark 2.3. *In particular, note that*

1) *If $\Phi(\tau) = \log(\tau)$, then ${}^C D_{0^+}^{\lambda,\Phi}$ is the Caputo-Hadamard fractional derivative.*
3) *If $\Phi(\tau) = \tau$, then ${}^C D_{0^+}^{\lambda,\Phi}$ is the Caputo fractional derivative.*

Remark 2.4. *In particular, if $\lambda \in]0,1[$, then we get*

$$^{C}D_{0^+}^{\lambda,\Phi}\varrho(\tau) = \frac{1}{\Gamma(\lambda)}\int_0^{\tau}(\Phi(\tau)-\Phi(s))^{\lambda-1}\varrho'(s)ds,$$

and

$$^{C}D_{0^+}^{\lambda,\Phi}\varrho(t) = I_{0^+}^{1-\lambda,\Phi}\left(\frac{\varrho'(\tau)}{\Phi'(\tau)}\right).$$

Proposition 2.5. *[2] Let $\varrho \in C^{n-1}(\Upsilon,\mathbb{R})$ and $\lambda > 0$, then*

1) $^{C}D_{0^+}^{\lambda,\Phi} I_{0^+}^{\lambda,\Phi}\varrho(\tau) = \varrho(\tau)$.

2) $I_{0^+}^{\lambda,\Phi}\, {}^{C}D_{0^+}^{\lambda,\Phi}\varrho(\tau) = \varrho(\tau) - \sum_{j=0}^{n-1}\frac{\varrho_{\Phi}^{[j]}(0)}{j!}(\Phi(\tau)-\Phi(0))^{j}$.

3) $I_{a^+}^{\lambda,\Phi} : C \to C$ *is linear and bounded.*

Remark 2.6. *[2] Let $\varrho \in C^2(\Upsilon,\mathbb{R})$, then we get*

1. $I_{0^+}^{\lambda,\Phi}\, {}^{C}D_{0^+}^{\lambda,\Phi}\varrho(\tau) = \varrho(\tau) + c_0$ *for all* $\lambda \in]0,1[$;
2. $I_{0^+}^{\lambda,\Phi}\, {}^{C}D_{0^+}^{\lambda,\Phi}\varrho(\tau) = \varrho(\tau) + c_0 + c_1(\Phi(\tau)-\Phi(0))$ *for all* $\lambda \in]1,2[$,
   where $c_0, c_1 \in \mathbb{R}$.

Proposition 2.7. *[2] Let $\tau \in \Upsilon$ and $\lambda > \eta > 0$, then*

1) $I_{0^+}^{\lambda,\Phi}(\Phi(\tau)-\Phi(0))^{\eta-1} = \frac{\Gamma(\eta)}{\Gamma(\lambda+\eta)}(\Phi(\tau)-\Phi(0))^{\lambda+\eta-1}$.

2) $D_{0^+}^{\lambda,\Phi}(\Phi(\tau)-\Phi(0))^{\eta-1} = \frac{\Gamma(\eta)}{\Gamma(\eta-\lambda)}(\Phi(\tau)-\Phi(0))^{\eta-\lambda-1}$.

3) $D_{0^+}^{\lambda,\Phi}(\Phi(\tau)-\Phi(0))^{j} = 0, \quad \forall j < n \in \mathbb{N}$.

We give also the notations and definitions used along this document.
Let $C(\Upsilon,\Sigma)$ be the space of Σ-valued conctinous functions with the uniform norm topology

$$\| y \|_C = \sup_{\tau\in\Upsilon} | y(\tau) | . \tag{4}$$

$L^p(\Upsilon,\Sigma)$ the space of Σ-valued Bochner intagrable functions endowed with the norm

$$\| \varrho \|_{L^p} = \Big[\int_{\Upsilon} | \varrho(\tau) |^p \, d\tau\Big]^{\frac{1}{p}}. \tag{5}$$

We consider the following set

$$C_\lambda(\Upsilon,\Sigma) = \Big\{ y \in C(\Upsilon^*,\Sigma), \quad \lim_{\tau\to 0^+}\tau^{1-\lambda}y(\tau) \text{ exists} \Big\}. \tag{6}$$

It is clear that $C_\lambda(\Upsilon,\Sigma)$ is a Banach space of continuous functions with the norm is given by

$$\| y \|_{C_\lambda} = \sup_{\tau\in\Upsilon}\{\tau^{1-\lambda} \mid y(\tau) \mid\}. \tag{7}$$

Let $\mathfrak{I}$ a subset of the space $C_\lambda(\Upsilon,\Sigma)$, we define $\mathfrak{I}_\lambda$ by

$$\mathfrak{I}_\lambda = \Big\{ y_\lambda, \quad y \in \mathfrak{I} \Big\}, \tag{8}$$

where

$$y_\lambda(\tau) = \begin{cases} \lim_{\tau\to 0^+}\tau^{1-\lambda}y(\tau), & \text{if } \tau = 0; \\ \tau^{1-\lambda}y(\tau), & \text{if } \tau \in \Upsilon^*. \end{cases} \tag{9}$$

Remark 2.8. *It is clear that $\Im_\lambda$ is a subset of $C(\Upsilon, \Sigma)$.*

Let

$$\psi_\lambda : \mathbb{R} \to \mathbb{R} \tag{10}$$

$$\tau \mapsto \psi_\lambda(\tau) = \begin{cases} \frac{\Phi(\tau)^{1-\lambda}}{\Gamma(\lambda)}), & \text{if } \tau > 0; \\ \\ 0, & \text{if } \tau \leq 0. \end{cases} \tag{11}$$

Then

$$I_{0^+}^{\lambda,\Phi} y(\tau) = \Big(\psi_\lambda * (\Phi'.y)\Big)(\tau); \tag{12}$$

$$D_{0^+}^{\lambda,\Phi} y(t) = \Big(\psi_\lambda * (\frac{y}{\Phi'})'\Big)(\tau), \tag{13}$$

where $*$ is convolution, $\Big(i.e, \quad \rho * \varrho(\tau) = \int_0^\tau \rho(\tau - \theta)\varrho(\theta)d\theta\Big)$.
The following result is a invariant of the Arzelà-Ascoli Theorem.

Lemma 2.9. *[13]*

$$\Im_\lambda \subset C(\Upsilon, \Sigma) \textit{ is relatively compact} \Leftrightarrow \Im \subset C_\lambda(\Upsilon, \Sigma) \textit{ is relatively compact} \tag{14}$$

In the rest of the article, we will need some definitions and results. For this we recall the following results and definitions. Denote by $\mathfrak{B}(\Sigma)$ be the space of all bounded subsets in Σ.

Definition 2.10. *[9, 10] Let a function $\xi : \mathfrak{B}(\Sigma) \to \mathbb{R}^+$. We say that ξ is a measure of noncompactness in Σ if, for every $\mathcal{K} \in \mathfrak{B}(\Sigma)$ we have*

$$\xi(\kappa) = \xi(\overline{co}\mathcal{K}), \tag{15}$$

where $\overline{co}\mathcal{K}$ is a closed convex hull of $\mathcal{K}$.

The Hausdorff measure of noncompactness is defined by:

$$\chi(\mathcal{K}) = \inf\Big\{\zeta > 0 \ : \mathcal{K} \text{ has a finite number of balls with radius} \leq \zeta\Big\}. \tag{16}$$

Proposition 2.11. *[9, 10]*

1) *if $\mathcal{K}_0 \subset \mathcal{K}_1$, then $\chi(\mathcal{K}_0) \leq \chi(\mathcal{K}_1)$, for all $\mathcal{K}_0, \mathcal{K}_1 \in \mathfrak{B}(\Sigma)$.*
2) *$\chi(\mathcal{K} \cup \{b\}) = \chi(\mathcal{K})$ for every $b \in \Sigma$, $\mathcal{K} \in \mathfrak{B}(\Sigma)$.*
3) *$\chi(\mathcal{K}) = 0$ is equivalent to the relative compactness of $\mathcal{K}$.*
4) *$\chi(\mathcal{K}_0 + \mathcal{K}_1) \leq \chi(\mathcal{K}_0) + \chi(\mathcal{K}_1)$.*
5) *$\chi(\mathcal{K} \cup \{K_c\}) = \chi(\mathcal{K})$ for every $K_c \in \mathfrak{B}_c(\Sigma), \mathcal{K} \in \mathfrak{B}(\Sigma)$.*

Lemma 2.12. *[4] Let $\{\mu_n\}_{n\in\mathbb{N}^*} \subset L^1(\Upsilon, \Sigma)$ such that*

$$\| \mu_n(\tau) \| \leq \kappa(\tau), \quad \textit{for all } n \in \mathbb{N}^* \textit{ and } \kappa \in L^1(\Upsilon, \mathbb{R}^+). \tag{17}$$

Then the function $\chi(\{\mu_n\}_{n\in\mathbb{N}^*}) \in L^1(\Upsilon, \mathbb{R}^+)$ and

$$\chi(\Big\{\int_0^\tau \mu_n(\theta)d\theta,\ n \in \mathbb{N}^*\Big\}) \leq 2\int_0^\tau \chi(\{\mu_n,\ , n \in \mathbb{N}^*\})d\theta. \tag{18}$$

Definition 2.13. *[3] The one-sided stable probability density is given by*

$$w_\lambda(\tau) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}(\Phi(\tau) - \Phi(0))^{-\lambda n-1}\frac{\Gamma(1 + \lambda n)}{n!}\sin(n\pi\lambda), \quad \tau \in (0, \infty). \tag{19}$$

Definition 2.14. *[6] Let $\mathcal{X}$ be a subset of Σ. We say that $\mathcal{X}$ is contractible, if there exists an element $\tilde{x} \in \mathcal{X}$ and a continuous function $\mathcal{P} : [0, 1] \times \mathcal{X} \to \mathcal{X}$ with*

$$\mathcal{P}(0, x) = \tilde{x} \quad \textit{and} \quad \mathcal{P}(1, x) = x, \quad \textit{for all } x \in \mathcal{X}. \tag{20}$$

Theorem 2.15. *[6] Let Ξ be a convex compact banach subset of a banach space. if $\psi : \Xi \to \Xi$ is with closed graph and compact contractive values, then there exists a fixed point $\tilde{\mu} \in \psi(\tilde{\mu})$.*

3. Main results

In this section we show the main results. Firstly we recall the lemma of a solution to the problem **(1)**. Let A generate a C_0-semigroup equicontinuous $Q(\tau),\ \tau \in \mathbb{R}$ and $L = \max_{\tau\in\Upsilon} \| Q(\tau) \|$.

Lemma 3.1. *[17] Let $\varrho \in C^1(\Upsilon, \Sigma)$. A function $\mu \in C^1(\Upsilon)$ is solution of problem*

$$\begin{cases} \left({}^{c}D^{\lambda,\Phi}_{0^+}\mu\right)(\tau) = \mathcal{A}\mu(\tau) + \varrho_\mu(\tau), \qquad \tau \in \Upsilon; \\ \varrho_\mu \in \Psi\Big(\tau, \mu(\tau)\Big), \\ \left(I^{1-\lambda,\Phi}_{0^+}\mu\right)(\tau) = \rho\Big(\mu(.)\Big)(:= \mu_0). \end{cases} \tag{21}$$

if and only if μ satisfies the following equation

$$\mu(\tau) = S_{\lambda,\Phi}(\tau)\mu_0 + \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau) - \Phi(\theta))\varrho_\mu(\theta)\Phi'(\theta)d\theta, \tag{22}$$

where

$$\begin{aligned} S_{\lambda,\Phi}(\tau) &= I^{1-\lambda,\Phi}R_{\lambda,\Phi}(\tau). \\ R_{\lambda,\Phi}(\tau) &= \tau^{\lambda-1}\int_0^\infty \lambda\Big(\Phi(\theta) - \Phi(0)\Big)\omega_\lambda(\theta)\ Q\Big(\tau^\lambda((\Phi(\theta) - \Phi(0))\Big)\Phi'(\theta)d\theta. \end{aligned}$$

Proposition 3.2. 1. *For any fixed $\tau > 0$, $\Big(S_{\lambda,\Phi}(\tau)\Big)_{\tau>0}$ and $\Big(R_{\lambda,\Phi}(\tau)\Big)_{\tau>0}$ are linear operators.*

2. *For $\mu \in \Sigma$, we have*

$$\begin{aligned} | R_{\lambda,\Phi}(\tau)\mu | &\leq \frac{\tau^{\lambda-1}L}{\Gamma(1+\lambda)} | \mu | . \\ | S_{\lambda,\Phi}(\tau)\mu | &\leq \frac{L}{\lambda} | \mu | . \end{aligned}$$

3. *$\Big(S_{\lambda,\Phi}(\tau)\Big)_{\tau>0}$ and $\Big(R_{\lambda,\Phi}(\tau)\Big)_{\tau>0}$ are strongly continuous for every $\tau > 0$, i.e for all $0 < \tau_1 < \tau_2 \leq \Lambda$, we get*

$$\begin{aligned} \lim_{\tau_1\to\tau_2} \| S_{\lambda,\Phi}(\tau_1)\mu - S_{\lambda,\Phi}(\tau_2)\mu \| &= \lim_{\tau_1\to\tau_2} \| R_{\lambda,\Phi}(\tau_1)\mu - R_{\lambda,\Phi}(\tau_2)\mu \| \\ &= 0. \end{aligned}$$

Proof. Let $\tau \in \Upsilon$, we have

$$R_{\lambda,\Phi}(\tau) = \tau^{\lambda-1}\int_0^\infty \lambda\Big(\Phi(\theta) - \Phi(0)\Big)\omega_\lambda(\theta)\ Q\Big(\tau^\lambda((\Phi(\theta) - \Phi(0))\Big)\Phi'(\theta)d\theta.$$

Then

$$\begin{aligned} | R_{\lambda,\Phi}(\tau)\mu | &= \tau^{\lambda-1} \left| \int_0^\infty \lambda\Big(\Phi(\theta) - \Phi(0)\Big)\omega_\lambda(\theta)\ Q\Big(\tau^\lambda((\Phi(\theta) - \Phi(0))\Big)\mu\Phi'(\theta)d\theta \right| \\ &\leq \tau^{\lambda-1}L | \mu | \left| \int_0^\infty \lambda\Big(\Phi(\theta) - \Phi(0)\Big)\omega_\lambda(\theta))\Phi'(\theta)d\theta \right| . \end{aligned}$$

On the other hand, we know that

$$\int_0^\infty \lambda\theta w_\lambda(\theta)\Phi'(\theta)d\theta = \frac{1}{\Gamma(1+\lambda)},$$

so

$$| R_{\lambda,\Phi}(\tau)\mu | \leq \frac{\tau^{\lambda-1}L}{\Gamma(1+\lambda)} | \mu | . \tag{23}$$

From **(23)** it follows that

$$\begin{aligned} \mid S_{\lambda,\Phi}\Big(\Phi(\tau)-\Phi(0)\Big)\mu \mid &= I^{1-\lambda,\Phi}R_{\lambda,\Phi}\Big(\Phi(\tau)-\Phi(0)\Big) \\ &\leq \frac{L}{\Gamma(1+\lambda)} I^{1-\lambda,\Phi}\Big(\Phi(\tau)-\Phi(0)\Big)^{\lambda-1} \mid \mu \mid . \end{aligned}$$

By Proposition **2.7**, we get

$$\begin{aligned} \mid S_{\lambda,\Phi}\Big(\Phi(\tau)-\Phi(0)\Big)\mu \mid &\leq \frac{L}{\Gamma(1+\lambda)}\Gamma(\lambda) \mid \mu \mid \\ &\leq \frac{L}{\lambda} \mid \mu \mid . \end{aligned}$$

Which imlies that

$$\mid S_{\lambda,\Phi}(\tau)\mu \mid \quad \leq \quad \frac{L}{\lambda} \mid \mu \mid, \quad \text{for all } \tau \in \Upsilon.$$

□

Next, we will need the following assumption.
(H1) The multivalued map $\Psi : \Upsilon \times \Sigma \to \Sigma$ has nonempty convex compact values, $\Psi(\tau, .)$ is upper semicontinuous and $\Psi(., \mu)$ has a strongly measurable selction.
(H2) There exist tow constants α and β such that

$$\| \psi(\tau,\mu) \| \leq \alpha + \beta\tau^{1-\lambda} \| \mu \|, \quad \text{for all } (\tau,\mu) \in \Upsilon \times \Sigma. \tag{24}$$

(H2′) There exists a function $\kappa \in L^q(\Upsilon, \mathbb{R}^+)$, $q > \frac{1}{\lambda}$ such that

$$\chi(\Psi(\tau,\mathcal{K}) \leq \kappa(\tau)\chi(\mathcal{K}), \quad \text{for all } \mathcal{K} \in \mathfrak{B}(\Sigma). \tag{25}$$

(H3) $\rho : C_\lambda \to \Sigma$ is compact and continuous map such that

$$\| \rho(\mu(.)) \| \leq \eta \ \| \mu \| + \gamma, \quad \text{for all } \mu \in C(\Upsilon, \Sigma), \tag{26}$$

where $\eta \in]0, \frac{\lambda}{L}[$ and $\gamma > 0$.

Theorem 3.3. *Let the assumptions **(H1)-(H3)**. Moreover if*

$$\Big(\frac{L\,\eta\Lambda^{1-\lambda}}{\lambda} + \frac{L\Lambda^{2(1-\lambda)}C_1}{\Gamma(1+\lambda)}\Big) < 1, \text{ where } C_1 = \int_0^{\Phi(\tau)-\Phi(0)} (\Phi(\tau)-\Phi(\theta))^{\lambda-1}\Phi'(\theta)d\theta,$$

*then the problem **(1)** has a solution.*

Proof. Let $y \in C_\lambda(\Upsilon, \Sigma)$ and $\mathcal{S}\mathfrak{L}(y)$ denote the set of all solutions of the following problem

$$\begin{cases} \Big({}^c D_{0^+}^{\lambda,\Phi}\mu\Big)(\tau) = \mathcal{A}\mu(\tau) + \varrho_\mu(\tau), & \tau \in \Upsilon; \\ \varrho_\mu \in \Psi\Big(\tau, y(\tau)\Big), \\ \mu_0 = \rho\Big(y(.)\Big). \end{cases} \tag{27}$$

The proof is divided in several steps.
Step1: We will prove $\mathcal{S}\mathfrak{L}(y)$ is bounded (if it is nonempty). Let $\mu \in \mathcal{S}\mathfrak{L}(y)$, then from Lemma **3.1** we have

$$\mu(\tau) = S_{\lambda,\Phi}(\tau)\mu_0 + \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_\mu(\theta)\Phi'(\theta)d\theta, \tag{28}$$

Taking into account the expression of μ_0 ab the fact $\varrho_\mu \in \Psi\Big(\tau, \mu(\tau)\Big)$ together with assumption **(H3)**, we get

$$\begin{aligned}\mu_\lambda(\tau) &= \tau^{1-\lambda}\mu(\tau)\\ &= \tau^{1-\lambda}S_{\lambda,\Phi}(\tau)\rho(\mu(.)) + \tau^{1-\lambda}\int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_\mu(\theta)\Phi'(\theta)d\theta.\end{aligned}$$

So,

$$\begin{aligned}\mid \mu_\lambda(\tau)\mid &\leq \tau^{1-\lambda}\mid S_{\lambda,\Phi}(\tau)\rho(\mu(.))\mid\\ &+ \tau^{1-\lambda}\int_0^{\Phi(\tau)-\Phi(0)}\mid R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_\mu(\theta)\Phi'(\theta)\mid d\theta.\end{aligned}$$

Using Prposition **3.2**, we derive

$$\begin{aligned}\mid \mu_\lambda(\tau)\mid &\leq \frac{L\,\tau^{1-\lambda}}{\lambda}\mid \rho(\mu(.))\mid\\ &+ \tau^{1-\lambda}\frac{L}{\Gamma(1+\lambda)}\int_0^{\Phi(\tau)-\Phi(0)}\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}\mid \varrho_\mu(\theta)\mid \Phi'(\theta)d\theta.\end{aligned}$$

And form **(H2)**, we have

$$\begin{aligned}\mid \mu_\lambda(\tau)\mid &\leq \frac{L\,\tau^{1-\lambda}}{\lambda}\Big(\eta\mid \mu(\tau)+\gamma\Big)\\ &+ \frac{L\tau^{1-\lambda}}{\Gamma(1+\lambda)}\int_0^{\Phi(\tau)-\Phi(0)}\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}\Big(\alpha+\beta\,\theta^{1-\lambda}\mid \mu(\theta)\mid\Big)\Phi'(\theta)d\theta\\ &\leq \frac{L\,\eta}{\lambda}\,\tau^{1-\lambda}\mid \mu(\tau)\mid + \frac{\gamma L\,\tau^{1-\lambda}}{\lambda}\\ &+ \frac{L\tau^{1-\lambda}}{\Gamma(1+\lambda)}\int_0^{\Phi(\tau)-\Phi(0)}\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}\Big(\alpha+\beta\,\theta^{1-\lambda}\mid \mu(\theta)\mid\Big)\Phi'(\theta)d\theta\\ &\leq \frac{L\,\eta}{\lambda}\,\tau^{1-\lambda}\mid \mu(\tau)\mid + \frac{\gamma L\,\tau^{1-\lambda}}{\lambda}\\ &+ \frac{\alpha L\tau^{1-\lambda}}{\Gamma(1+\lambda)}\int_0^{\Phi(\tau)-\Phi(0)}\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}\Phi'(\theta)d\theta\\ &+ \frac{\beta\,L\tau^{1-\lambda}}{\Gamma(1+\lambda)}\int_0^{\Phi(\tau)-\Phi(0)}\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}\theta^{1-\lambda}\mid \mu(\theta)\mid \Phi'(\theta)d\theta\\ &\leq \frac{L\,\eta}{\lambda}\mid \mu_\lambda(\tau)\mid + \frac{\gamma L\,\tau^{1-\lambda}}{\lambda}\\ &+ \frac{\alpha L\tau^{1-\lambda}}{\lambda\Gamma(1+\lambda)}C_1\\ &+ \frac{\beta\,L\tau^{1-\lambda}}{\Gamma(1+\lambda)}\int_0^{\Phi(\tau)-\Phi(0)}\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}\mid \mu_\lambda(\theta)\mid \Phi'(\theta)d\theta.\end{aligned}$$

So,

$$\begin{aligned} |\mu_\lambda(\tau)| &\leq \frac{L\,\tau^{1-\lambda}}{\lambda - L\eta}\Big[\gamma + \frac{\alpha C_1}{\Gamma(1+\lambda)}\Big] \\ &+ \frac{\beta\, L\lambda\tau^{1-\lambda}}{(\lambda - L\eta)\Gamma(1+\lambda)}\int_0^{\Phi(\tau)-\Phi(0)}\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}|\mu_\lambda(\theta)|\,\Phi'(\theta)d\theta \\ &\leq \frac{L\,\Lambda^{1-\lambda}}{\lambda - L\eta}\Big[\gamma + \frac{\alpha C_1}{\Gamma(1+\lambda)}\Big] \\ &+ \frac{\beta\, L\lambda\Lambda^{1-\lambda}}{(\lambda - L\eta)\Gamma(1+\lambda)}\int_0^{\Phi(\tau)-\Phi(0)}\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}|\mu_\lambda(\theta)|\,\Phi'(\theta)d\theta \end{aligned}$$

Set

$$\xi(\tau) = \frac{L\,\Lambda^{1-\lambda}}{\lambda - L\eta}\Big[\gamma + \frac{\alpha C_1}{\Gamma(1+\lambda)}\Big],$$

and

$$\Delta = \frac{\beta\, L\lambda\Lambda^{1-\lambda}}{(\lambda - L\eta)\Gamma(1+\lambda)}.$$

By Gronwall's lemma for singular kernaels, ther existe N_λ such that

$$\begin{aligned} |\mu_\lambda(\tau)| &\leq \xi + \Delta N_\lambda \int_0^{\Phi(\tau)-\Phi(0)} \xi(\theta)\Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1}|\,\Phi'(\theta)d\theta \\ &\leq \xi + \Delta N_\lambda\, C_1 \frac{L\,\Lambda^{1-\lambda}}{\lambda - L\eta}\Big[\gamma + \frac{\alpha C_1}{\Gamma(1+\lambda)}\Big]. \end{aligned}$$

Finally

$$\|\mu\|_{C_\lambda} \leq \xi + \Delta N_\lambda\, C_1 \frac{L\,\Lambda^{1-\lambda}}{\lambda - L\eta}\Big[\gamma + \frac{\alpha C_1}{\Gamma(1+\lambda)}\Big]. \tag{29}$$

Setp 2: We have to show that $\mathcal{S}\mathfrak{L}(y)$ is compcat contractive and closed graph. Let $y \in C_\lambda(\Upsilon, \Sigma))$ and let μ with pseudoderivative ϱ_μ be a solution of

$$\begin{cases} \Big({}^c D_{0^+}^{\lambda,\Phi}\mu\Big)(\tau) = \mathcal{A}\mu(\tau) + \varrho_\mu(\tau), & \tau \in \Upsilon; \\ \varrho_\mu \in \Psi\Big(\tau, y(\tau)\Big), \\ I_{0^+}^{1-\lambda}\mu(t) = \rho\Big(y(.)\Big)(:= \mu_0). \end{cases}$$

if ν with pseudoderivative ϱ_ν be another solution and $\mu_0 = \nu_0$.
Let $\delta \in [0,1]$. We defined the following function

$$\mathcal{P}(\delta,\nu)(\tau) = \begin{cases} \nu_\lambda(\tau), & \tau \in [0, \delta\Lambda]; \\ \vartheta_\lambda(\tau, \delta\Lambda)(\tau), & \tau \in [\delta\Lambda, \Lambda], \end{cases}$$

Where

$$\nu_\lambda(\tau) = \tau^{1-\lambda} S_{\lambda,\Phi}(\tau)\mu_0 + \int_0^{\Phi(\delta\Lambda)-\Phi(0)} \tau^{1-\lambda} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_\nu(\theta)\Phi'(\theta)d\theta$$

and

$$\begin{aligned} \vartheta_\lambda(\tau,\delta\Lambda)(\tau) &= \tau^{1-\lambda} S_{\lambda,\Phi}(\tau)\mu_0 \\ &+ \int_0^{\Phi(\delta\Lambda)-\Phi(0)} \tau^{1-\lambda} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_\vartheta^\delta(\theta)\Phi'(\theta)d\theta \end{aligned}$$

with

$$\varrho_{\vartheta}^{\delta}(\tau)=\begin{cases}\varrho_{\nu}(\tau), & \tau\in[0,\delta\Lambda];\\ \varrho_{\mu}(\tau), & \tau\in[\delta\Lambda,\Lambda].\end{cases}$$

Now, we have to prove that $\mathcal{P}(.,.)$ is continuous. Let

$$\begin{aligned}\nu_{\lambda}(\tau+h) &= \tau^{1-\lambda}S_{\lambda,\Phi}(\tau)\mu_0\\ &+ \int_0^{\Phi(\tau)-\Phi(0)}\tau^{1-\lambda}R_{\lambda,\Phi}(\Phi(\tau+h)-\Phi(\theta))\varrho_{\nu}(\theta)\Phi'(\theta)d\theta\\ &+ \int_{\Phi(\tau)-\Phi(0)}^{\Phi(\tau+h)-\Phi(0)}\tau^{1-\lambda}R_{\lambda,\Phi}(\Phi(\tau+h)-\Phi(\theta))\varrho_{\tilde{\nu}}(\theta)\Phi'(\theta)d\theta\end{aligned}$$

and

$$\begin{aligned}\tilde{\nu}_{\lambda}(\tau+h) &= \tau^{1-\lambda}S_{\lambda,\Phi}(\tau)\mu_0\\ &+ \int_0^{\Phi(\tau+h)-\Phi(0)}\tau^{1-\lambda}R_{\lambda,\Phi}(\Phi(\tau+h)-\Phi(\theta))\varrho_{\nu}(\theta)\Phi'(\theta)d\theta.\end{aligned}$$

Then

$$\begin{aligned}\nu_{\lambda}(\tau+h) &- \tilde{\nu}_{\lambda}(\tau+h)=\int_0^{\Phi(\tau)-\Phi(0)}\tau^{1-\lambda}R_{\lambda,\Phi}(\Phi(\tau+h)-\Phi(\theta))\varrho_{\nu}(\theta)\Phi'(\theta)d\theta\\ &+ \int_{\Phi(\tau)-\Phi(0)}^{\Phi(\tau+h)-\Phi(0)}\tau^{1-\lambda}R_{\lambda,\Phi}(\Phi(\tau+h)-\Phi(\theta))\varrho_{\tilde{\nu}}(\theta)\Phi'(\theta)d\theta\\ &- \int_0^{\Phi(\tau+h)-\Phi(0)}\tau^{1-\lambda}R_{\lambda,\Phi}(\Phi(\tau+h)-\Phi(\theta))\varrho_{\nu}(\theta)\Phi'(\theta)d\theta\\ &= \int_{\Phi(\tau)-\Phi(0)}^{\Phi(\tau+h)-\Phi(0)}\tau^{1-\lambda}R_{\lambda,\Phi}(\Phi(\tau+h)-\Phi(\theta))\Big[\varrho_{\nu}(\theta)-\varrho_{\tilde{\nu}}(\theta)\Big]\Phi'(\theta)d\theta.\end{aligned}$$

And thus

$$\begin{aligned}\mid\nu_{\lambda}(\tau+h) &- \tilde{\nu}_{\lambda}(\tau+h)\mid\leq\Lambda^{1-\lambda}\\ &\int_{\Phi(\tau)-\Phi(0)}^{\Phi(\tau+h)-\Phi(0)}\mid R_{\lambda,\Phi}(\Phi(\tau+h)-\Phi(\theta))\Big[\varrho_{\nu}(\theta)-\varrho_{\tilde{\nu}}(\theta)\Big]\mid\Phi'(\theta)d\theta.\end{aligned}$$

and from Proposition **3.2**, we get

$$\begin{aligned}\mid\nu_{\lambda}(\tau+h) &- \tilde{\nu}_{\lambda}(\tau+h)\mid\leq\frac{L\Lambda^{1-\lambda}}{\Gamma(1+\lambda)}\\ &\int_{\Phi(\tau)-\Phi(0)}^{\Phi(\tau+h)-\Phi(0)}(\Phi(\tau+h)-\Phi(\theta))^{\lambda-1}\mid\varrho_{\nu}(\theta)-\varrho_{\tilde{\nu}}(\theta)\mid\Phi'(\theta)d\theta.\end{aligned}$$

So,

$$\|\nu_{\lambda}(\tau+h) - \tilde{\nu}_{\lambda}(\tau+h)\|\to 0,\ \text{as } h\to 0.$$

Consequently $\mathcal{P}(.,.)$ is continuous.
On the other hand, it is clear

$$\mathcal{P}(1,\nu)(\tau)=\nu_{\lambda}(t)$$

and

$$\mathcal{P}(0,\nu)(\tau)=\mu_{\lambda}(t),$$

where

$$\mu_\lambda(\tau) = \tau^{1-\lambda} S_{\lambda,\Phi}(\tau)\mu_0 + \int_0^{\Phi(\tau)-\Phi(0)} \tau^{1-\lambda} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_\mu(\theta)\Phi'(\theta)d\theta$$

Finaaly $\mathcal{S}\mathfrak{L}(y)$ is contractive.
We have to show that $\mathcal{S}\mathfrak{L}(y)$ is closed. From **(H2)**, we get

$$\begin{aligned} \mid \varrho_y(\tau) \mid &\leq \alpha + \beta \mid \tau^{1-\lambda} y(\tau) \mid \\ &\leq \alpha + \beta \parallel y \parallel_{C_{1-\lambda}} . \end{aligned}$$

Let

$$\mu_n(\tau) = S_{\lambda,\Phi}(\tau)\mu_0 + \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_{n,\mu}(\theta)\Phi'(\theta)d\theta.$$

Suppose that

$$\mu_n(\tau) \longrightarrow \mu(\tau), \text{ uniformly on } \Upsilon.$$

Since

$$\mid \varrho_{n,y}(\tau) \mid \quad \leq \quad \alpha + \beta \parallel y \parallel_{C_{1-\lambda}} .$$

Then $\varrho_{n,y} \in L^\lambda(\Upsilon, \Sigma)$ is precompact for any τ, and thus

$$\varrho_{n,y}(\tau) \longrightarrow \varrho_y(\tau), \text{ weakly in } L^\lambda(\Upsilon, \Sigma),$$

because $\Psi(., y(.))$ has compact values and it is almost USC. So it is easy to see that

$$\mu(\tau) = S_{\lambda,\Phi}(\tau)\mu_0 + \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_\mu(\theta)\Phi'(\theta)d\theta.$$

Thus μ is also a solution of

$$\begin{cases} \left({}^c D_{0^+}^{\lambda,\Phi}\mu\right)(\tau) = \mathcal{A}\mu(\tau) + \varrho_\mu(\tau), \qquad \tau \in \Upsilon; \\ \varrho_\mu \in \Psi\Big(\tau, y(\tau)\Big), \\ I_{0^+}^{1-\lambda}\mu(t) = \rho\Big(y(.)\Big)(:= \mu_0). \end{cases}$$

Therfore $\mathcal{S}\mathfrak{L}(y)$ is compcat contractive. One can use the similare arguments as above to prove that $\mathcal{S}\mathfrak{L}(y)$ has closed graph.
Step 3 : Existance an convex and compact $\Bbbk \subset C(\Upsilon, \Sigma)$ such that $\mathcal{S}\mathfrak{L}(.) : \Bbbk \to \Bbbk$.
According to what is preceded, we have

$$\mu_\lambda(\tau) = \tau^{1-\lambda} S_{\lambda,\Phi}(\tau)\rho(\mu(.)) + \tau^{1-\lambda} \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\varrho_\mu(\theta)\Phi'(\theta)d\theta.$$

And thus

$$
\begin{aligned}
\mid \mu_\lambda(\tau) \mid \; &\leq \; \frac{L\,\tau^{1-\lambda}}{\lambda} \mid \rho(\mu(.)) \mid \\
&+ \; \tau^{1-\lambda} \frac{L}{\Gamma(1+\lambda)} \int_0^{\Phi(\tau)-\Phi(0)} \Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1} \mid \varrho_\mu(\theta) \mid \Phi'(\theta) d\theta \\
&\leq \; \frac{L\,\tau^{1-\lambda}}{\lambda} \mid \gamma + \eta\, k) \mid \\
&+ \; \tau^{1-\lambda} \frac{L}{\Gamma(1+\lambda)} \int_0^{\Phi(\tau)-\Phi(0)} \Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1} \mid \alpha + \beta\ \theta^{1-\lambda}\, k \mid \Phi'(\theta) d\theta \\
&\leq \; \gamma \frac{L\,\tau^{1-\lambda}}{\lambda} + k \frac{L\,\eta\tau^{1-\lambda}}{\lambda} \\
&+ \; \tau^{1-\lambda} \frac{L}{\Gamma(1+\lambda)} \alpha\, C_1 \\
&+ \; \tau^{2(1-\lambda)} \frac{L}{\Gamma(1+\lambda)} \beta\, k \int_0^{\Phi(\tau)-\Phi(0)} \Big(\Phi(\tau)-\Phi(\theta)\Big)^{\lambda-1} \Phi'(\theta) d\theta.
\end{aligned}
$$

So,

$$
\| \mu_\lambda(\tau) \|_C \;\; \leq \;\; \Lambda^{1-\lambda} L\Big(\frac{\gamma}{\lambda} + \frac{\alpha C_1}{\Gamma(1+\lambda)}\Big) + k\Big(\frac{L\,\eta\Lambda^{1-\lambda}}{\lambda} + \frac{L\Lambda^{2(1-\lambda)} C_1}{\Gamma(1+\lambda)}\Big).
$$

Since $\left(\frac{L\,\eta\Lambda^{1-\lambda}}{\lambda} + \frac{L\Lambda^{2(1-\lambda)}C_1}{\Gamma(1+\lambda)}\right) < 1$, then for $\xi > \frac{\Lambda^{1-\lambda} L\left(\frac{\gamma}{\lambda} + \frac{\alpha C_1}{\Gamma(1+\lambda)}\right)}{1-\left(\frac{L\eta\Lambda^{1-\lambda}}{\lambda} + \frac{L\Lambda^{2(1-\lambda)}C_1}{\Gamma(1+\lambda)}\right)}$, we have that $\mathcal{S}\mathfrak{L}(.) : \mathfrak{F}_0 \to \mathfrak{F}_0$, where $\mathfrak{F}_0 = \xi\overline{\mathfrak{B}}$ and $\mathfrak{B}$ is the open unit ball in $C_\lambda(\Upsilon, \Sigma)$.
Let $\mathfrak{F}_{n+1} = \overline{\mathcal{S}\mathfrak{L}}(\mathfrak{F}_n)$, then

$$
\mathfrak{F}_{n+1} \subset \mathfrak{F}_n.
$$

Now we define the following set $\Re_n(\tau) = \Big\{\mu(\tau) \;:\; \mu \in \mathfrak{F}_n\Big\}$, so

$$
\Re_{n+1}(\tau) = S_{\lambda,\Phi}(\tau)\rho(\mathfrak{F}_n) + \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\overline{\Big(\bigcup_{k=1}^{\infty} \varrho_k(\theta)\Big)}\Phi'(\theta)d\theta.
$$

Moreover according to the Properties **2.11** and **(H2′)**, we obtain

$$
\chi\Big(\Re_{n+1}(\tau)\Big) \leq \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\kappa(\theta)\chi(\Re_n(\theta))\Phi'(\theta)d\theta,
$$

since $\Re_{n+1} \subset \Re_n$ and χ is a Hausdorff measure of noncompactness, then $\chi\Big(\Re_{n+1}(\tau)\Big) \leq \chi\Big(\Re_n(\tau)\Big)$. And thus

$$
\int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\kappa(\theta)\chi(\Re_n(\theta))\Phi'(\theta)d\theta
$$

has a limit as well as $\chi\Big(\Re_0(\tau)\Big) = \Re(\tau)$, and this

$$
\chi\Big(\Re(\tau)\Big) \leq \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\kappa(\theta)\chi(\Re(\theta))\Phi'(\theta)d\theta.
$$

Consequently $\chi\Big(\Re(\tau)\Big) \leq d(\tau)$, where

$$
d(\tau) = \int_0^{\Phi(\tau)-\Phi(0)} R_{\lambda,\Phi}(\Phi(\tau)-\Phi(\theta))\kappa(\theta)d(\theta)\Phi'(\theta)d\theta. \tag{30}
$$

Since **(30)** has a unique solution $d(\tau) = 0$. So, $\Re_n$ is a bounded convex and compact subsets of $C(\Upsilon, \Sigma)$ and from Lemma **2.9**, then $\mathfrak{F}_n$ is a bounded convex and compact subsets of $C_\lambda(\Upsilon, \Sigma)$. Evidently $\mathcal{S}\mathfrak{L}(y) : \Bbbk \to \Bbbk$, where

$$\Bbbk = \bigcap_{n=1}^{\infty} \mathfrak{F}_n.$$

Finally from Theorem **2.15**, then $\mathcal{S}\mathfrak{L}(.)$ admet a fixed point

$$y \in \mathcal{S}\mathfrak{L}(y(.)).$$

Which is a solution of **(1)**. □

4. Examples

we give a nontrivial example to illustrate our main result. Let $\Sigma \subset \mathbb{R}^n$ with $\partial\Sigma$ is smooth boundary, $\Upsilon = [0, 1]$ and Lebesgue measure $\sigma(\Sigma)$. Consider the Following problem

$$\begin{cases} \frac{\partial^{\frac{1}{2}}}{\partial\tau^{\frac{1}{2}}}\mu(\tau, y) - \sum_{j=1}^{n} \hbar_j \frac{\partial\mu}{\partial y_j} \in \Big[H_1(\tau, y, \mu(\tau, y)), H_2(\tau, y, \mu(\tau, y))\Big], \quad (\tau, y) \in \Upsilon^* \times \Sigma; \\ I_{0^+}^{1-\lambda}\mu(t) = \int_\Upsilon \Big(\int_\Sigma \varphi(\tau, y, \ell, \mu(\theta, \ell))d\ell\Big)d\theta, \\ \mu(\tau, y) = 0, \qquad y \in \partial\Sigma, \ \tau \in \Upsilon \diagdown \{0, 1\}. \end{cases} \tag{31}$$

Where $\mu : \Upsilon \times L^q(\Sigma) \to \mathbb{R}^+$, $\vec{\hbar} = \begin{bmatrix} \hbar_1 \\ \vdots \\ x_m \end{bmatrix}$ is a vector such that $\sum_{j=1}^{n} | \hbar_j |= \frac{1}{\Gamma(\frac{1}{2})}$, $H_i : \Upsilon \times \Sigma \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} H_1(\tau, y, \mu(\tau, y)) &\leq H_2(\tau, y, \mu(\tau, y)), \\ \exists\alpha, \beta > 0, \quad | H_i(\tau, y, \mu(\tau, y)) | &\leq \alpha | \mu | +\beta, \end{aligned}$$

$\nabla_y\mu = \big(\frac{\partial\mu}{\partial y_1}, \cdots, \frac{\partial\mu}{\partial y_n}\big)$ and φ is continuous such that

$$\exists \ \gamma, \eta > 0; \quad | \varphi(\tau, y, z, \mu) |\leq \eta | \mu | +\gamma.$$

We define an operator $\mathcal{A}$ by

$$\mathcal{A}\mu = \vec{\hbar}\nabla_y\mu = \sum_{j=1}^{n} \hbar_j \frac{\partial\mu}{\partial y_j}.$$

with th domain

$$\mathcal{D}(\mathcal{A}) = \Big\{\mu \in \Sigma, \quad \vec{\hbar}\nabla_y\mu\Big\}$$

Thon from [18] the operator $\mathcal{A}$ is the infinitesimal generator of the equicontinuous C_0-semigroup $\mathcal{P}(\tau)$ define by

$$\mathcal{P}(\tau)\nu(y) = \nu(y + \tau \ \vec{\hbar}).$$

The equations **(31)** can be reformulation as the problem **(21)**, where $\Phi(\tau) = \tau$,

$$\Psi(\tau, \mu) = \bigcup_{k\in[0,1]} k \, H_1\Big(\tau, y, \mu(\tau, y)ig) \pm (1 - k)H_2\Big(\tau, y, \mu(\tau, y)\Big).$$

and

$$\rho(\mu)(y) = \int_\Upsilon \Big(\int_\Sigma h(\tau, y, \ell, \mu(\theta, \ell))d\ell\Big)d\theta, \quad y \in \Sigma.$$

Then Ψ has nonempty convex and compact values. Moreover the hypothesis **(H2')** is verified.
On the other hand if we suppose that for every $\zeta > 0$, there exists a function $f_\zeta \in L^1(\Upsilon \times \Sigma \times \Sigma \times \mathbb{R}^+, \mathbb{R}^+)$ such that

$$| \varphi(\tau, y_1, z, \mu) - \varphi(\tau, y_2, z, \mu) | \leq f_\zeta(\tau, y_1, y_2, \mu),$$

for all (τ, y_1, z, μ), $(\tau, y_2, z, \mu) \in \Upsilon \times \Sigma \times \Sigma \times \mathbb{R}^+$ with $| \mu | \leq \zeta$ and

$$\lim_{y_1 \to y_2} \int_\Upsilon \int_\Sigma f_\zeta(\tau, y_1, y_2, \ell) \tau d\ell d\tau = 0, \text{ uniformly on } y_2 \in \Sigma.$$

Then, by using Lemma **4.1** and Lemma **4.2** of chapiter **5** in [18], we show that $\rho : L^q \to L^q$ is defined and completely continuous. So, the assumptions **(H1)-(H3)** are satisfied. Finally let $\left(\eta + \frac{1}{\Gamma(3/2)}\right) < \frac{\Gamma(1/2)}{2}$, then from Theorem **3.3**, then the prblem **(31)** has a solution.

Acknowledgements

The authors are thankful to the referee for her/his valuable suggestions towards the improvement of the paper.

Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability

The data used to support the findings of this study are included in the references within the article.

References

[1] R. Almeida, *Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul., **44**, 460–481 (2017)
[2] R. Almeida, A. B. Malinowska, M. T. T. Monteiro, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Mathematical Methods in the Applied Sciences., **41**(1), 336–352 (2018)
[3] F. Mainardi, *On the initiale value problem for the fraction diffusion-wave equation*, Waves and Stability in Continuous Media World Scientific W. Schick. Singapore., 246–251 (1994)
[4] D. Bothe, *Multivalued perturbations of m-accretive differential incluions*, Israel J. Math., **108**(1), 109–138 (1998)
[5] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach spaces*, Lect. Notes Pure Appl. Math., Dekker, New York, **21**(1), 131–143 (1980)
[6] L. Gorniewicz, A. Granas, W. Kryszewski, *Sur la methode de l'homotopie dans la theorie des points fixes pour les applications multivoques*, Partie 1: Transversalite topologique. CR Acad. Sci. Paris Sér. I Math., **307**, 489–492 (1988).
[7] G. Darbo, *Punti uniti dans transformazioni un condomino non compatto*, Rend. Univ Padova, **24**, 84–92 (1955)
[8] K. Kuratowski, *Sur les espaces complets*, Fund, Math, **15**, 301–309 (1930)
[9] M. Kamenski, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Space*, De Gruyter, Berlin, (2011)
[10] J. Toledano, T. Benavides, G. Azedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhauser, Basel, Springer Science Business Media (1997)
[11] H. Afshari, S. Kalantari, D. Baleanu, *Solution of fractional differential equations via α-φ-geraghty type mappings*, Adv. Differ. Equ.(1), 2017–347 (2018)
[12] S. Abbas, M. Benchohra, S. Krim, *initial value proplems for Caputo-Fabrizio implicit fractional dierential equations in b-metrice spaces*, Bull. Transilv. Univ. Bras., **1**(63), 1–12 (2021)
[13] M. Ziane, *On the solution set for weightened fractional differential equations in a Banach space*, Difer. Eqn. Dyn. Syst., **28**(2), 419–430 (2020)
[14] H. M. Ahmed, M. A. Ragusa , *Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential*, Bull. of the Malaysian Mathematical Sciences Society, 45(6), 3239–3253 (2022)
[15] A. Alsaedi, A. Assolami, B. Ahmad, *Existence results for nonlocal Hilfer-type integral-multipoint boundary value problems with mixed nonlinearities*, Filomat, 36(14), 4751–4766(2022)
[16] V. Gafiychuk, B. Datsun, V. Meleshko, *Mathematical modeling of time fractional reactiondiffusion systems*, J. Comp. Appl. Math., **220**(1-2), 215–225 (2008)
[17] A. El mfadel, S. Melliani, A. Kassidi, M. Elomaro, *Existence mild solutions for nonlocal $\Phi-$Caputo-type fractional evolution equations with nondense domain*, Dynamical Systems., **9**(1), 272–289 (2022)
[18] R. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, (Robert H. Martin, Jr.) Revue SIAM ,**20**(1), 202–204 (1978)

[19] K. Diethelm, A. D. Freed, *On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity*, "Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties" (F. Keil, W. Mackens, H. Voss, and J. Werther, Eds) Springer- Verlag, Heidelberg, 217–224 (1999)
[20] R. Hilfer, *Applications of Fractional Calculus in Physics*, Singapore, World Scientific (2000)