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Existence and Mittag-Leffler-Ulam-stability results of sequential fractional hybrid pantograph equations

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Abstract. In this present work, the existence and uniqueness of solutions for fractional pantograph differential equations involving Riemann-Liouville and Caputo fractional derivatives are established by applying contraction mapping principle and Leray-Schauder's alternative. The Mittag-Leffler-Ulam stability results are also obtained via generalized singular Gronwall's inequality. Finally, we give an illustrative example.

1. Introduction and fractional calculus

Fractional differential equations have recently been studied by many scholars, these equations will be used to describe phenomena of real world problems. For more information, see the monographs [1, 18, 20] and the references therein. Many interesting and important area concerning of research for fractional differential equations are devoted to the existence theory and analysis of the solutions, see works [1, 3, 17, 27, 29, 31]. Hybrid differential equations using fractional calculus have also been studied by several scholars, for instance, see, [2, 10, 13, 22, 25]. Recently, several researchers have discussed the existence, uniqueness and Ulam-stability of solutions for hybrid differential equations involving fractional derivatives, for instance, see [2, 14, 19, 33]. In recent years, considerable attention has been given to the study of existence, uniqueness and Ulam stability of solutions for sequential fractional differential equations, we refer the reader to the monographs [24, 26, 28, 34] and thereference therein. The pantograph type equation is considered as a special delay differential equation, it arises in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, electro dynamics and quantum mechanics, see [9, 12, 32]. The classical form of the pantograph equation is given by the following differential equation [6]:

$$\begin{cases} z'(t) = Az(t) + Bz(\eta t), \\ \qquad z(0) = z_0, \\ 0 \leq t \leq T,\ 0 < \eta < 1. \end{cases} \tag{1.1}$$

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In recent years, there are many scholars have discussed the existence, uniqueness and different types of Ulam stability of solutions of the above equation invoving different fractional derivatives. For some recent work on pantograph equation of fractional-order, we refer to [4, 5, 11, 15, 16, 30] and references therein. In [6] the authors established existence and uniqueness of fractional pantograph equation with Caputo fractional derivative of the form:

$$\begin{cases} {}^{C}D^{\delta}z(t) = \psi(t, z(t), z(\mu t)), \\ z(0) = z_0, \\ 0 \le t \le T, 0 < \delta < 1, 0 < \mu < 1, \end{cases} \tag{1.2}$$

where ${}^{C}D^{\delta}$ is the Caputo fractional derivative. On the other hand, in [7], the authors discussed the existence and uniqueness for the fractional hybrid pantograph equation involving Riemann-Liouville fractional derivative:

$$\begin{cases} {}^{RL}D^{\delta}\left[\dfrac{z(t)}{\varphi(t, z(t), z(\eta t))}\right] = \psi(s, z(t), z(\mu t)), \\ z(0) = 0, \\ 0 \le t \le 1,\ 0 < \delta \le 1,\ 0 < \eta, \mu < 1, \end{cases} \tag{1.3}$$

where ${}^{RL}D^{\delta}$ denote the Riemann-Liouville fractional derivative, $\varphi \in C\left([0,1] \times \mathbb{R}^2, \mathbb{R}\backslash\{0\}\right)$ and $\psi \in C\left([0,1] \times \mathbb{R}^2, \mathbb{R}\right)$.

Motivated by the aforementioned papers, in this article, we study the existence, uniqueness and Mittag-Leffler-Ulam-stability of solutions for the following fractional hybrid pantograph equation with mixed Riemann-Liouville and Caputo fractional derivatives:

$$\begin{cases} {}^{RL}D^{\delta}\left[{}^{C}D^{\lambda}\left[\dfrac{z(t)}{\varphi(t, z(t), z(\eta t))}\right]\right] = \psi(s, z(t), z(\mu t)), \\ z(0) = 0, \quad z(1) = 0, \\ 0 \le t \le 1,\ 0 < \eta, \mu < 1,\ 0 < \delta, \lambda \le 1, \end{cases} \tag{1.4}$$

where ${}^{RL}D^{\delta}$ and ${}^{C}D^{\lambda}$ denote the Riemann-Liouville and Caputo fractional derivatives, $\varphi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}\backslash\{0\}$ and $\psi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions. The operator ${}^{RL}D^{\delta}$ is the fractional derivative in the sense of Riemann-Liouville [20, 23], defined by

$${}^{RL}D^{\delta}z(t) = \frac{1}{\Gamma(n-\delta)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\delta-1} z(s)\,ds, n = [\delta] + 1,$$

where $\Gamma(.)$ is the Euler gamma function given by

$$\Gamma(\delta) = \int_0^{\infty} e^{-z} z^{\delta-1} dz.$$

The operator ${}^{C}D^{\lambda}$ is the fractional derivative in the sense of Caputo [20, 23], defined by

$${}^{C}D^{\lambda}z(t) = \frac{1}{\Gamma(n-\lambda)} \int_0^t (t-s)^{n-\lambda-1} z^{(n)}(s)\,ds, n = [\lambda] + 1,$$

and the Riemann-Liouville fractional integral [20, 23] of order $\alpha > 0$, defined by

$$I^{\alpha} z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds, t > 0.$$

We recall the following lemmas [18, 23].

Lemma 1.1. *Let $\alpha > \varepsilon > 0$ and $\varphi \in L^1([a,b])$. Then $D^{\varepsilon} I^{\alpha} z(t) = I^{\alpha-\varepsilon} z(t)$, $t \in [a,b]$.*

Lemma 1.2. *For $\alpha > 0$ and $\varepsilon > -1$, we have*

$$I^{\alpha}\left[(t-w)^{\varepsilon}\right] = \frac{\Gamma(\varepsilon+1)}{\Gamma(\alpha+\varepsilon+1)} (t-w)^{\varepsilon+\alpha}.$$

Also we recall the following lemmas

Lemma 1.3. *[18] Let $\delta > 0$ and $z \in C(0,1) \cap L^1(0,1)$. Then the fractional differential equation ${}^{RL}D^{\delta} z(t) = 0$ has a unique solution*

$$z(t) = \sum_{i=1}^{n} c_i t^{\delta-i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, .., n$ and $n-1 < \delta < n$.

Lemma 1.4. *[18] Let $\delta > 0$. Then for $z \in C(0,1) \cap L^1(0,1)$ and ${}^{RL}D^{\delta} z \in C(0,1) \cap L^1(0,1)$, we have*

$$I^{\delta}\left[{}^{RL}D^{\delta} z(t)\right] = z(t) + \sum_{i=1}^{n} c_i t^{\delta-i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, $n = [\delta] + 1$.

Lemma 1.5. *[18] For $\lambda > 0$, the general solution of the fractional differential equation ${}^{C}D^{\lambda} z(t) = 0$ is given by*

$$z(t) = \sum_{i=0}^{n-1} c_i t^i, c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1 \text{ and } n-1 < \lambda < n.$$

Lemma 1.6. *[18] Let $\lambda > 0$. Then*

$$I^{\lambda}\left[{}^{C}D^{\lambda} z(t)\right] = z(t) + \sum_{i=0}^{n-1} c_i t^i, c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\lambda] + 1.$$

In what follow, we need an important singular type Gronwall inequality.

Theorem 1.7. *[21] For any $t \in [0,1)$. If*

$$y(t) \leq m(t) + \sum_{i=1}^{n} k_i(t) \int_0^t (t-s)^{\nu_i - 1} y(s)\, ds,$$

where all the functions are not negative and continuous. The costants $\nu_i > 0$, k_i $(i = 1, 2, ..., n)$ are the bounded and monotonic increasing functions on $[0,1)$, then

$$y(t) \leq m(t) + \sum_{j=1}^{\infty} \left(\sum_{1',2',...,j'=1}^{n} \frac{\prod_{i=1}^{j} \left[k_{i'}(t)\Gamma(\nu_{i'})\right]}{\Gamma\left(\sum_{i=1}^{j} \nu_{i'}\right)} \int_0^t (t-s)^{\sum_{i=1}^{j} \nu_{i'} - 1} m(s)\, ds \right).$$

Remark 1.8. *For $n = 2$, if $\nu_1, \nu_2 > 0, k_1, k_2 \geq 0, m(t)$ is nonnegative and locally integrable on $[0,1)$ and $y(t)$ is nonnegative and locally integrable on $[0,1)$ with*

$$y(t) \leq m(t) + k_1 \int_0^t (t-s)^{\nu_1 - 1} y(s)\, ds + k_2 \int_0^t (t-s)^{\nu_2 - 1} y(s)\, ds,$$

then

$$\begin{aligned} y(t) \quad &\leq \quad m(t) + \sum_{j=1}^{\infty} \left(\frac{(k_1 \Gamma(\nu_1))^j}{\Gamma(j\nu_1)} \int_0^t (t-s)^{j\varepsilon_1 - 1} m(s)\, ds \right. \\ &\qquad \left. + \frac{(k_2 \Gamma(\nu_2))^j}{\Gamma(j\nu_2)} \int_0^t (t-s)^{j\varepsilon_2 - 1} m(s)\, ds \right). \end{aligned}$$

Remark 1.9. *Under the conditions of Remark 1.8, let $m(t)$ is a nondecreasing function on $[0,1)$. Then we have*

$$y(t) \leq m(t) \left(E_{\nu_1} [k_1 \Gamma(\nu_1) t^{\nu_1}] + E_{\nu_2} [k_2 \Gamma(\nu_2) t^{\nu_2}] \right),$$

where E_ν is the Mittag-Leffler function [1] defined by: $E_\nu [w] = \sum_{j=1}^{\infty} \frac{w^\nu}{\Gamma(j\nu + 1)}, w \in \mathbb{C}$.

The following proposition includes important findings that form the cornerstone of deriving the main results in the current article.

Proposition 1.10. *Let $0 < \delta, \lambda < 1$. If $\varphi \in C\left([0,1] \times \mathbb{R}^2, \mathbb{R} - \{0\}\right)$ and $h \in C([0,1], \mathbb{R})$, then, the solution of the problem*

$$\begin{cases} {}^{RL}D^{\delta} \left[{}^{C}D^{\lambda} z(t) \left[\dfrac{z(t)}{\varphi(t, z(t), z(\eta t))} \right] \right] = h(t), \\ \qquad z(0) = 0, \quad z(1) = 0, \\ 0 \leq t \leq 1,\ 0 < \eta, \mu < 1,\ 0 < \delta, \lambda < 1, \end{cases} \tag{1.5}$$

is given by

$$\begin{aligned} z(t) \quad &= \quad \frac{\varphi(t, z(t), z(\eta t))}{\Gamma(\delta + \lambda)} \int_0^t (t-s)^{\delta + \lambda - 1} h(s)\, ds \\ &\qquad - t^{\delta + \lambda - 1} \frac{\varphi(t, z(t), z(\eta t))}{\Gamma(\delta + \lambda)} \int_0^1 (t-s)^{\delta + \lambda - 1} h(s)\, ds. \end{aligned} \tag{1.6}$$

Proof. Let z be a solution of the problem (1.5). Then, we have

$${}^{RL}D^{\delta} \left[{}^{C}D^{\lambda} z(t) \left[\frac{z(t)}{\varphi(t, z(t), z(\eta t))} \right] \right] = h(t).$$

Now, applying the operator I^δ to both sides of the above equation and by Lemma 1.4, we can write

$${}^{C}D^{\lambda} z(t) \left[\frac{z(t)}{\varphi(t, z(t), z(\eta t))} \right] = I^{\delta} [h(t)] + a t^{\delta - 1}, \tag{1.7}$$

where $a \in \mathbb{R}$. Taking the Riemann-Liouville fractional $q-$integral of order λ to both sides of (1.7) and using Lemma 1.6, we get

$$z(t) = \varphi(t, z(t), z(\eta t)) \left[I^{\delta + \lambda} [h(t)] + \frac{\Gamma(\delta) a}{\Gamma(\delta + \lambda)} t^{\delta + \lambda - 1} + b \right], \tag{1.8}$$

where $b \in \mathbb{R}$. Using the conditions $z(0) = 0$ and $z(1) = 0$, we find that

$$a = -\frac{\Gamma(\delta + \lambda)}{\Gamma(\delta)} I^{\delta+\lambda} [h(1)] \text{ and } b = 0.$$

Inserting the values of a and b in (1.8) yields the solution (1.6). □

2. Existence results

Let $Z = C([0,1], \mathbb{R})$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|z\| = \sup\{|z(t)| : t \in [0,1]\},$$

and the multiplication is defined by $(zw)(t) = z(t) w(t)$ for all $w, z \in Z$.
By $L^1([0,1], \mathbb{R})$ we denote the space of Lebesgue-integrable functions $z : [0,1] \to \mathbb{R}$ with the norm

$$\|z\|_{L^1} = \int_0^1 |z(t)| \, dt.$$

By Proposition 1.10, we define an operator $Q : Z \to Z$ by

$$\begin{aligned} Qz(t) &= \frac{\varphi(t, z(t), z(\eta t))}{\Gamma(\delta + \lambda)} \int_0^t (t-s)^{\delta+\lambda-1} \psi(s, z(s), z(\mu s)) \, ds \\ &\quad - t^{\delta+\lambda-1} \frac{\varphi(t, z(t), z(\eta t))}{\Gamma(\delta + \lambda)} \int_0^1 (t-s)^{\delta+\lambda-1} \psi(s, z(s), z(\mu s)) \, ds. \end{aligned} \tag{2.1}$$

Obviously, the fixed points of operator Q are solutions of the fractional hybrid pantograph problem (1.4).

Before starting and proving the main results, we introduce the following conditions.
(C_1) $\varphi : [0,1] \times \mathbb{R}^2 \to \mathbb{R} - \{0\}$ is continuous function and there exists a constant $\omega > 0$, such that for all $t \in [0,1]$ and $z_j, w_j \in \mathbb{R}$, $j = 1, 2$,

$$\left|\varphi(t, z_1, z_2) - \varphi(t, w_1, w_2)\right| \le \omega(|z_1 - w_1| + |z_2 - w_2|).$$

(C_2) $\psi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function and there exists a constant $\vartheta > 0$, such that for all $t \in [0,1]$ and $z_j, w_j \in \mathbb{R}$, $j = 1, 2$,

$$\left|\psi(t, z_1, z_2) - \psi(t, w_1, w_2)\right| \le \vartheta(|z_1 - w_1| + |z_2 - w_2|),$$

(C_3) There exists $A, B \in \mathbb{R}_+^*$, where $\left|\varphi(t, z_1, z_2)\right| \le A$ and $\left|\psi(t, z_1, z_2)\right| \le B$, for all $t \in [0,1]$ and $z_j \in \mathbb{R}$, $j = 1, 2$.
(C_4) There exists a function $\phi \in L^1([0,1], \mathbb{R})$, such that for all $t \in [0,1]$ and $z_j \in \mathbb{R}$, $j = 1, 2$,

$$\left|\psi(t, z_1, z_2)\right| \le \phi(t).$$

The following uniqueness result is based on Banach's fixed point theorem.

Theorem 2.1. *Assume that conditions (C_i), $i = 1, ..., 3$ hold. If the inequality*

$$4(A\omega + B\vartheta) < \Gamma(\delta + \lambda + 1), \tag{2.2}$$

is valid, then the fractional hybrid pantograph problem (1.4) has a unique solution on $[0,1]$.

Proof. We show that the operator Q is a contraction. Let $z, w \in Z$, then we have

$$\begin{aligned} &\left|Qz(t)-Qw(t)\right| \\ \leq\ & \frac{\left|\varphi(t,z(t),z(\eta t))\right|}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}\left|\psi(s,z(s),z(\mu s))-\psi(s,w(s),w(\mu s))\right|ds \\ &+t^{\delta+\lambda-1}\frac{\left|\varphi(t,z(t),z(\eta t))\right|}{\Gamma(\delta+\lambda)}\int_0^1 (t-s)^{\delta+\lambda-1}\left|\psi(s,z(s),z(\mu s))-\psi(s,w(s),w(\mu s))\right|ds \\ &+\frac{\left|\varphi(t,z(t),z(\eta t))-\varphi(t,w(t),w(\eta t))\right|}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}\left|\psi(s,z(s),z(\mu s))\right|ds \\ &+t^{\delta+\lambda-1}\frac{\left|\varphi(t,z(t),z(\eta t))-\varphi(t,w(t),w(\eta t))\right|}{\Gamma(\delta+\lambda)}\int_0^1 (t-s)^{\delta+\lambda-1}\left|\psi(s,z(s),z(\mu s))\right|ds. \end{aligned}$$

So, thanks to $(C_i), i = 1, ..., 3$, we get

$$\|Q(z)-Q(w)\| \leq \frac{4(A\omega+B\vartheta)}{\Gamma(\delta+\lambda+1)}\|z-w\|.$$

In view of condition (2.2), we infer that Q is a contraction operator. The proof is complete.

In the next result, we show the existence of solutions for the hybrid fractional pantograph problem (1.4) by applying the following theorem. □

Theorem 2.2. *[8] Suppose that W is a non-empty subset of Z, which closed convex and bounded, $Q_1 : Z \to Z$, and $Q_2 : W \to Z$ are two operators satisfying the following conditions:*

1. Q_1 is Lipschitizian with a constant ω,
2. Q_2 is completely continuous,
3. $z = Q_1 z Q_2 w \Rightarrow z \in W$ for all $w \in W$, and
4. $wB < 1$, where $B = \|Q_2(W)\| = \sup\{\|Q_2(z)\| : z \in W\}$.

Then the operator equation $z = Q_1 z Q_2 z$ has a solution.

Theorem 2.3. *Suppose that conditions (C_1) and (C_4) are valid. Further, if*

$$4\omega\left\|\phi\right\|_{L^1} < \Gamma(\delta+\lambda+1), \tag{2.3}$$

then the fractional hybrid pantograph problem (1.4) has at least one solution on $[0,1]$.

Proof. We consider a subset W of Z given by

$$W = \{z \in Z : \|z\| \leq \sigma\},$$

where $\sigma = \frac{2\Lambda_\varphi\|\phi\|_{L^1}}{\Gamma(\delta+\lambda+1)-4\omega\|\phi\|_{L^1}}$ and $\Lambda_\varphi = \sup_{t\in[0,1]}\left|\varphi(t,0,0)\right|$.

Then in order to transform problem (1.4) into the operator equation $z = Q_1 z Q_2 z$, we need to define Q_1 and Q_2 as

$$Q_1 z(t) = \varphi(t, z(t), z(\eta t)), \tag{2.4}$$

and

$$\begin{aligned} Q_2 z(t) &= \frac{1}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}\psi(s,z(s),z(\mu s))\,ds \\ &\quad -\frac{t^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)}\int_0^1 (t-s)^{\delta+\lambda-1}\psi(s,z(s),z(\mu s))\,ds. \end{aligned} \tag{2.5}$$

We shall show that the operators $Q_i, i = 1, 2$ satisfy all the conditions of Theorem 2.2.

We start by showing that Q_1 is a ω Lipschitzian operator on Z. Let $w, z \in Z$, then by (C_1), we have

$$\begin{aligned} |Q_1 z(t) - Q_1 w(t)| &= \left|\varphi(t, z(t), z(\eta t)) - \varphi(t, w(t), w(\eta t))\right| \\ &\leq \omega\left(|z(t) - w(t)| + \left|z(\eta t) - w(\eta t)\right|\right) \\ &\leq 2\omega |z(t) - w(t)| \leq 2\omega \|z - w\|, \end{aligned}$$

Then

$$\|Q_1(z) - Q_1(w)\| \leq 2\omega \|z - w\|, \text{ for all } w, z \in Z.$$

Next, Q_2 is completely continuous on W. We begin by ensuring the continuity of Q_2 on W. Let $\{z_n\}$ be a sequence in W converging to a point $z \in W$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} Q_2 z_n(t) &= \lim_{n\to\infty}\left(\frac{1}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}\psi(s, z_n(s), z_n(\mu s))\,ds \right.\\ &\quad \left. -\frac{t^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)}\int_0^1 (t-s)^{\delta+\lambda-1}\psi(s, z_n(s), z_n(\mu s))\,ds\right) \\ &= \frac{1}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}\lim_{n\to\infty}\psi(s, z_n(s), z_n(\mu s))\,ds \\ &\quad -\frac{t^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)}\int_0^1 (t-s)^{\delta+\lambda-1}\lim_{n\to\infty}\psi(s, z_n(s), z_n(\mu s))\,ds \\ &= \frac{1}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}\psi(s, z(s), z(\mu s))\,ds \\ &\quad -\frac{t^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)}\int_0^1 (t-s)^{\delta+\lambda-1}\psi(s, z(s), z(\mu s))\,ds \\ &= Q_2 z(t). \end{aligned}$$

Moreover, we show that $Q_2(W)$ is a uniformly bounded and equicontinuous set in Z. First we prove the uniform boundedness of the set $Q_2(W)$ in Z.

For $z \in W$ and $t \in [0, 1]$, using (C_4), we can write

$$\begin{aligned} |Q_2 z(t)| &\leq \frac{1}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}\left|\psi(s, z(s), z(\mu s))\right| ds \\ &\quad +\frac{t^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)}\int_0^1 (t-s)^{\delta+\lambda-1}\left|\psi(s, z(s), z(\mu s))\right| ds \\ &\leq \frac{2\left\|\phi\right\|_{L^1}}{\Gamma(\delta+\lambda+1)}, \end{aligned}$$

which implies that

$$\|Q_2(z)\| \leq \frac{2\left\|\phi\right\|_{L^1}}{\Gamma(\delta+\lambda+1)}.$$

On the other hand, we demonstrate that Q_2 is an equicontinuous set in W. Let $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$ and

$z \in W$. Then we have

$$\begin{aligned}
& \left|Q_2 z(t_1) - Q_2 z(t_2)\right| \\
\leq & \left|\frac{1}{\Gamma(\delta+\lambda)}\int_0^{t_1}\left[(t_2-s)^{\delta+\lambda-1}-(t_1-s)^{\delta+\lambda-1}\right]\psi(s,z(s),z(\mu s))\,ds\right. \\
& \left.+\frac{1}{\Gamma(\delta+\lambda)}\int_0^{t_2}(t_2-s)^{\delta+\lambda-1}\psi(s,z(s),z(\mu s))\,ds\right| \\
& +\frac{\left|t_1^{\delta+\lambda-1}-t_2^{\delta+\lambda-1}\right|}{\Gamma(\delta+\lambda)}\int_0^1(t-s)^{\delta+\lambda-1}\left|\psi(s,z(s),z(\mu s))\right|ds.
\end{aligned}$$

Thanks to condition (C_4), we get

$$\begin{aligned}
\left|Q_2 z(t_1) - Q_2 z(t_2)\right| \leq & \frac{\left\|\phi\right\|_{L^1}}{\Gamma(\delta+\lambda+1)}\left[(t_2-t_1)^{\delta+\lambda}+t_2^{\delta+\lambda}-t_1^{\delta+\lambda}\right] \\
& +\frac{\left\|\phi\right\|_{L^1}}{\Gamma(\delta+\lambda+1)}\left|t_1^{\delta+\lambda-1}-t_2^{\delta+\lambda-1}\right|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $z \in W$ as $t_2 - t_1 \to 0$. Therefore, it follows from the Arzela-Ascoli theorem that Q_2 is a completely continuous operator on W.

Next, we show that hypothesis (3) of Theorem 2.2 is satisfied. For $z \in Z$ and $w \in W$, where $z = Q_1 z Q_2 w$, we get

$$\begin{aligned}
|z(t)| = & |Q_1 z(t)|\,|Q_2 w(t)| \\
\leq & \left|\varphi(t,z(t),z(\eta t))\right|\left[\frac{1}{\Gamma(\delta+\lambda)}\int_0^t(t-s)^{\delta+\lambda-1}\left|\psi(s,w(s),w(\mu s))\right|ds\right. \\
& \left.+\frac{t^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)}\int_0^1(t-s)^{\delta+\lambda-1}\left|\psi(s,w(s),w(\mu s))\right|ds\right] \\
\leq & \left|\varphi(t,z(t),z(\eta t))-\varphi(t,0,0)\right|+\left|\varphi(t,0,0)\right| \\
& \times\left[\frac{1}{\Gamma(\delta+\lambda)}\int_0^t(t-s)^{\delta+\lambda-1}\left|\psi(s,w(s),w(\mu s))\right|ds\right. \\
& \left.+\frac{t^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)}\int_0^1(t-s)^{\delta+\lambda-1}\left|\psi(s,w(s),w(\mu s))\right|ds\right] \\
\leq & \left(2\omega\,|z(t)|+\Lambda_\varphi\right)\frac{2\left\|\phi\right\|_{L^1}}{\Gamma(\delta+\lambda+1)}.
\end{aligned}$$

which implies that

$$\|z\| \leq \frac{2\Lambda_\varphi\left\|\phi\right\|_{L^1}}{\Gamma(\delta+\lambda+1)-4\omega\left\|\phi\right\|_{L^1}} = \sigma.$$

This shows that condition (3) of Theorem 2.2 is satisfied. Finally, we have

$$N = \|Q_2(W)\| = \sup\{Q_2(z)\, z \in W\} \leq \frac{2\left\|\phi\right\|_{L^1}}{\Gamma(\delta+\lambda+1)}$$

From above estimate, we obtain

$$\omega N = \omega\frac{2\left\|\phi\right\|_{L^1}}{\Gamma(\delta+\lambda+1)} < 1.$$

Thus, all the conditions of Theorem 2.2 are satisfied and so, the operator equation $z = Q_1 z Q_2 z$ has a solution in W. In consequence, problem (1.4) has a solution on $[0,1]$. □

3. Mittag-Leffler-Ulam-stability results

In this section, we consider the Mittag-Leffler-Ulam-Hyers stability and Mittag-Leffler-Ulam-Hyers-Rassias stability for the hybrid fractional pantograph problem (1.4). For $t \in [0,1]$, we give the following inequalities:

$$\left| {}^{RL}D^{\delta}\left[{}^{C}D^{\lambda} z(t)\left[\frac{z(t)}{\varphi(t,z(t),z(\eta t))}\right]\right] - \psi(s,z(s),z(\mu s))\,ds\right| \leq \beta, \tag{3.1}$$

and

$$\left| {}^{RL}D^{\delta}\left[{}^{C}D^{\lambda} z(t)\left[\frac{z(t)}{\varphi(t,z(t),z(\eta t))}\right]\right] - \psi(s,z(s),z(\mu s))\,ds\right| \leq \beta\theta(t), \tag{3.2}$$

where β is positive real number and $\theta : [0,1] \to \mathbb{R}^{+}$ is continuous function.

Definition 3.1. *Problem (1.4) is Mittag-Leffler-Ulam-Hyers stable, with respect to $E_{\delta+\lambda}$ if there exists a real number ϖ such that for each $\beta > 0$ and for each solution $w \in Z$ of the inequality (3.1), there exists a solution $z \in Z$ of the problem (1.4) with*

$$|w(t) - z(t)| \leq \varpi\beta E_{\delta+\lambda}[t],\ t \in [0,1].$$

Definition 3.2. *Problem (1.4) is Mittag-Leffler-Ulam-Hyers-Rassias stable, with respect to $\theta E_{\delta+\lambda}$ if there exists a real number $\gamma_{\theta} > 0$ such that for each $\beta > 0$ and for each solution $w \in Z$ of the inequality (3.2), there exists a solution $z \in Z$ of problem (1.4) with*

$$|w(t) - z(t)| \leq \gamma_{\theta}\beta\theta(t) E_{\delta+\lambda}[t],\ t \in [0,1].$$

Remark 3.3. *A function $w \in Z$ is a solution of the inequality (3.1) if and only if there exists a function $f \in C([0,1],\mathbb{R})$ (which depend on w) such that*

$$|f(t)| \leq \beta, \quad t \in [0,1],$$

and

$${}^{RL}D^{\delta}\left[{}^{C}D^{\lambda} z(t)\left[\frac{z(t)}{\varphi(t,z(t),z(\eta t))}\right]\right] = \psi(s,z(s),z(\mu s))\,ds + f(t), t \in [0,1].$$

Theorem 3.4. *If conditions $(C_i), i = 1, ..., 3$ are satisfied, then the problem (1.4) is Mittag-Leffler-Ulam-Hyers stable*

Proof. Let $w \in Z$ be a solution of the inequality (3.1) and let us denote by $z \in Z$ the unique solution of the problem

$$\begin{cases} {}^{RL}D^{\delta}\left[{}^{C}D^{\lambda} z(t)\left[\frac{z(t)}{\varphi(t,z(t),z(\eta t))}\right]\right] = \psi(s,z(s),z(\mu s))\,ds, \\ \qquad z(0) = w(0), \quad z(1) = w(1), \\ \qquad 0 \leq t \leq 1,\ 0 < \eta, \mu < 1,\ 0 < \delta, \lambda \leq 1, \end{cases}$$

Applying Proposition 1.10, we can write

$$\frac{z(t)}{\varphi(t,z(t),z(\eta t))} = I^{\delta+\lambda}[h_z(t)] + a\frac{\Gamma(\delta)}{\Gamma(\delta+\lambda)}t^{\delta+\lambda-1} + b.$$

By integration of the inequality (3.1), we obtain

$$\begin{aligned}
&\left|\frac{w(t)}{\varphi(t,w(t),w(\eta t))}-I^{\delta+\lambda}[h_w(t)]-c\frac{\Gamma(\delta)}{\Gamma(\delta+\lambda)}t^{\delta+\lambda-1}-d\right| \\
\leq\ & \frac{\beta}{\Gamma(\delta+\lambda+1)}t^{\delta+\lambda}\leq\frac{\beta}{\Gamma(\delta+\lambda+1)}. \qquad (3.3)
\end{aligned}$$

From these relations, we obtain that

$$\begin{aligned}
&|w(t)-z(t)| \\
\leq\ & A\left|\frac{w(t)}{\varphi(t,w(t),w(\eta t))}-\frac{z(t)}{\varphi(t,z(t),z(\eta t))}\right| \\
\leq\ & A\left|\frac{w(t)}{\varphi(t,w(t),w(\eta t))}-I^{\delta+\lambda}[h_w(t)]-c\frac{\Gamma(\delta)}{\Gamma(\delta+\lambda)}t^{\delta+\lambda-1}-d\right. \\
& \left.+I^{\delta+\lambda}[h_w(t)-h_z(t)]\right| \\
\leq\ & A\left|\frac{w(t)}{\varphi(t,w(t),w(\eta t))}-I^{\delta+\lambda}[h_w(t)]-c\frac{\Gamma(\delta)}{\Gamma(\delta+\lambda)}t^{\delta+\lambda-1}-d\right| \\
& +A\left|I^{\delta+\lambda}[h_w(t)-h_z(t)]\right|,
\end{aligned}$$

where

$$h_z(t)=\psi(s,z(s),z(\mu s)) \text{ and } h_w(t)=\psi(s,w(s),w(\mu s)),$$

and the constant A is defined in (C_3).

Thanks to (C_2) and (3.3), we get

$$\begin{aligned}
&|w(t)-z(t)| \\
\leq\ & \frac{A\beta}{\Gamma(\delta+\lambda+1)}+\frac{2A\vartheta}{\Gamma(\delta+\lambda)}\int_0^t(t-s)^{\delta+\lambda-1}|w(t)-z(t)|\,ds.
\end{aligned}$$

Now, applying Remark 1.8 and Remark 1.9, we obtain

$$|w(t)-z(t)|\leq\frac{A\beta}{\Gamma(\delta+\lambda+1)}\left(E_{\delta+\lambda}\left[2A\vartheta t^{\delta+\lambda}\right]\right),\ t\in[0,1].$$

So, the problem (1.4) is Mittag-Leffler-Ulam-Hyers stable. □

Theorem 3.5. *If conditions (C_i), $i=1,...,3$ are satisfied. Suppose there exists a function $\theta\in C([0,1],\mathbb{R}_+)$ is increasing and there exists $\gamma_\theta>0$ such that for any $t\in[0,1]$*

$$\frac{1}{\Gamma(\delta+\lambda)}\int_0^t(t-s)^{\delta+\lambda-1}\theta(s)ds\leq\gamma_\theta\theta(t). \qquad (3.4)$$

Then the problem (1.4) is Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to $\theta E_{\delta+\lambda}$.

Proof. Let $w\in Z$ be a solution of the inequality (3.2). Using Remark 3.3, we can write

$$\begin{aligned}
&\left|\frac{w(t)}{\varphi(t,w(t),w(\eta t))}-I^{\delta+\lambda}[h_w(t)]-c\frac{\Gamma(\delta)}{\Gamma(\delta+\lambda)}t^{\delta+\lambda-1}-d\right| \\
\leq\ & \frac{\beta}{\Gamma(\delta+\lambda)}\int_0^t(t-s)^{\delta+\lambda-1}\theta(s)ds.
\end{aligned}$$

Let us denote by $z \in Z$ the unique solution of the problem

$$\begin{cases} {}^{RL}D^{\delta}\left[{}^{C}D^{\lambda}z(t)\left[\dfrac{z(t)}{\varphi(t,z(t),z(\eta t))}\right]\right] = \psi(s,z(s),z(\mu s))\,ds, \\ z(0) = w(0), \quad z(1) = w(1), \\ 0 \leq t \leq 1,\ 0 < \eta,\mu < 1,\ 0 < \delta,\lambda \leq 1. \end{cases}$$

We have

$$\frac{z(t)}{\varphi(t,z(t),z(\eta t))} = I^{\delta+\lambda}[h_z(t)] + a\frac{\Gamma(\delta)}{\Gamma(\delta+\lambda)}t^{\delta+\lambda-1} + b.$$

By (3.4) and (C_2), we have

$$\begin{aligned} & |w(t) - z(t)| \\ \leq\ & A\left|\frac{w(t)}{\varphi(t,w(t),w(\eta t))} - \frac{z(t)}{\varphi(t,z(t),z(\eta t))}\right| \\ \leq\ & A\left|\frac{w(t)}{\varphi(t,w(t),w(\eta t))} - I^{\delta+\lambda}[h_w(t)] - c\frac{\Gamma(\delta)}{\Gamma(\delta+\lambda)}t^{\delta+\lambda-1} - d\right. \\ & \left.+I^{\delta+\lambda}[h_w(t) - h_z(t)]\right| \\ \leq\ & \frac{A\beta}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}\theta(s)ds \\ & +\frac{2A\vartheta}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}|w(t) - z(t)|\,ds \\ \leq\ & A\beta\gamma_\theta\theta(t) + \frac{2A\vartheta}{\Gamma(\delta+\lambda)}\int_0^t (t-s)^{\delta+\lambda-1}|w(t) - z(t)|\,ds. \end{aligned}$$

So, using Remark 1.8 and Remark 1.9, we get

$$|w(t) - z(t)| \leq A\beta\gamma_\theta\theta(t)\left(E_{\delta+\lambda}\left[2A\vartheta t^{\delta+\lambda}\right]\right),\ t \in [0,1].$$

Then, the problem (1.4) is Mittag-Leffler-Ulam-Hyers-Rassias stable. □

4. Example

Let us consider the following problem

$$\begin{cases} {}^{RL}D^{\frac{4}{5}}\left[{}^{C}D^{\frac{6}{7}}\left[\dfrac{z(t)}{\frac{1}{13}\tan^{-1}(t)z(t) + \frac{1}{13}\cos z\left(\frac{2}{3}t\right) + \frac{2}{9}}\right]\right] \\ = \dfrac{t}{11}\cos z(t) + \dfrac{t}{11}\sin z\left(\frac{3}{5}t\right) + \frac{1}{11}t,\ t \in [0,1], \\ z(0) = 0, \quad z(1) = 0, \end{cases} \tag{4.1}$$

and the following inequalities

$$\left|{}^{RL}D^{\frac{4}{5}}\left[{}^{C}D^{\frac{6}{7}}z(t)\left[\frac{z(t)}{\varphi\left(t,z(t),z\left(\frac{2}{3}t\right)\right)}\right]\right] - \psi\left(s,z(s),z\left(\frac{3}{5}s\right)\right)ds\right| \leq \beta,$$

and

$$\left|{}^{RL}D^{\frac{4}{5}}\left[{}^{C}D^{\frac{6}{7}}z\left(t\right)\left[\frac{z\left(t\right)}{\varphi\left(t,z\left(t\right),z\left(\frac{2}{3}t\right)\right)}\right]\right]-\psi\left(s,z\left(s\right),z\left(\frac{3}{5}s\right)\right)ds\right|\leq\beta\theta\left(t\right),$$

where

$$\begin{aligned}\varphi\left(t,z\left(t\right),z\left(\frac{2}{3}t\right)\right) &= \frac{1}{13}\tan^{-1}z\left(t\right)+\frac{1}{13}\cos z\left(\frac{2}{3}t\right)+\frac{2}{3},\\ \psi\left(t,z\left(t\right),z\left(\frac{3}{5}t\right)\right) &= \frac{t}{11}\cos z\left(t\right)+\frac{t}{11}\sin z\left(\frac{3}{5}t\right)+\frac{1}{11}t,\end{aligned}$$

For all $t\in[0,1]$ and $z_j,w_j\in\mathbb{R}$, $j=1,2$, we have

$$\begin{aligned}\left|\varphi\left(t,z_1\left(t\right),z_2\left(\lambda t\right)\right)-\varphi\left(t,w_1\left(t\right),w_2\left(\lambda t\right)\right)\right| &\leq \frac{1}{13}\left(\left|z_1-w_1\right|+\left|z_2-w_2\right|\right),\\ \left|\psi\left(t,z_1\left(t\right),z_2\left(\lambda t\right)\right)-\psi\left(t,w_1\left(t\right),w_2\left(\lambda t\right)\right)\right| &\leq \frac{1}{11}\left(\left|z_1-w_1\right|+\left|z_2-w_2\right|\right),\end{aligned}$$

and

$$\left|\varphi\left(t,z_1\left(t\right),z_2\left(\lambda t\right)\right)\right|\leq\frac{32}{39}=A,\quad\psi\left(t,z_1\left(t\right),z_2\left(\lambda t\right)\right)\leq\frac{3}{11}=B.$$

Hence conditions (C_1) and (C_2) hold with $\omega=\frac{1}{13}$ and $\vartheta=\frac{1}{11}$ respectively.

Thus condition (2.2) reads

$$4\left(\omega A+\vartheta B\right)=0.351\,64<\Gamma\left(\delta+\lambda+1\right)=1.4934,$$

It follows from Theorem 2.1, that the problem problem (4.1) has a unique solution on $[0,1]$, and is Mittag-Leffler-Ulam-Hyers stable with

$$\left|w\left(t\right)-z\left(t\right)\right|\leq1.2254\beta\left(E_{\frac{58}{35}}\left[0.14918t^{\frac{58}{35}}\right]\right),\ t\in\left[0,1\right].$$

Let $\theta\left(t\right)=t^{\frac{1}{2}}$. Then

$$I^{\frac{4}{5}+\frac{6}{7}}\left[\theta(t)\right]=I^{\frac{4}{5}+\frac{6}{7}}\left[t^{\frac{1}{2}}\right]=\frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{117}{70}\right)}t^{\frac{1}{2}+\frac{4}{5}+\frac{6}{7}}\leq\frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{117}{70}\right)}t^{\frac{1}{2}}=\gamma_\theta\theta(t).$$

Thus condition (3.4) is satisfied with $\theta\left(t\right)=t^{\frac{1}{2}}$ and $\gamma_\theta=\frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{117}{70}\right)}$. It follows from Theorem 3.5 problem (4.1) is Mittag-Leffler-Ulam-Hyers stable with

$$\left|w\left(t\right)-z\left(t\right)\right|\leq\frac{16\sqrt{\pi}}{39\Gamma\left(\frac{117}{70}\right)}\beta t^{\frac{1}{2}}\left(E_{\frac{58}{35}}\left[0.14918t^{\frac{58}{35}}\right]\right),\ t\in\left[0,1\right].$$

References

[1] M. I. Abbas, Existence and uniqueness of Mittag-Leffler-Ulam stable solution for fractional integrodifferential equations with nonlocal initial conditions, Eur. J. Pure Appl. Math. 8(4), (2015), 478-498.

[2] M. I. Abbas, Existence results and the Ulam stability for fractional differential equations with hybrid proportional-Caputo derivatives, J. Nonlinear Funct. Anal. 2020: Article ID 48 (2020), 1-14.

[3] A. O. Akdemir, A. Karaoblan, M. A. Ragusa, E. Set, Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions, Journal of Function Spaces.2021: Article ID 1055434, 1055434, (2021), 1-10.

[4] G. Ali, K. Shah and G. Rahman, Investigating a class of pantograph differential equations under multi-points boundary conditions with fractional order, Int. J. Appl. Comput. Math. 7(2) (2021), 1-13.

[5] A. Ali, I. Mahariq, K. Shah, T. Abdeljawad and B. Al-Sheikh, Stability analysis of initial value problem of pantograph-type implicit fractional differential equations with impulsive conditions, Adv. Difference Equ. 2021, 55 (2021), 1-17.
[6] K. Balachandran, S. Kiruthika and J. J. Trujillo, Existence of solutions of nonlinear fractional pantograph equations, Acta Mathematica Scientia. 33B (2013), 1-9.
[7] M. A. Darwish and K. Sadarangani, Existence of solutions for hybrid fractional pantograph equations, Appl. Anal. Discrete Math. 9 (2015), 150-167.
[8] B. C. Dhage, On a fixed point theorem in Banach algebras with applications. Appl Math Lett. 18 (2005), 273-280.
[9] G. Derfel, A. Iserles, The pantograph equation in the complex plane, J Math Anal Appl. 213(1), (1997), 117-132.
[10] M. A. E. Herzallah and D. Baleanu, On fractional order hybrid differential equations, Abstr. Appl. Anal. 2014 (2014), 1-8.
[11] M. Houas, Existence and stability results for hybrid fractional $q-$differential pantograph equations, Asia Mathematika. 5(2) (2021), 20-35.
[12] A. Iserles, Exact and discretized stability of the pantograph equation, Appl. Numer. Math. 24 (1997), 295-308.
[13] M. Jamil, R.A. Khan and K. Shah, Existence theory to a class of boundary value problems of hybrid fractional sequential integro-differential equations, Bound. Value Probl. 2019: 77 (2019), 1-12.
[14] R. A. Khan, S. Gul, F. Jarad and H. Khan, Existence results for a general class of sequential hybrid fractional differential equations, Adv.Difference. Equ. 2021: 284 (2021), 1-14.
[15] M. B. A. Khan, T. Abdeljawad, K. Shah, G. Ali, H. Khan and A. Khan, Study of a nonlinear multi-terms boundary value problem of fractional pantograph differential equations, Adv. Difference Equ. 2021: 143 (2021), 1-15.
[16] B. Khaminsou, C. Thaiprayoon, W. Sudsutad and S. A. Jose, Qualitative analysis of a proportional Caputo fractional pantograph differential equation with mixed nonlocal conditions, Nonlinear Functional Analysis and Applications. 26(1) (2021), 197-223.
[17] V. V. Kharat, S. Tate, A. R. Reshimkar, Some existence results on implicit fractional differential equations, Filomat, 35 (12) (2021), 4257-4265.
[18] A.A.Kilbas, H.M. Srivastava, J.J.Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies., 204. Elsevier Science B.V. Amsterdam. (2006).
[19] A.Lachouri, A. Ardjouni and A. Djoudi , Existence and Ulam stability results for nonlinear hybrid implicit Caputo fractional differential equations, Mathematica Moravica. 24(1) (2020), 109-122.
[20] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, Nonlinear. Anal. 69 (2008), 2677-2682.
[21] S.Y. Lin. Generalized Gronwall inequalities and their applications to fractional differential equations, Journal of Inequalities and Applications 2013, 549 (2013), 1-9.
[22] H. Mohammadi, S. Rezapour, S. Etemad and D. Baleanu, Two sequential fractional hybrid differential inclusions, Adv. Difference. Equ. 2020: 385 (2020), 1-24.
[23] L. Podlubny, Fractional differential equations, Academic Press, New York. 1999.
[24] A.Salem, M. Alnegga, Measure of noncompactness for hybrid Langevin fractional differential equations, Axioms. 9(2): 59 2020, 1-12.
[25] S. Sitho, S.K. Ntouyas and J. Tariboon, Existence results for hybrid fractional integro-differential equations. Bound. Value Probl. 2015: 113 (2015), 1-13.
[26] X. Su, S. Zhang and L. Zhang, Periodic boundary value problem involving sequential fractional derivatives in Banach space, AIMS Mathematics, 5(6) (2020), 7510-7530.
[27] J. Wang, L. Lv and Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, Electron. J. Qual. Theory Differ. Equ. 2011 (63) (2011), 1-10.
[28] Z. Wei, W. Dong, Periodic boundary value problems for Riemann-Liouville sequential fractional differential equations, Electron. J. Qual. Theory Differ. Equ. 87 (2011), 1-13.
[29] J. Wang, Y. Zhang, Ulam-Hyers-Mittag-Leffer stability of fractional-order delay differential equations, Optimization. 63(8) (2014), 1181-1190.
[30] A. Wongcharoen, S.K. Ntouyas and J. Tariboon, Nonlocal boundary value problems for Hilfer-type pantograph fractional differential equations and inclusions, Adv. Difference. Equ. 2020: 279 (2020), 1-21.
[31] I. Yaslan, M. Gunendi, Positive solutions for higher-order multi-point fractional boundary value problems, Miskolc Mathematical Notes, 22 (2) (2021), 1013-1026.
[32] Z.H Yu, Variational iteration method for solving the multi-pantograph delay equation, Phys Lett A. 372 (2008), 6475-6479.
[33] Y. Zhao, S. Sun, Z. Han, Z and Q. Li, Theory of fractional hybrid differential equations, Comput. Math. Appl. 62(3), (2011), 1312-1324.
[34] H. Zhang, Y. Li and J. Yang, New sequential fractional differential equations with mixed-type boundary conditions, Journal of Function Spaces. 2020: Article ID 6821637 (2020), 1-9.