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Existence results for some elliptic systems with perturbed gradient

Said Ait Temghart[a], Hasnae El Hammar[a], Chakir Allalou[a], Khalid Hilal[a]

[a]*LMACS, FST of Beni Mellal, Sultan Moulay Slimane University, Morocco*

Abstract. In this paper, we consider the existence of weak solutions for some nonlinear elliptic problems with perturbed gradient under homogeneous Dirichlet boundary conditions. We apply the Galerkin approximation and the convergence in term of Young measure combined with the theory of Sobolev spaces to obtain the existence of at least one weak solution $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

1. Introduction:

Let Ω denote a bounded open domain of $\mathbb{R}^{n\geq 2}$. We denote by $\mathbb{M}^{m\times n}$, the set of $m\times n$ matrices with reduced $\mathbb{R}^{mn}$ topology, i.e., if $\xi \in \mathbb{M}^{m\times n}$ then $|\xi|$ is the norm of ξ when regarded as a vector of $\mathbb{R}^{mn}$. We endow $\mathbb{M}^{m\times n}$ with the product

$$\xi : \eta = \sum_{i,j} \xi_{ij}\eta_{ij}.$$

The main aim of the this paper is to prove the existence of weak solutions to a class of nonlinear elliptic problems of the following prototype

$$\begin{cases} -\operatorname{div}(a(x, Du - \Theta(u))) = f(x,u,Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ is a function assumed to satisfy some conditions (see below) and the function $a : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ satisfies the following assumptions

(H_0) $a : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function.

(H_1) The function a is strictly monotone, that is

$\Big(a(x,\eta-\Theta(s)) - a(x,\xi-\Theta(s))\Big)(\eta-\xi) > 0, \forall \eta,\xi \in \mathbb{M}^{m\times n}, \eta \neq \xi.$

(H_2) As well as the growth and the coercivity assumptions

$|a(x,\eta-\Theta(s))| \leq \mathcal{M}(x) + |\eta-\Theta(s)|^{p-1};$

$a(x,\eta-\Theta(s)) : \eta \geq \alpha|\eta-\Theta(s)|^p - b_0(x).$

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Email addresses: saidotmghart@gmail.com (Said Ait Temghart), hasnaeelhammar11@gmail.com (Hasnae El Hammar), chakir.allalou@yahoo.fr (Chakir Allalou), hilalkhalid2005@yahoo.fr (Khalid Hilal)

Here, $\mathcal{M} \in L^1(\Omega)$, $b_0 \in L^{p'}(\Omega)$ and α is positive constant. The function $\Theta : \mathbb{R}^m \to \mathbb{R}^{m\times n}$ is continuous such that

$$\Theta(0) = 0 \text{ and } |\Theta(x) - \Theta(y)| \leq C_\Theta |x - y| \quad \forall x, y \in \mathbb{R}^m,$$

where C_Θ is a positive constant related to the exponent p and the diameter of Ω by

$$C_\Theta \leq \frac{1}{2diam(\Omega)}.$$

We assume that f satisfies the following assumption

(H_3) $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ is a Carathéodory function.

Moreover, the function f satisfies one of the following condtions:

(1) There exist $0 < \gamma < p - 1,\ 0 \leq \mu < p - 1,\ d_0 \in L^{p'}(\Omega)$
there holds $|f(x, s, \xi)| \leq d_0(x) + |s|^\gamma + |\xi|^\mu$,
for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$.

(2) The function f is independent of the third variable, $x \in \Omega$
or, for almost and all $s \in \mathbb{R}^m$, the mapping $\xi \mapsto f(x, s, \xi)$ is linear.

The problem (1) models several natural phenomena which appear in area of oceanography, turbulent fluid flows, induction heating and electrochemical problems. We cite for example the following parabolic model (see also [2]):
- Fluid flow through porous media: this model is governed by the following equation,

$$\frac{\partial\theta}{\partial t} - \operatorname{div}\Big(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\Big) = 0,$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.
A known prototype of the operator a is defined by $a(x, Du - \Theta(u)) = |Du - \Theta(u)|^{p-2}(Du - \Theta(u))$ and is called the generalized p-Laplacian operator. The problem (1) is a generalization of the following nonlinear problem

$$\begin{cases} -\operatorname{div}(|Du - \Theta(u)|^{p-2}(Du - \Theta(u))) = f & in\ \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

studied in [3] by E. Azroul and F. Balaadich, they proved the existence of weak solutions when $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$ by using Young measures and without any Leray-Lions type growth conditions. Dolzmann et al. [15] investigated the existence of a distributional solution and Lorentz estimate for some p-harmonic systems with a measure-valued right hand side , i.e., $f = \mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$, under the condition $2 - \frac{1}{n} < p < n$. Cianchi and Maz'ya in their works [13, 14] discussed a global Lipschitz regularity, and they obtained a sharp estimate for the decreasing length of the gradient for Dirichlet and Neumann problems associated to $-\operatorname{div}\Big(|Du|^{p-2}Du\Big) = f$ in Ω. In [19], Hungerbühler considered the following quasilinear elliptic system under certain natural conditions on the function σ

$$-\operatorname{div} \sigma(x,\ u,\ Du) = f \quad \text{in } \Omega, \tag{3}$$

the author got some existence result by using the tool of Young measures and weak monotonicity over σ. Many papers were written to investigate the existence of solutions to elliptic problems of the type (3) by using classical monotone operator methods (See [1, 10–12, 17, 20, 21, 23] and references therein). We address the reader to see [4–6] where Azroul and Balaadich have used the theory of Young measures for different kinds of nonlinear elliptic systems. The problem (1) with f independent of u and Du was treated in [8], where the authors proved the existence of weak solutions under some conditions on the operator A. The gol of the present paper is to establish the existence of solutions to the problem (1) and extend the result of [8] by considering a general source term. The main tools are Galerkin method to construct the approximating solutions and the theory of Young measures to identify weak limits when passing to the limit.

Remark 1.1. *A particular case when $a(x, \xi - \Theta(v)) = |\xi - \Theta(v)|^{p-2}(\xi - \Theta(v))$ and $f(x, v, \xi) = d(x) + |v|^{p-1} + |\xi|^{p-1}$, our problem is reduced to the following generalized p-Laplacian system*

$$-\operatorname{div}\left(|Du - \Theta(u)|^{p-2}(Du - \Theta(u))\right) = d(x) + |u|^{p-1} + |Du|^{p-1}, \text{ in } \Omega.$$

Remark 1.2. *The hypothesis (H_1) can be replaced by one of the following hypotheses:*
$(H_1)'$ For all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $\xi \mapsto a(x, \xi - \Theta(u))$ is a C^1-function and is monotone, that is,

$$(a(x, \xi - \Theta(u)) - a(x, \eta - \Theta(u))) : (\xi - \eta) \geq 0, \quad \forall \xi, \eta \in \mathbb{M}^{m\times n}.$$

$(H_1)''$ There exists a function (potential) $B : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $a(x, \xi - \Theta(u)) = (\partial B/\partial \xi)(x, \xi - \Theta(u)) := D_\xi B(x, \xi - \Theta(u))$, and $\xi \mapsto B(x, \xi - \Theta(u))$ is convex and C^1-function for all $x \in \Omega$ and $u \in \mathbb{R}^m$.
$(H_1)'''$ The operator a is strictly quasimonotone, that is, there exists $c_0 > 0$ such that

$$\int_\Omega (a(x, Du - \Theta(u)) - a(x, Dv - \Theta(u))) : (Du - Dv)dx \geq c_0 \int_\Omega |Du - Dv|^p dx.$$

In the sequel, we need the following two technical lemmas

Lemma 1.3. *[7]. For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 1.4. *For $a \geq 0, b \geq 0$ and $1 \leq p < \infty$, we have*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

Before stating the main result, we give a definition of weak solutions for the elliptic problem (1).

Definition 1.5. *A function $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ is said to be a weak solution of (1) if*

$$\int_\Omega a(x, Du - \Theta(u)) : D\varphi dx = \int_\Omega f(x, u, Du) \cdot \varphi dx$$

holds for all $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$.

Theorem 1.6. *Assume that $(H_0) - (H_3)$ hold. Then the Dirichlet problem (1) has a weak solution in the sense of Definition 1.5.*

Lemma 1.7. *For $a \geq 0$, $b \geq 0$ and $1 \leq p < \infty$, we have*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

The following is a breakdown of the current paper's structure. In Section 2, we go through some basic information about Young measures. Section 3 is dedicated to the existence of weak solution for approximation problem by applying Galerkin method, while Section 4 concerns the proof of the main theorem based on the convergence in term of Young measure.

2. Young measures and nonlinear weak-* convergence:

Throughout the paper, we denote by δ_c the Dirac measure on $\mathbb{R}^n$ ($n \in \mathbb{N}$), concentrated at the point $c \in \mathbb{R}^n$, $C_0(\mathbb{R}^m)$ denotes the set of functions $\varphi \in C(\mathbb{R}^m)$ satisfying $\lim\limits_{|\lambda|\to\infty} \varphi(\lambda) = 0$. We know that $(C_0(\mathbb{R}^m))' = \mathcal{M}(\mathbb{R}^m)$ and the duality pairing is given for $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ by

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda).$$

As stated in the introduction, the tool we use to prove the needed result is the Young measure. For the reader not familiar with this concept we recall some basic notions and properties (See [9] and [18]).

Lemma 2.1. *[18]. Let $(z_k)_k$ be a bounded sequence in $L^\infty(\Omega; \mathbb{R}^m)$. Then there exists a subsequence $\Big($ denoted again by $(z_k)\Big)$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for each $\varphi \in C_0(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \rightarrow^* \overline{\varphi} \text{ weakly in } L^\infty(\Omega; \mathbb{R}^m),$$

where $\overline{\varphi}(x) = \langle \nu_x, \varphi \rangle$ for a.e. $x \in \Omega$.

Definition 2.2. *We call $\nu = \{\nu_x\}_{x\in\Omega}$ the family of Young measures associated to (z_k). In [9], it is shown that if for all $R > 0$*

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_R(0) \ : \ |z_k(x)| \geq L\}| = 0,$$

then the Young measure ν_x generated by z_k is a probability measure, i.e., $\|\nu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in \Omega$.
The following properties build the basic tools used in the sequel. If $|\Omega| < \infty$, then there holds

$$z_k \to z \text{ in measure } \Leftrightarrow \nu_x = \delta_{z(x)} \text{ for a.e. } x \in \Omega. \tag{4}$$

If we choose $z_k = Dw_k$ for $w_k : \Omega \to \mathbb{R}^m$, the above results remain valid.

Lemma 2.3. *[3]. Assume that Dw_k is bounded in $L^p(\Omega; \mathbb{M}^{m\times n})$, then the Young measure ν_x generated by Dw_k satisfies:*

1. *ν_x is a probability measure.*
2. *The weak L^1- limit of Dw_k is given by $\langle \nu_x, id \rangle$.*
3. *The identification $\langle \nu_x, id \rangle = Dw(x)$ holds for a.e. $x \in \Omega$.*

We conclude this section by recalling the following Fatou-type inequality.

Lemma 2.4. *[16]. Let $\varphi : \mathbb{M}^{m\times n} \to \mathbb{R}$ be a continuous function and $w_k : \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that Dw_k generates the Young measure ν_x, with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$. Then*

$$\liminf_{k\to\infty} \int_\Omega \varphi(Dw_k) dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \varphi(\lambda) d\nu_x(\lambda) dx,$$

provided that the negative part of $\varphi(Dw_k)$ is equiintegrable.

3. Existence of weak solution:

The aim of this section is to use the well-known Galerkin method to construct approximating solutions. Firstly, the Hölder inequality and the following Poincaré inequality (See [19], Lemma 2.2) are central to establish the required estimates to prove the desired results. There exists a positive constant α such that

$$\|v\|_p \leq \frac{\alpha}{2} \|Dv\|_p, \quad \forall v \in W_0^{1,p}(\Omega; \mathbb{R}^m). \tag{5}$$

Now, consider the mapping $T : W_0^{1,p}(\Omega;\mathbb{R}^m) \to W^{-1,p'}(\Omega;\mathbb{R}^m)$ given for arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ by

$$\langle T(u),\varphi\rangle = \int_\Omega a(x,Du-\Theta(u))Du : D\varphi dx - \int_\Omega f(x,u,Du)\varphi dx.$$

Lemma 3.1. *$T(u)$ is well defined, linear and bounded.*

Proof. For arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, $T(u)$ is trivially linear and (without loss of generality, we may assume that $\gamma = p-1 = \mu$) we can obtain,

$$\begin{aligned}
|I_1| &\le \int_\Omega |a(x,Du-\Theta(u))||D\varphi|dx \\
&\le \int_\Omega \mathcal{M}(x)|D\varphi|dx + \int_\Omega |Du-\Theta(u)|^{p-1}|D\varphi|dx \\
&\le \|\mathcal{M}\|_{p'}\,\|D\varphi\|_p + \left(\int_\Omega |Du-\Theta(u)|^p dx\right)^{\frac{1}{p'}} \|D\varphi\|_p \\
&\le \|\mathcal{M}\|_{p'}\,\|D\varphi\|_p + 2^{\frac{(p-1)^2}{p}} \left(\|Du\|_p^p + \|\Theta(u)\|_p^p\right)^{\frac{p-1}{p}} \|D\varphi\|_p \\
&= \left(\|\mathcal{M}\|_{p'} + 2^{\frac{(\varphi-1)^2}{p}} \left(\|Du\|_p^p + \|\Theta(u)\|_p^p\right)^{\frac{p-1}{p}}\right)\|D\varphi\|_p.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|I_2| &\le \int_\Omega |f(x,u,Du)||\varphi|dx \\
&\le \left(\|d_0\|_{p'} + \|Du\|_p^{p-1} + \|u\|_p^{p-1}\right)\|\varphi\|_p.
\end{aligned}$$

Since these two expressions are finite by our assumptions, $T(u)$ is well defined. Finally we have

$$\begin{aligned}
|\langle T(u),\varphi\rangle| &\le |I_1| + |I_2| \\
&\le C_1\|D\varphi\|_p + C_2\|\varphi\|_p \\
&\le C_3\|D\varphi\|_p,
\end{aligned}$$

thus T is well defined and bounded.

Lemma 3.2. *The restriction of T to a finite linear subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuous.*

Proof. Let X be a finite subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ with $\dim X = r$ and $(x_i)_{i=1,\cdots,r}$ a basis of X. We consider in X, the sequence $\left(u_k = a_k^i x_i\right)$ which converges to $u = a^i x_i$ in X. Hence $u_k \to u$ and $Du_k \to Du$ almost everywhere for a subsequence still denoted by $(u_k)_k$. From the continuity of a and f, one can obtain that

$$a\,(x, Du_k - \Theta(u_k)) \to a\,(x, Du - \Theta(u)) \text{ a.e. in } \Omega$$

and

$$f\,(x,u_k,Du_k) \to f(x,u,Du) \text{ a.e. in } \Omega.$$

Using the strong convergence of u_k to u in X ana Lemma 1.7, we can infer that $\|u_k\|_p$ and $\|Du_k\|_p$ are bounded. Now, in order to apply the Vitali Theorem, we need to show the equi-integrability of the sequences $(a\,(x,Du_k-\Theta(u_k)) : D\varphi)$ and $(f\,(x,u_k,Du_k)\cdot\varphi)$. To do this, let $E \subset \Omega$ a measurable subset, then by the growth condition in (H_0), we have

$$\int_E \left| a\left(x, Du_k - \Theta(u_k)\right) : D\varphi \right| dx \leq \left(\int_E |\mathcal{M}(x)|^{p'} + |Du_k - \Theta(u_k)|^p dx \right)^{\frac{1}{p'}} \left(\int_E |D\varphi|^p dx \right)^{\frac{1}{p}}$$

$$\leq \left(\|\mathcal{M}\|_{p'}^{p'} + 2^{p-1} \left(\underbrace{\|Du_k\|_p^p}_{\leq C} + c^p \underbrace{\|u_k\|_p^p}_{\leq C} \right) \right)^{\frac{1}{p'}} \left(\int_E |D\varphi|^p dx \right)^{\frac{1}{p}}$$

and

$$\int_E \left| f\left(x, u_k, Du_k\right) \cdot \varphi \right| dx \leq C \left(\|d_0\|_{p'} + \underbrace{\|u_k\|_p^{p-1}}_{\leq C} + \underbrace{\|Du_k\|_p^{p-1}}_{\leq C} \right) \left(\int_E |D\varphi|^p dx \right)^{\frac{1}{p}}.$$

Since $\int_E |D\varphi|^p dx$ is arbitrary small if the measure of E is chosen small enough, then the equiintegrability of $\left(a\left(x, Du_k - \Theta(u_k)\right) : D\varphi\right)$ and $\left(f\left(x, u_k, Du_k\right) \cdot \varphi\right)$ follows. From Vitali's Theorem, we conclude the continuity of mapping T.

Lemma 3.3. *The operator T defined above is coercive.*

Proof. By taking $\varphi = u$ in the definition of T, we have

$$\begin{aligned}\langle T(u), u\rangle &= \int_\Omega a(x, u, Du)Du : Du dx - \int_\Omega f(x, u, Du)u dx \\ &\geq \alpha \int_\Omega |Du - \Theta(u)|^p dx - \int_\Omega b_0(x) dx - \int_\Omega f(x, u, Du)u dx.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2^{p-1}}|Du|^p &= \frac{1}{2^{p-1}}|Du - \Theta(u) + \Theta(u)|^p \\ &\leq \frac{1}{2^{p-1}} \left[2^{p-1} \left(|Du - \Theta(u)|^p + |\Theta(u)|^p \right) \right] \\ &\leq |Du - \Theta(u)|^p + |\Theta(u)|^p.\end{aligned}$$

Using Hölder inequality, (5) and the assumption (H_3), we deduce that

$$\begin{aligned}&\left| \int_\Omega f(x, u, Du) \cdot u dx \right| \leq \int_\Omega d_0(x)|u| dx + \int_\Omega |u|^\gamma |u| dx + \int_\Omega |Du|^\mu |u| dx \\ &\leq \|d_0\|_{p'} \|u\|_p + \|u\|_{\gamma p'}^\gamma \|u\|_p + \|Du\|_{\mu p'}^\mu \|u\|_p \\ &\leq \frac{\alpha}{2} \|d_0\|_{p'} \|Du\|_p + \left(\frac{\alpha}{2}\right)^{\gamma+1} \|Du\|_p^{\gamma+1} + \frac{\alpha}{2} \|Du\|_p^{\mu+1}.\end{aligned}$$

From (5) and the choice of the constant C_Θ in the assumption on Θ, we obtain

$$\begin{aligned}&\langle T(u), u\rangle \geq \frac{\alpha}{2^{p-1}} \int_\Omega |Du|^p dx - \alpha \int_\Omega |\Theta(u)|^p dx - \int_\Omega b_0(x) dx - \frac{\alpha}{2} \|d_0\|_{p'} \|Du\|_p - \left(\frac{\alpha}{2}\right)^{\gamma+1} \|Du\|_p^{\gamma+1} - \frac{\alpha}{2} \|Du\|_p^{\mu+1} \\ &\geq \frac{\alpha}{2^{p-1}} \int_\Omega |Du|^p dx - \alpha C_\Theta^p \int_\Omega |u|^p dx - \int_\Omega b_0(x) dx - \frac{\alpha}{2} \|d_0\|_{p'} \|Du\|_p - \left(\frac{\alpha}{2}\right)^{\gamma+1} \|Du\|_p^{\gamma+1} - \frac{\alpha}{2} \|Du\|_p^{\mu+1} \\ &\geq \frac{\alpha}{2^p} \int_\Omega |Du|^p dx - \int_\Omega b_0(x) dx - \frac{\alpha}{2} \|d_0\|_{p'} \|Du\|_p - \left(\frac{\alpha}{2}\right)^{\gamma+1} \|Du\|_p^{\gamma+1} - \frac{\alpha}{2} \|Du\|_p^{\mu+1}.\end{aligned}$$

Hence

$$\langle T(u), u\rangle \longrightarrow \infty \quad \text{as } \|u\|_{1,p} \to \infty,$$

since $p > \max\{1, \gamma + 1, \mu + 1\}$.

In what follows, let us fix some k and assume that X_k has the dimension r and $e_1, \ldots, e_r$ is a basis of X_k. We define the map

$$G : \mathbb{R}^r \to \mathbb{R}^r$$

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^r \end{pmatrix} \mapsto \begin{pmatrix} \langle T(\beta^i e_i), e_1\rangle \\ \langle T(\beta^i e_i), e_2\rangle \\ \vdots \\ \langle T(\beta^i e_i), e_r\rangle \end{pmatrix}$$

Lemma 3.4. G is continuous and $G(\beta).\beta \to \infty$ as $\|\beta\|_{\mathbb{R}^r} \to \infty$, where $\beta = (\beta^1, \ldots, \beta^r)^t$ and the dot is the inner product of two vectors of $\mathbb{R}^r$.

Proof. Let $u_j = \beta_i^j e_i \in X_k,\ u_0 = \beta_i^0 e_i \in X_k$. Then $\|\beta^j\|_{\mathbb{R}^r}$ is equivalent to $\|u_j\|_{1,p}$ and $\|\beta^0\|_{\mathbb{R}^r}$ is equivalent to $\|u_0\|_{1,p}$ and

$$G(\beta) \cdot \beta = \langle T(u), u\rangle.$$

Lemma 3.3 gives $G(\beta).\beta \to \infty$ when $\|\beta\|_{\mathbb{R}^r} \to \infty$.

Lemma 3.5. *For all $k \in \mathbb{N}$ there exists $u_k \in X_k$ such that*

$$\langle T(u_k)\,,\,\varphi\rangle = 0 \text{ for all } \varphi \in X_k. \tag{6}$$

and there is a constant $R > 0$ such that

$$\|u_k\|_{1,p} \le R \quad \text{for all} \quad k \in \mathbb{N}. \tag{7}$$

Proof. From Lemma 3.4, it follows the existence of a constant $R > 0$ such that for any $\beta \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(\beta).\beta > 0$ and the topological argument [22] gives that $G(x) = 0$ has a solution $x \in B_R(0)$. Therefore, for each $k \in \mathbb{N}$ there exists $u_k \in X_k$ such that (6) holds.

4. The convergence in term of Young measure:

Assertion 1

The sequence (u_k) is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ for some $p > 1$, thus a subsequence converges weakly in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ to an element denoted by u.

Proof.

we have $\langle T(u)\,,\,u\rangle \to \infty$ as $\|u\|_{1,p} \to \infty$. Hence, there exists $R > 0$ with the property, that $\langle T(u)\,,\,u\rangle > 1$ whenever $\|u\|_{1,p} > R$. Consequently, for the sequence of Galerkin approximations $u_k \in X_k$ which satisfy (6) with φ replaced by u_k, we get that (u_k) is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Assertion 2

The sequence a_k defined by $a_k := a(x, Du_k - \Theta(u_k))$ is uniformly bounded in $L^{p'}(\Omega;\mathbb{R}^m)$ and therefore equi-integrable on Ω.

Proof.

By using the growth assumption (H_2), we get

$$\int_\Omega |a(x, Du_k - \Theta(u_k))|^{p'} \le \int_\Omega \mathcal{M}(x)dx + \int_\Omega |Du_k - \Theta(u_k)|^p < \infty, \tag{8}$$

by the boundedness of $(u_k)_k$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Hence $a_k(x)$ is uniformly bounded in $L^{p'}(\Omega;\mathbb{R}^m)$.

Assertion 3

The sequence $(a_k(x) : Du_k)^-$ is equi-integrable on Ω. Moreover, there exists a sequence (v_k) such that $v_k \to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ and

$$\int_\Omega a_k(x) : (Du_k - Dv_k)dx \to 0 \; as \; k \to \infty.$$

Proof.
For any measurable subset E of Ω and by the coercivity assumption, we have

$$\int_\Omega |\min(a(x, Du_k - \Theta(u_k)) \; : \; Du_k, 0)|dx \leq \frac{\alpha}{2^{p-1}}\int_E |Du_k|^p dx + \alpha \int_E |\Theta(u_k)|^p dx + \int_E |b_0(x)|dx < \infty.$$

Then $(a_k(x) : Du_k)^-$ is equi-integrable.
We choose a subsequence v_k which belongs to the same finite dimensional space X_k as u_k such that $v_k \to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. By taking $u_k - v_k$ as a test function in (6), we deduce that

$$\begin{aligned}&\int_\Omega a(x, Du_k - \Theta(u_k)) : \; (Du_k - Dv_k)dx = \int_\Omega f(x, u_k, Du_k)(u_k - v_k)dx \\ &\leq \|f(x, u_k, Du_k)\|_{p'}\|u_k - v_k\|_p \\ &\leq C\|u_k - v_k\|_p.\end{aligned}$$

Since $u_k - v_k \to 0 \quad \text{in} \quad W_0^{1,p}(\Omega;\mathbb{R}^m)$, then

$$\int_\Omega a(x, Du_k - \Theta(u_k)) : (Du_k - Dv_k)dx \to 0 \; as \; k \to \infty.$$

Assertion 4
The following div-curl inequality holds:

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} a(x, \lambda - \Theta(u)) : (\lambda - Du)d\nu_x(\lambda)dx \leq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x, Du - \Theta(u)) : (\lambda - Du)d\nu_x(\lambda)dx. \tag{9}$$

Proof.
We define the sequence

$$\begin{aligned}J_k &:= (a(x, Du_k - \Theta(u)) - a(x, Du - \Theta(u))) \; : \; (Du_k - Du) \\ &= a(x, Du_k - \Theta(u)) : (Du_k - Du) - a(x, Du - \Theta(u)) : (Du_k - Du)\end{aligned}$$

By using the growth condition in (H_1), (H_0) and the Poincaré's inequality, we get

$$\int_\Omega |a(x, Du - \Theta(u))|^{p'} dx \leq C + C' \int_\Omega |Du|^p dx < \infty \tag{10}$$

for arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, hence $a(x, Du - \Theta(u)) \in L^{p'}(\Omega;\mathbb{M}^{m\times n})$. According to the weak convergence described in Lemma 2.3, one can obtain

$$\liminf_{k\to\infty} \int_\Omega J_{k,2}dx = \int_\Omega a(x, Du - \Theta(u)) : \left(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - Du\right)dx = 0. \tag{11}$$

Next, from Assertion 1, there exits a subsequence u_k such that $u_k \to u$ in measure. Since Θ is continuous then $\Theta(u_k) \to \Theta(u)$ almost everywhere in Ω. In view of Lemma 2.4, one can conclude that

$$\begin{aligned}J : &= \liminf_{k\to\infty} \int_\Omega J_k dx \\ &= \liminf_{k\to\infty} \int_\Omega J_{k,1} dx \\ &\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x, \lambda - \Theta(u)) : (\lambda - Du)d\nu_x(\lambda)dx.\end{aligned}$$

Showing (9) is equivalent to proving that $J \leq 0$. By virtue of Assertion 3, we deduce that

$$\begin{aligned} A &= \liminf_{k\to\infty} \int_\Omega a(x, Du_k - \Theta(u_k)) : (Du_k - Du)dx \\ &= \liminf_{k\to\infty} \int_\Omega a(x, Du_k - \Theta(u_k)) : (Dv_k - Du)dx \\ &\leq \liminf_{k\to\infty} \|\|a(x, Du_k - \Theta(u_k))\|\|_{p'} \, \|v_k - u\|_{1,p} = 0. \end{aligned}$$

It follows that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (a(x, \lambda - \Theta(u)) - a(x, Du - \Theta(u))) : (\lambda - Du)d\nu_x(\lambda)dx \leq 0.$$

Moreover, the monotonicity of the function a implies that the above integral must vanish with respect to the product measure $d\nu_x(\lambda) \otimes dx$, hence

$$(a(x, \lambda - \Theta(u)) - a(x, Du - \Theta(u))) : (\lambda - Du) = 0 \text{ on } supp\nu_x.$$

Assertion 5
The sequence a_k converges weakly in the space $L^1(\Omega; \mathbb{M}^{m\times n})$ as $k \to +\infty$ to the weak limit $\bar{a}$ given by

$$\bar{a}(x) = a(x, Du - \Theta(u))$$

and Du_k converges to Du in measure on Ω, as $k \to +\infty$.
Proof.
Using (9) and the strict monotonicity assumption (H_4), we deduce that

$$(a(x, \lambda - \Theta(u)) - a(x, Du - \Theta(u))) : (\lambda - Du) = 0 \text{ a.e. } x \in \Omega, \; \lambda \in \mathbb{R}^N.$$

Then $\lambda = Du(x)$ a.e. $x \in \Omega$ with respect to the measure ν_x on $\mathbb{R}^N$. Therefore, the measure ν_x reduces to the Dirac measure $\delta_{Du(x)}$. By virtue of Theorem ii), we deduce that $Du_k \to Du$ in measure, then $u_k \to u$ and $Du_k \to Du$ almost everywhere (up to a subsequence) in Ω. From the continuity of Θ and a one can deduce that

$$a(x, Du_k - \Theta(u_k)) \to a(x, Du - \Theta(u)) \text{ a.e. } x \in \Omega.$$

From Assertion 2, a_k is equiintegrable, then one can apply Vitali's Theorem to get

$$a(x, Du_k - \Theta(u_k)) \to a(x, Du - \Theta(u)) \text{ in } L^1(\Omega; \mathbb{M}^{m\times n}).$$

Lemma 4.1. *The function u is a weak solution to problem* (1).

Now, we have all ingredients to pass to the limit and so to prove the main result. From the Assertion 5, we have

$$\lim_{k\to+\infty} \int_\Omega a(x, Du_k - \Theta(u_k)) : D\varphi dx = \int_\Omega a(x, Du - \Theta(u)) : D\varphi dx,$$

for all $\varphi \bigcup_{k\in\mathbb{N}} X_k$.
Now, we focus our attention on the source term. Let start with the case $(H_3)(1)$, the continuity of f permit to deduce that

$$f(x, u_k, Du_k) \cdot \varphi \to f(x, u, Du) \cdot \varphi$$

for all $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$. From the growth condition in $(H_5)(i)$, we deduce the equiintegrability of $(f(x, u_k, Du_k) \cdot \varphi(x))$, which implies by Vitali Convergence Theorem that, $f(x, u_k, Du_k) \cdot \varphi(x) \to f(x, u, Du) \cdot \varphi(x)$ in $L^1(\Omega)$. Therefore

$$\lim_{k\to\infty} \int_\Omega f(x, u_k, Du_k) \cdot \varphi(x)dx = \int_\Omega f(x, u, Du) \cdot \varphi(x)dx, \; \forall\varphi \in \bigcup_{k\in\mathbb{N}} X_k.$$

Next, we consider the case $(H_3)(2)$, if the function f is independent of the third variable, then we can obtain

$$f(x, u_k) \rightharpoonup f(x, u) \quad \text{in} \quad L^{p'}(\Omega).$$

On the other hand, we assume that the mapping $A \mapsto f(x,u,A)$ is linear, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^m$. Since $f(x,u_k,Du_k)$ is equiintegrable. We deduce that

$$\begin{aligned} f(x,u_k,Du_k) \rightarrow \langle \nu_x, f(x,u,.)\rangle &= \int_{\mathbb{M}^{m\times n}} f(x,u,\lambda)d\nu_x(\lambda) \\ &= f(x,u,.)o \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=:Du(x)} \\ &= f(x,u,Du), \end{aligned}$$

by the linearity of f.

It remains to show that $\langle T(u),\varphi\rangle = 0$ for any $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, to complete the proof of Theorem 1.6. Let $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, the density of $\bigcup_{k\in\mathbb{N}} X_k$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ implies the existence of a sequence $\{\varphi_k\} \subset \bigcup_{k\in\mathbb{N}} X_k$ such that $\varphi_k \rightarrow \varphi$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ as k goes to $+\infty$. We conclude that

$$\begin{aligned} &\langle T(u_k),\varphi_k\rangle - \langle T(u),\varphi\rangle = \int_\Omega a(x,Du_k-\Theta(u_k)) : D\varphi_k dx - \int_\Omega a(x,Du-\Theta(u)) : D\varphi dx - \int_\Omega f(x,u_k,Du_k)\cdot\varphi_k dx \\ &+ \int_\Omega f(x,u,Du)\cdot\varphi dx \\ &= \int_\Omega a(x,Du_k-\Theta(u_k)) : (D\varphi_k - D\varphi)dx + \int_\Omega (a(x,Du_k-\Theta(u_k)) - a(x,Du-\Theta(u))) : D\varphi dx \\ &- \int_\Omega f(x,u_k,Du_k)(\varphi_k-\varphi)\,dx - \int_\Omega (f(x,u_k,Du_k) - f(x,u,Du))\varphi dx. \end{aligned}$$

We take the limit as k goes to $+\infty$, it follows that

$$\lim_{k\rightarrow+\infty} \langle T(u_k),\varphi_k\rangle = \langle T(u),\varphi\rangle.$$

From Lemma 3.5, we deduce that $\langle T(u),\varphi\rangle = 0$ for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

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