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Some binomial identities related to the Catalan triangles and the halves of the Pascal matrix

Lin Yang[a], **Sheng-Liang Yang**[a,*]

[a]*Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, 730050, Gansu, PR China*

Abstract. In this paper, five binomial sums with two additional parameters are shown to be equipollent by using the Riordan array method and the generalized Catalan matrices, as well as the halves of the Pascal matrix.

1. Introduction

For the following binomial sums

$$\begin{aligned}
\Omega_n &= \sum_{k=0}^{n} \binom{3(n-k)}{n-k}\binom{3k}{k}, \\
A_n &= \sum_{k=0}^{n} 3^k \binom{3n-k}{2n}, \\
B_n &= \sum_{k=0}^{n} 2^k \binom{3n+1}{2n+k+1}, \\
E_n &= \sum_{k=0}^{n} 3^k \binom{3n-k}{n-k}\frac{2k(k+1)}{3n-k}, \\
F_n &= \sum_{k=0}^{n} 2^k \binom{3n+2}{n-k}\frac{(3k+2)(k+1)}{3n+2},
\end{aligned}$$

Alzer and Prodinger [1], Bai and Chu [2], Duarte and Guedes de Oliveira [8], and Kilic and Arikan [14] found, by using different methods respectively, the following combinatorial identities

$$\Omega_n = A_n = B_n = E_n = F_n. \tag{1}$$

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* Corresponding author: Sheng-Liang Yang
Email addresses: yanglinmath@163.com (Lin Yang), slyang@lut.edu.cn (Sheng-Liang Yang)

In this paper, we define five new kinds of binomial sums with two additional parameters m and r:

$$\begin{aligned}
\Omega_n^{(m,r)} &= \sum_{k=0}^{n} \binom{(m+1)(n-k)}{n-k}\binom{(m+1)k+r}{k},\\
A_n^{(m,r)} &= \sum_{k=0}^{n} (m+1)^k \binom{(m+1)n-k+r}{mn+r},\\
B_n^{(m,r)} &= \sum_{k=0}^{n} m^k \binom{(m+1)n+r+1}{mn+k+r+1},\\
E_n^{(m,r)} &= \sum_{k=0}^{n} \frac{mk+r}{mn+n-k+r}\binom{mn+n-k+r}{n-k}(k+1)(m+1)^k,\\
F_n^{(m,r)} &= \sum_{k=0}^{n} \frac{mk+k+r+2}{(m+1)n+r+2}\binom{(m+1)n+r+2}{n-k}(k+1)m^k.
\end{aligned}$$

As an extension of the identities (1), some new identities are provided in this paper, the main results of which can be stated as following theorem.

Theorem 1.1. *For $n, m, r \geq 0$,*

$$\Omega_n^{(m,r)} = A_n^{(m,r)} = B_n^{(m,r)} = E_n^{(m,r)} = F_n^{(m,r)}.$$

In the following of this section, we will briefly recall the Riordan arrays [3–5, 7, 10, 16, 19, 21] and the (m, r, s)-half of the Pascal matrix [4, 11, 24, 25] which are two important tools in the sequel. An infinite lower triangular matrix $G = (g_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has generating function $d(t)h(t)^k$, where $d(t) = \sum_{n=0}^{\infty} d_n t^n$ and $h(t) = \sum_{n=1}^{\infty} h_n t^n$ are formal power series with $d_0 \neq 0$ and $h_1 \neq 0$. If $(b_n)_{b\in\mathbb{N}}$ is any sequence having $b(t) = \sum_{n=0}^{\infty} b_n t^n$ as its generating function, then for every Riordan array $(d(t), h(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$

$$\sum_{k=0}^{n} g_{n,k} b_k = [t^n] d(t) b(h(t)). \tag{2}$$

This is called the fundamental theorem of Riordan arrays (FTRA) and it can be rewritten as

$$(d(t), h(t))b(t) = d(t)b(h(t)). \tag{3}$$

For example, the Pascal matrix $P = \left(\binom{n}{k}\right)_{n,k\geq 0} = (\frac{1}{1-t}, \frac{t}{1-t})$ is a Riordan array, which is registered as sequence A007318 in OEIS [20]. In the sequel, sequences are frequently referred to by their OEIS number.

The vertical halves of Riordan arrays and the horizontal halves of Riordan arrays were introduced in Yang et al. [11, 24, 25] and Barry [4, 15], respectively. In recently, we introduce the (m, r, s)-half of a Riordan array [22], which can be viewed as a skew half of a Riordan antecedent [4].

Definition 1.2. *Let $G = (u(t), tv(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array, and let m, r be nonnegative integers and s a positive fractional number such that ms is integral number. The (m, r, s)-half of G is defined as the matrix $G^{(m,r,s)}$ with general (n, k)-th term $g_{(m+1)n+(ms-m-1)k+r,mn+(ms-m)k+r}$.*

In [22], using the generating function $\mathcal{B}_{m+1}(t) = \sum_{n=0}^{\infty} \frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n} t^n$, the (m, r, s)-half of the Pascal matrix $G = (\frac{1}{1-t}, \frac{t}{1-t})$ have been found:

Lemma 1.3. *The (m, r, s)-half of the Pascal matrix $G = (\frac{1}{1-t}, \frac{t}{1-t})$ is*

$$G^{(m,r,s)} = \left(\frac{\mathcal{B}_{m+1}(t)^{r+1}}{1 - mt\mathcal{B}_{m+1}(t)^{m+1}}, t\mathcal{B}_{m+1}(t)^{ms}\right).$$

For giving another form of the halves of the Pascal matrix, let us recall some equations for Fuss-Catalan numbers $\frac{1}{pn+1}\binom{pn+1}{n}$, see [6, 9, 17]. The generating function $\mathcal{B}_p(t) = \sum_{n=0}^{\infty} \frac{1}{pn+1}\binom{pn+1}{n} t^n$ satisfies the functional equation

$$\mathcal{B}_p(t) = 1 + t\mathcal{B}_p(t)^p, \tag{4}$$

and the powers of $\mathcal{B}_p(t)$ admit quite nice Taylor expansion:

$$\mathcal{B}_p(t)^s = \sum_{n=0}^{\infty} \frac{s}{pn+s}\binom{pn+s}{n} t^n. \tag{5}$$

The coefficients $\frac{s}{pn+s}\binom{pn+s}{n}$ have also combinatorial interpretations, see also [9], and are called generalized Fuss-Catalan numbers or Raney numbers. Formulas (4), (5) remain true if the parameters p, s are real. It can be checked in [9, 12, 13, 17, 23] that the following identities are valid

$$\mathcal{B}_p(t\mathcal{B}_{p+s}(t)^s) = \mathcal{B}_{p+s}(t), \tag{6}$$

$$\frac{\mathcal{B}_p(t)^{s+1}}{1-(p-1)t\mathcal{B}_p(t)^p} = \sum_{n=0}^{\infty} \binom{pn+s}{n} t^n, \tag{7}$$

and the derivative of $\mathcal{B}_p(t)$ is given by

$$\mathcal{B}'_p(t) = \frac{\mathcal{B}_p(t)^{p+1}}{1-(p-1)t\mathcal{B}_p(t)^p} = \sum_{n=0}^{\infty} \binom{pn+p}{n} t^n. \tag{8}$$

By the Lemma 1.3 and equation (8), we have

$$\frac{\mathcal{B}_{m+1}(t)^{r+1}}{1-mt\mathcal{B}_{m+1}(t)^{m+1}} = \mathcal{B}'_{m+1}(t)\mathcal{B}_{m+1}(t)^{r-m-1}.$$

Hence, we can obtain the following result.

Theorem 1.4. *The (m, r, s)-half of the Pascal matrix $G = (\frac{1}{1-t}, \frac{t}{1-t})$ is given by*

$$G^{(m,r,s)} = \left(\mathcal{B}'_{m+1}(t)\mathcal{B}_{m+1}(t)^{r-m-1}, t\mathcal{B}_{m+1}(t)^{ms}\right). \tag{9}$$

2. Proofs

In this section, we will give the proof for the Theorem 1.1.

Proof. **Step 1:** $\mathbf{A_n^{(m,r)} = \Omega_n^{(m,r)}}$.

The generating function of the sequence $(\Omega_n^{(m,r)})_{n\geq 0}$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} \Omega_n^{(m,r)} t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{(m+1)(n-k)}{n-k}\binom{(m+1)k+r}{k} t^n \\
&= \left(\sum_{n=0}^{\infty} \binom{(m+1)n}{n} t^n\right)\left(\sum_{n=0}^{\infty} \binom{(m+1)n+r}{n} t^n\right) \\
&= \frac{\mathcal{B}_{m+1}(t)}{1-mt\mathcal{B}_{m+1}(t)^{m+1}} \cdot \frac{\mathcal{B}_{m+1}(t)^{r+1}}{1-mt\mathcal{B}_{m+1}(t)^{m+1}} \\
&= \frac{\mathcal{B}_{m+1}(t)^{r+2}}{(1-mt\mathcal{B}_{m+1}(t)^{m+1})^2}.
\end{aligned}$$

From (9), we know that the $(m, r, 1)$-half of the Pascal matrix is

$$G^{(m,r,1)} = \left(\mathcal{B}'_{m+1}(t)\mathcal{B}_{m+1}(t)^{r-m-1}, t\mathcal{B}_{m+1}(t)^m\right),$$

and its general term is $\binom{(m+1)n-k+r}{mn+r}$. By the FTRA and equation (8), we have

$$\begin{aligned}
& G^{(m,r,1)} \cdot \frac{1}{1-(m+1)t} \\
= & \frac{\mathcal{B}'_{m+1}(t)\mathcal{B}_{m+1}(t)^{r-m-1}}{1-(m+1)t\mathcal{B}_{m+1}(t)^m} \\
= & \frac{\mathcal{B}_{m+1}(t)^{m+1}}{1-(m+1)t\mathcal{B}_{m+1}(t)^m} \frac{\mathcal{B}_{m+1}(t)^{r-m-1}}{1-(m+1)t\mathcal{B}_{m+1}(t)^m} \\
= & \frac{\mathcal{B}_{m+1}(t)^{m+2}}{\mathcal{B}_{m+1}(t)-(m+1)t\mathcal{B}_{m+1}(t)^{m+1}} \frac{\mathcal{B}_{m+1}(t)^{r-m}}{\mathcal{B}_{m+1}(t)-(m+1)t\mathcal{B}_{m+1}(t)^{m+1}} \\
= & \frac{\mathcal{B}_{m+1}(t)^{r+2}}{(1-mt\mathcal{B}_{m+1}(t)^{m+1})^2} = \sum_{n=0}^{\infty} \Omega_n^{(m,r)} t^n,
\end{aligned}$$

which is equivalent to $A_n^{(m,r)} = \Omega_n^{(m,r)}$.
Step 2: $\mathbf{B_n^{(m,r)} = \Omega_n^{(m,r)}}$.

By Theorem 1.4, we know that the $(m, r+1, \frac{m+1}{m})$-half of the Pascal matrix is

$$G^{(m,r+1,\frac{m+1}{m})} = \left(\mathcal{B}'_{m+1}(t)\mathcal{B}_{m+1}(t)^{r-m}, t\mathcal{B}_{m+1}(t)^{m+1}\right),$$

and the general term is $\binom{(m+1)n+r+1}{mn+k+r+1}$. By the FTRA and equation (8), we have

$$\begin{aligned}
& G^{(m,r+1,\frac{m+1}{m})} \cdot \frac{1}{1-mt} \\
= & \left(\mathcal{B}'_{m+1}(t)\mathcal{B}_{m+1}(t)^{r-m}, t\mathcal{B}_{m+1}(t)^{m+1}\right) \cdot \frac{1}{1-mt} \\
= & \frac{\mathcal{B}'_{m+1}(t)\mathcal{B}_{m+1}(t)^{r-m}}{1-mt\mathcal{B}_{m+1}(t)^{m+1}} \\
= & \frac{\mathcal{B}_{m+1}(t)^{r+2}}{(1-mt\mathcal{B}_{m+1}(t)^{m+1})^2} \\
= & \sum_{n=0}^{\infty} \Omega_n^{(m,r)} t^n,
\end{aligned}$$

which is equivalent to $B_n^{(m,r)} = \Omega_n^{(m,r)}$.
Step 3: $\mathbf{E_n^{(m,r)} = \Omega_n^{(m,r)}}$, $\mathbf{F_n^{(m,r)} = \Omega_n^{(m,r)}}$.

Consider the following two Riordan matrices

$$E = (\mathcal{B}_{m+1}(t)^r, t\mathcal{B}_{m+1}(t)^m) \text{ and } F = \left(\mathcal{B}_{m+1}(t)^{r+2}, t\mathcal{B}_{m+1}(t)^{m+1}\right),$$

which are the generalized Catalan matrices considered by several authors [6, 13, 25]. Their generic elements are given by

$$\begin{aligned}
E_{n,k} &= [t^n]\mathcal{B}_{m+1}(t)^r \left(t\mathcal{B}_{m+1}(t)^m\right)^k \\
&= [t^{n-k}]\mathcal{B}_{m+1}(t)^{mk+r} \\
&= \frac{mk+r}{mn+n-k+r}\binom{mn+n-k+r}{n-k},
\end{aligned}$$

$$\begin{aligned} F_{n,k} &= [t^n]\mathcal{B}_{m+1}(t)^{r+2}\left(t\mathcal{B}_{m+1}(t)^{m+1}\right)^k \\ &= [t^{n-k}]\mathcal{B}_{m+1}(t)^{mk+k+r+2} \\ &= \frac{mk+k+r+2}{(m+1)n+r+2}\binom{(m+1)n+r+2}{n-k}. \end{aligned}$$

From the following two matrix equations

$$\begin{aligned} (\mathcal{B}_{m+1}(t)^r, t\mathcal{B}_{m+1}(t)^m)\cdot\frac{1}{(1-(m+1)t)^2} &= \frac{\mathcal{B}_{m+1}(t)^r}{(1-(m+1)t\mathcal{B}_{m+1}(t)^m)^2} \\ &= \frac{\mathcal{B}_{m+1}(t)^{r+2}}{(1-mt\mathcal{B}_{m+1}(t)^{m+1})^2} \\ &= \sum_{n=0}^{\infty}\Omega_n^{(m,r)}t^n, \end{aligned}$$

$$\begin{aligned} \left(\mathcal{B}_{m+1}(t)^{r+2}, t\mathcal{B}_{m+1}(t)^{m+1}\right)\cdot\frac{1}{(1-mt)^2} &= \frac{\mathcal{B}_{m+1}(t)^{r+2}}{(1-mt\mathcal{B}_{m+1}(t)^{m+1})^2} \\ &= \sum_{n=0}^{\infty}\Omega_n^{(m,r)}t^n, \end{aligned}$$

we can obtain that $E_n^{(m,r)} = \Omega_n^{(m,r)}$ and $F_n^{(m,r)} = \Omega_n^{(m,r)}$. □

Note that if we set $m = 1$ and $r = 0$, then

$$E = (1, t\mathcal{B}_2(t)) \text{ and } F = \left(\mathcal{B}_2(t)^2, t\mathcal{B}_2(t)^2\right),$$

where $\mathcal{B}_2(t) = \sum_{n=0}^{\infty}\frac{1}{2n+1}\binom{2n+1}{n}t^n$ is the generating function of the Catalan numbers. Thus, $E = (1, t\mathcal{B}_2(t))$ is the Ballot matrix [12, 23] and $F = \left(\mathcal{B}_2(t)^2, t\mathcal{B}_2(t)^2\right)$ is the Shapiro's Catalan triangle [18].

3. Special cases

In this section, we will give some identities for some special m and r.

Example 3.1. If $m = 2$ and $r = 0$, then we have

$$\Omega_n^{(2,0)} = A_n^{(2,0)} = B_n^{(2,0)} = E_n^{(2,0)} = F_n^{(2,0)}.$$

It means that

$$\Omega_n = A_n = B_n = E_n = F_n,$$

where the sequence $(\Omega_n)_{n\geq 0}$ begins (A006256)

$$1, 6, 39, 258, 1719, 11469, \cdots$$

These results are also obtained in [1, 2, 8, 14].

Example 3.2. In the case $m = 1$,

$$\begin{aligned} \sum_{n=0}^{\infty}\Omega_n^{(1,r)}t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2(n-k)}{n-k}\binom{2k+r}{k}t^n \\ &= \frac{C(t)^{r+2}}{(1-tC(t)^2)^2} \\ &= B(t)^2C(t)^r, \end{aligned}$$

where $C(t) = \mathcal{B}_2(t) = \sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{2n+1}{n} t^n$ is the generating function of the Catalan numbers, and $B(t) = \frac{C(t)}{1-tC(t)^2}$ is the generating function of the central binomial coefficients $\binom{2n}{n}$.

(i) If $m = 1$ and $r = 0$, then we have

$$\Omega_n^{(1,0)} = A_n^{(1,0)} = B_n^{(1,0)} = E_n^{(1,0)} = F_n^{(1,0)} = 4^n,$$

where

$$\begin{aligned}
\Omega_n^{(1,0)} &= \sum_{k=0}^{n} \binom{2(n-k)}{n-k}\binom{2k}{k}, \\
A_n^{(1,0)} &= \sum_{k=0}^{n} 2^k \binom{2n-k}{n}, \\
B_n^{(1,0)} &= \sum_{k=0}^{n} \binom{2n+1}{n+k+1}, \\
E_n^{(1,0)} &= \sum_{k=0}^{n} \frac{k(k+1)}{2n-k}\binom{2n-k}{n-k} 2^k, \\
F_n^{(1,0)} &= \sum_{k=0}^{n} \frac{(k+1)^2}{n+1}\binom{2n+2}{n-k}.
\end{aligned}$$

(ii) If $m = 1$ and $r = 1$, then we can get that

$$\Omega_n^{(1,1)} = A_n^{(1,1)} = B_n^{(1,1)} = E_n^{(1,1)} = F_n^{(1,1)} = 2^{2n+1} - \binom{2n+1}{n},$$

where

$$\begin{aligned}
\Omega_n^{(1,1)} &= \sum_{k=0}^{n} \binom{2(n-k)}{n-k}\binom{2k+1}{k}, \\
A_n^{(1,1)} &= \sum_{k=0}^{n} 2^k \binom{2n-k+1}{n+1}, \\
B_n^{(1,1)} &= \sum_{k=0}^{n} \binom{2n+2}{n+k+2}, \\
E_n^{(1,1)} &= \sum_{k=0}^{n} \frac{(k+1)^2}{2n-k+1}\binom{2n-k+1}{n-k} 2^k, \\
F_n^{(1,1)} &= \sum_{k=0}^{n} \frac{(2k+3)(k+1)}{2n+3}\binom{2n+3}{n-k},
\end{aligned}$$

and the sequence $(\Omega_n^{(1,1)})_{n\geq 0}$ begins (A000346)

$$1, 5, 22, 93, 386, 1586, \cdots$$

(iii) If $m = 1$ and $r = 2$, then we have

$$\Omega_n^{(1,2)} = 4^{n+1} - \binom{2n+3}{n+1},$$

and the sequence $(\Omega_n^{(1,2)})_{n\geq 0}$ begins (A008549)

$$1, 6, 29, 130, 562, 2380, 9949, 41226, \cdots$$

(iv) If $m = 1$ and $r = 3$, then we have

$$\Omega_n^{(1,3)} = 2^{2n+3} - \binom{2n+3}{n} - \binom{2n+4}{n+2},$$

and the sequence $(\Omega_n^{(1,3)})_{n\geq 0}$ begins (A006419)

$$1, 7, 37, 176, 794, 3473, 14893, 63004, \cdots$$

Example 3.3. If $m = 2$ and $r = 1$, then we can obtain that

$$\Omega_n^{(2,1)} = A_n^{(2,1)} = B_n^{(2,1)} = E_n^{(2,1)} = F_n^{(2,1)},$$

where

$$\begin{aligned}
\Omega_n^{(2,1)} &= \sum_{k=0}^{n} \binom{2(n-k)}{n-k}\binom{3k+1}{k}, \\
A_n^{(2,1)} &= \sum_{k=0}^{n} 3^k \binom{3n-k+1}{2n+1}, \\
B_n^{(2,1)} &= \sum_{k=0}^{n} 2^k \binom{3n+2}{2n+k+2}, \\
E_n^{(2,1)} &= \sum_{k=0}^{n} \frac{(2k+1)(k+1)}{3n-k+1} \binom{3n-k+1}{n-k} 3^k, \\
F_n^{(2,1)} &= \sum_{k=0}^{n} \frac{(k+1)^2}{n+1} \binom{3n+3}{n-k} 2^k,
\end{aligned}$$

and the sequence $(\Omega_n^{(2,1)})_{n\geq 0}$ begins (A036829)

$$1, 7, 48, 327, 2221, 15060, 102012, \cdots$$

Example 3.4. If $m = 3$ and $r = 0$, then we have

$$\Omega_n^{(3,0)} = A_n^{(3,0)} = B_n^{(3,0)} = E_n^{(3,0)} = F_n^{(3,0)},$$

where

$$\begin{aligned}
\Omega_n^{(3,0)} &= \sum_{k=0}^{n} \binom{4(n-k)}{n-k}\binom{4k}{k}, \\
A_n^{(3,0)} &= \sum_{k=0}^{n} 4^k \binom{4n-k}{3n}, \\
B_n^{(3,0)} &= \sum_{k=0}^{n} 3^k \binom{4n+1}{3n+k+1}, \\
E_n^{(3,0)} &= \sum_{k=0}^{n} \frac{3k(k+1)}{4n-k} \binom{4n-k}{n-k} 4^k, \\
F_n^{(3,0)} &= \sum_{k=0}^{n} \frac{(2k+1)(k+1)}{2n+1} \binom{4n+2}{n-k} 3^k,
\end{aligned}$$

and the sequence $(\Omega_n^{(3,0)})_{n\geq 0}$ begins (A078995)

$$1, 8, 72, 664, 6184, 57888, \cdots$$

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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