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Revised algorithm for finding a common solution of variational inclusion and fixed point problems

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Abstract. Recent research has uncovered an algorithm for locating the common solution to variational inclusion problems with multivalued maximal monotone mapping and α-inverse strongly monotone mapping, as well as the points that are invariant under non-expansive mapping. In their algorithm, Zhang et al. [S. Zhang, J. H. W. Lee, C. K. Chan, *Algorithms of common solutions to quasi-variational inclusion and fixed point problems*, Appl. Math. Mech. **29(5)** (2008), 571–581.], λ must satisfy a very strict condition, namely $\lambda \in [0, 2\alpha]$; thus, it cannot be used for all Lipschitz continuous mappings, despite the fact that inverse strongly monotone implies Lipschitz continuous. This manuscript aims to define a new algorithm that addresses the flaws of the previously described algorithm. Our algorithm is used to solve minimization problems involving the fixed point set of a non-expansive mapping. In addition, we support all of our claims with numerical examples derived from computer simulation.

1. Introduction and preliminaries

Everywhere, in the paper $\mathcal{V}$ designates a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The variational inclusion problem consists of finding $v \in \mathcal{V}$ such that

$$0 \in (\mathcal{C} + \mathcal{B})v, \tag{1}$$

where $\mathcal{C} : \mathcal{V} \to \mathcal{V}$ and $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ are single and multivalued mappings, respectively. The solution set of problem (1) is denoted by $(\mathcal{C} + \mathcal{B})^{-1}0$.

Many researchers working in the field have given their algorithms for solving problems (1), like Lions and Mercier [12]. Their algorithm is as follows:

$$v_{m+1} = \mathcal{J}_{\lambda}^{\mathcal{B}}(v_m - \lambda \mathcal{C} v_m), \quad m = 0, 1, 2, \cdots . \tag{2}$$

Here $\mathcal{J}_{\lambda}^{\mathcal{B}}$ is known as resolvent operator for $\mathcal{B}$ and is defined as $\mathcal{J}_{\lambda}^{\mathcal{B}} := [I + \lambda\mathcal{B}]^{-1}$. The drawback of (2) is that it converges weakly to the solution of problem (1) when $\mathcal{C}$ is inverse strongly monotone.

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The drawback of weak convergence of (2) was overwhelmed by Takahashi et al. [21] by proposing the following algorithm: for $w \in \mathcal{K}$ (closed and convex subset of $\mathcal{V}$) and

$$v_{m+1} = \alpha_m w + (1-\alpha_m)\mathcal{J}^{\mathcal{B}}_{\lambda_m}(v_m - \lambda_m C(v_m)), \;\; m = 0,1,2,\cdots. \tag{3}$$

They proved strong convergence of the algorithm by imposing some restrictions on the sequences $\{\lambda_m\}$ and $\{\alpha_m\}$.
In the case of Banach spaces, Lopez et al. [13] gave a strong convergent algorithm for solving the problem (1). Their algorithm has the following form: $v_1, w \in E$(Banach space) and

$$v_{m+1} = \alpha_m w + (1-\alpha_m)\left(\mathcal{J}^{\mathcal{B}}_{\lambda_m}(v_m - r_m(Cv_m + a_m) + b_m\right), \;\; m = 0,1,2,\cdots. \tag{4}$$

By imposing some restrictions on sequences $\{\lambda_m\}$, $\{\alpha_m\}$, $\{a_m\}$ and $\{b_m\}$, they proved its strong convergence. Recently many researchers worked on different algorithms dealing with such problems. For more synthesis on this topic, one may refer to [2, 8–10, 16–18, 20, 23, 24, 27].
On the other hand, the fixed point problem consists of finding

$$v^* \in \mathcal{K}: \;\; v^* = Av^*, \tag{5}$$

where $\mathcal{K}$ is a nonempty, closed, and convex subset of a Hilbert space $\mathcal{V}$ and $A : \mathcal{K} \to \mathcal{K}$ is a nonexpansive mapping. The solution set of (5) is denoted by $P(A)$. There are many iterative procedures for approximating fixed points of problem (5), like Mann's iteration [14], which is as follows: $v_1 \in \mathcal{K}$ and

$$v_{m+1} = \alpha_m v_m + (1-\alpha_m)Av_m, \;\; m = 0,1,2,\cdots.$$

Here $\{\alpha_m\} \subseteq [0,1]$. A detailed synthesis of fixed point problems and their applications can be found in the noteworthy manuscripts [3, 15, 25].
Halpern [7] also gave an iteration scheme which is defined as $v_1 = v \in K$

$$v_{m+1} = \alpha_m v + (1-\alpha_m)Av_m, \;\; m = 0,1,2,\cdots.$$

Here also $\{\alpha_m\} \subseteq [0,1]$.
For the last several years, researchers are finding a common solution to problems (1) and (5) like Zhang et al. [26]. They designed the following algorithm:

$$\begin{cases} v_{m+1} = \beta_m v + (1-\beta_m)Aw_m, \\ w_m = \mathcal{J}^{\mathcal{B}}_{\lambda}(v_m - \lambda C v_m), \;\; m = 0,1,2,\cdots. \end{cases} \tag{6}$$

Here A is α-inversely strongly monotone and $\lambda \in (0, 2\alpha]$ and sequence $\beta_m \subseteq [0,1]$ has the following restrictions:

(i) $\beta_m \to 0$, $\sum_{m=0}^{\infty} \beta_m = \infty$,

(ii) $\sum_{m=0}^{\infty} |\beta_{m+1} - \beta_m| < \infty$.

They show that $\{v_m\}$ converges strongly to $P(A) \cap (C + \mathcal{B})^{-1}0$.

In this paper, we have modified algorithm (6). The merits of our algorithm over algorithm (6) are as follows:

Remark 1.1. *(i) Our algorithm can be used for all types of Lipschitz continuous functions while algorithm* (6) *cannot be used for all types of Lipschtiz continuous functions, see Example 3.1 and Figures 1 and 2.*

(ii) Our algorithm can be used for $\lambda \in \mathbb{R}^+$ while algorithm (6) *has a very strict condition on λ, that is, $\lambda \in (0, 2\alpha]$, see Example 3.2 and Figures 3 and 4.*

Let us go back to some earlier definitions and results which we use in this paper.

Definition 1.2. *[11] If there exists a constant $\alpha \in \mathbb{R}^+$ such that*

$$\alpha\|Cv - Cw)\|^2 \leq \langle Cv - Cw, v - w\rangle\,, \ \forall \ v, w \in \mathcal{V}.$$

Then $C : \mathcal{V} \to \mathcal{V}$ is known as an α-inverse strongly monotone mapping.
If $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ satisfies,

$$0 \leq \langle x - y, v - w\rangle\,, \ \forall \, v, w \in \mathcal{V}, \ x \in \mathcal{B}v \ \text{and} \ \ y \in \mathcal{B}w.$$

Then it is called monotone, and if it is monotone and $(I + \lambda\mathcal{B})v = \mathcal{V}$ for $\lambda \in \mathbb{R}^+$, where I is the identity mapping, then it is called a maximal monotone.

Definition 1.3. *[11] The single-valued mapping $\mathcal{J}_\lambda{}^{\mathcal{B}} : \mathcal{V} \to \mathcal{V}$ defined by*

$$\mathcal{J}_\lambda{}^{\mathcal{B}}(v) = [I + \lambda\mathcal{B}]^{-1}v, \ \forall \, v \in \mathcal{V},$$

is known as resolvent operator for $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$.

Definition 1.4. *[19] If the mapping $A : \mathcal{V} \to \mathcal{V}$ satisfies*

$$\|v - w\| \geq \|Av - Aw\|, \ \forall \ v, w \in \mathcal{V}.$$

Then it is called nonexpansive.
The resolvent operator is nonexpansive, that is,

$$\|v - w\| \geq \|\mathcal{J}_\lambda{}^{\mathcal{B}}(v) - \mathcal{J}_\lambda{}^{\mathcal{B}}(w)\|.$$

Definition 1.5. *[19] Let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{V}$. Then for any $v \in \mathcal{V}$, there exists one and only one nearest point in $\mathcal{K}$, known as a metric projection of v on $\mathcal{K}$ and is denoted by $proj_{\mathcal{K}}v$ i.e.*

$$\|v - proj_{\mathcal{K}}v\| \leq \|v - w\|, \ \ \forall w \in \mathcal{K}.$$

Remark 1.6. *The following characteristic properties are owned by metric projection $proj_{\mathcal{K}}$:*

(i) $proj_{\mathcal{K}} : \mathcal{V} \to \mathcal{K}$ is nonexpansive, that is,

$$\|proj_{\mathcal{K}}(v) - proj_{\mathcal{K}}(w)\| \leq \|v - w\|, \ \ \forall \, v, w \in \mathcal{V};$$

(ii) $proj_{\mathcal{K}}$ is firmly nonexpansive, that is,

$$\|proj_{\mathcal{K}}(v) - proj_{\mathcal{K}}(w)\|^2 \leq \langle proj_{\mathcal{K}}(v) - proj_{\mathcal{K}}(w), v - w\rangle, \ \ \forall \, v, w \in \mathcal{V};$$

(iii) For each $v \in \mathcal{V}$, $u = proj_{\mathcal{K}}(v)$ if and only if

$$\langle v - u, u - w\rangle \geq 0, \ \ \forall \, w \in \mathcal{K}.$$

Lemma 1.7. *[4] A mapping $\mathcal{B} + C : \mathcal{V} \to 2^{\mathcal{V}}$ is maximal monotone, if $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ is maximal monotone and $C : \mathcal{V} \to \mathcal{V}$ is Lipschitz continuous.*

With the help of the above lemma, we define a new resolvent operator as follows:

Definition 1.8. *Let $C : \mathcal{V} \to \mathcal{V}$ be a Lipschitz continuous mapping and $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ be a maximal monotone operator. Then a new resolvent operator of the maximal monotone operator $\mathcal{B} + C$ can be defined by:*

$$\mathcal{J}_\lambda^{\mathcal{B}+C}(v) = [I + \lambda(\mathcal{B} + C)]^{-1}v, \ \ \forall \, v \in \mathcal{V}. \tag{7}$$

Remark 1.9. *The resolvent operator given by* (7) *is nonexpansive and* 1*-inverse strongly monotone.*

Lemma 1.10. *[26] Let $\{\alpha_m\}, \{\beta_m\}$ and $\{\gamma_m\}$ be three nonnegative real sequences satisfying the following condition:*

$$\alpha_{m+1} \leq (1-\lambda_m)\alpha_m + \beta_m + \gamma_m, \ \ \forall\, m \geq m_0,$$

where m_0 is some nonnegative integer, $\{\lambda_m\}$ is a sequence in $(0,1)$ with $\sum_{m=0}^{\infty} \lambda_m = \infty$, $\beta_m = o(\lambda_m)$ and $\sum_{m=0}^{\infty} \gamma_m < \infty$. Then $\lim_{m\to\infty} \alpha_m = 0$.

Lemma 1.11. *[5] If $\mathcal{V}$ is a real Hilbert space, then*

$$\|v_1 + v_2\|^2 \leq \|v_1\|^2 + 2\langle v_2, v_1 + v_2\rangle, \ \ \forall\, v_1, v_2 \in \mathcal{V}.$$

2. Main Result

In this section, we put forward a new algorithm and use it to get a solution that is common to both variational inclusion problem (1) and problem (5). In order to prove the main result we need the following lemma:

Lemma 2.1. *$v = \mathcal{J}_\lambda^{C+\mathcal{B}}(v)$ for all $\lambda \in \mathbb{R}^+$ if and only if $v \in \mathcal{V}$ satisfies* (1).

Proof. If $v \in \mathcal{V}$ is a solution of problem (1), then for $\lambda \in \mathbb{R}^+$, $0 \in \lambda(C+\mathcal{B})v$, and hence $v \in [I + \lambda(C+\mathcal{B})v]$. Therefore, we have

$$v = [I + \lambda(C+\mathcal{B})]^{-1}v = \mathcal{J}_\lambda^{C+\mathcal{B}}(v).$$

The converse implication is also obvious. □

Theorem 2.2. *Let $C : \mathcal{V} \to \mathcal{V}$, $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ and $A : \mathcal{V} \to \mathcal{V}$ be Lipschitz continuous, maximal monotone and non-expansive mappings, respectively. Suppose that $P(A) \cap (C+\mathcal{B})^{-1}0 \neq \emptyset$. Let $v = v_0 \in \mathcal{V}$ and $\{v_m\}$ be the sequence generated by*

$$\begin{cases} v_{m+1} = \beta_m v + (1-\beta_m)Aw_m, \\ w_m = \mathcal{J}_\lambda^{C+\mathcal{B}}(v_m), \ \ m = 0,1,2,\cdots. \end{cases} \tag{8}$$

satisfying the following conditions:

(i) $\beta_m \to 0$, $\sum_{m=0}^{\infty} \beta_m = \infty$,

(ii) $\sum_{m=0}^{\infty} |\beta_{m+1} - \beta_m| < \infty$.

Then $\{v_m\}$ converges strongly to a point of $P(A) \cap (C+\mathcal{B})^{-1}0$.

Proof. The result is proved in six steps:

Step 1. First, we show that the sequences $\{v_m\}$ and $\{w_m\}$ are bounded.
For $z \in P(A) \cap (C+\mathcal{B})^{-1}0$ and from Lemma 2.1, we have

$$z = \mathcal{J}_\lambda^{C+\mathcal{B}}(z).$$

So, we have

$$\begin{aligned} \|w_m - z\| &= \|\mathcal{J}_\lambda^{C+\mathcal{B}}(v_m) - \mathcal{J}_\lambda^{C+\mathcal{B}}(z)\| \\ &\leq \|v_m - z\|, \ \ \forall m \geq 0. \end{aligned} \tag{9}$$

Using (8) and (9), we can write

$$\begin{aligned}
\|v_{m+1} - z\| &= \|\beta_m(v - z) + (1 - \beta_m)(A_{w_m} - z)\| \\
&\leq \beta_m\|v - z\| + (1 - \beta_m)\|w_m - z\| \\
&\leq \beta_m\|v - z\| + (1 - \beta_m)\|v_m - z\| \\
&\leq \max\{\|v - z\|, \|v_m - z\|\} \\
&\vdots \\
&\leq \max\{\|v - z\|, \|v_0 - z\|\} \\
&= \|v - z\|. \qquad (10)
\end{aligned}$$

From above inequality (10) we conclude that the sequences $\{v_m\}$ and $\{w_m\}$ are bounded. Since A is nonexpansive and C is Lipschitz continuous, $\{Cv_m\}$ and $\{Aw_m\}$ are also bounded in $\mathcal{V}$.

Step 2. We prove that

$$\|v_{m+1} - v_m\| \to 0 \text{ and } \|w_{m+1} - w_m\| \to 0 \text{ as } m \to 0. \qquad (11)$$

We note that

$$\begin{aligned}
\|w_{m+1} - w_m\| &= \|\mathcal{J}_\lambda^{C+\mathcal{B}}(v_{m+1}) - \mathcal{J}_\lambda^{C+\mathcal{B}}(v_m)\| \\
&\leq \|v_{m+1} - v_m\|. \qquad (12)
\end{aligned}$$

Hence from (8) and (12), we obtain

$$\begin{aligned}
\|v_{m+1} - v_m\| &= \|\beta_m v + (1 - \beta_m)Aw_m - (\beta_{m-1}v + (1 - \beta_{m-1})Aw_{m-1})\| \\
&= \|(\beta_m - \beta_{m-1})(v - Aw_{m-1}) + (1 - \beta_m)(Aw_m - Aw_{m-1})\| \\
&\leq |\beta_m - \beta_{m-1}|\|v - Aw_{m-1}\| + (1 - \beta_m)\|Aw_m - Aw_{m-1}\| \\
&\leq M|\beta_m - \beta_{m-1}| + (1 - \beta_m)\|w_m - w_{m-1}\| \\
&\leq M|\beta_m - \beta_{m-1}| + (1 - \beta_m)\|v_m - v_{m-1}\|, \qquad (13)
\end{aligned}$$

where $M = \sup_{m\geq 1}\|v - Aw_{m-1}\|$. We see that all the conditions of Lemma 1.10 are satisfied by taking $e_m = \|v_m - v_{m-1}\|$, $f_m = 0$ and $g_m = M|\beta_m - \beta_{m-1}|$ and it is clear from Lemma 1.10 that $\|v_{m+1} - v_m\| \to 0$ as $m \to 0$. And also, from (12) we have $\|w_{m+1} - w_m\| \to 0$ as $m \to 0$.

Step 3. We prove that for $z \in P(A) \cap (C + \mathcal{B})^{-1}0$,

$$\|v_m - Aw_m\| \to 0 \text{ as } m \to \infty. \qquad (14)$$

We note that

$$\begin{aligned}
\|v_m - Aw_m\| &\leq \|v_m - Aw_{m-1}\| + \|Aw_{m-1} - Aw_m\| \\
&\leq \beta_{m-1}\|v - Aw_{m-1}\| + \|w_{m-1} - w_m\|. \qquad (15)
\end{aligned}$$

Since $\beta_m \to 0$ and $\|w_{m-1} - w_m\| \to 0$, we have that $\|v_m - Aw_m\| \to 0$.

Step 4. We prove that

$$\|v_m - w_m\| \to 0 \text{ and } \|Aw_m - w_m\| \to 0. \qquad (16)$$

For $z \in P(A) \cap (C + \mathcal{B})^{-1}0$ and using Remark 1.9, Lemma 2.1 and equation (8), we obtain

$$\begin{aligned} \|w_m - z\|^2 &= \|\mathcal{J}_\lambda^{C+\mathcal{B}}(v_m) - \mathcal{J}_\lambda^{C+\mathcal{B}}(z)\|^2 \\ &\leq \langle v_m - z, \mathcal{J}_\lambda^{C+\mathcal{B}}(v_m) - \mathcal{J}_\lambda^{C+\mathcal{B}}(z)\rangle \\ &= \langle v_m - z, w_m - z\rangle \\ &= \frac{1}{2}\left\{\|v_m - z\|^2 + \|w_m - z\|^2 - \|v_m - z - (w_m - z)\|^2\right\} \\ &\leq \frac{1}{2}\left\{\|v_m - z\|^2 + \|v_m - z\|^2 - \|v_m - w_m\|^2\right\}. \end{aligned}$$

So, we get

$$\|w_m - z\|^2 \leq \|v_m - z\|^2 - \frac{1}{2}\|v_m - w_m\|^2. \tag{17}$$

So, Using (8) and (17), we have

$$\begin{aligned} \|v_{m+1} - z\|^2 &= \|\beta_m(v - z) - (1 - \beta_m)(Aw_m - z)\|^2 \\ &\leq \beta_m\|v - z\|^2 + (1 - \beta_m)\|Aw_m - z\|^2 \\ &\leq \beta_m\|v - z\|^2 + (1 - \beta_m)\|w_m - z\|^2 \\ &\leq \beta_m\|v - z\|^2 + (1 - \beta_m)\left\{\|v_m - z\|^2 - \frac{1}{2}\|v_m - w_m\|^2\right\}. \end{aligned}$$

This implies that

$$\frac{(1 - \beta_m)}{2}\|v_m - w_m\|^2 \leq \beta_m\|v - z\|^2 + (\|v_m - z\|^2 - \|v_{m+1} - z\|^2). \tag{18}$$

Since $\beta_m \to 0$ and

$$|\|v_m - z\|^2 - \|v_{m+1} - z\|^2| \leq \|v_{m+1} - v_m\|(\|v_m\| + \|v_{m+1}\|) \to 0,$$

from (18), $\|v_m - w_m\| \to 0$. Also from (14) we obtain

$$\|Aw_m - w_m\| \leq \|Aw_m - v_m\| + \|v_m - w_m\| \to 0.$$

Step 5. We prove that

$$\limsup_{m\to\infty}\langle v - q, Aw_m - q\rangle \leq 0, \tag{19}$$

where $q = proj_{P(A)\cap(C+\mathcal{B})^{-1}0}(v)$.

Since $\{w_m\}$ is a bounded sequence in $\mathcal{V}$, there exists a subsequence $\{w_{m_i}\} \subset \{w_m\}$ such that $w_{m_i} \rightharpoonup w \in \mathcal{V}$ and

$$\limsup_{m\to\infty}\langle v - q, Aw_m - q\rangle = \lim_{m_i\to\infty}\langle v - q, Aw_{m_i} - q\rangle. \tag{20}$$

Since $\|Aw_m - w_m\| \to 0$, $\|Aw_{m_i} - w_{m_i}\| \to 0$ and A is nonexpansive, $I - A : \mathcal{V} \to \mathcal{V}$ is demiclosed, so we have $Aw = w$, that is, $w \in P(A)$.

Now we prove that

$$w \in (C + \mathcal{B})^{-1}0. \tag{21}$$

Since C is Lipschitz continuous and $\mathcal{B}$ is maximal monotone, by Lemma 1.7, $C + \mathcal{B}$ is maximal monotone. Let $(a, b) \in Graph(C + \mathcal{B})$, that is, $b \in (C + \mathcal{B})a$. Since $w_{m_i} = \mathcal{J}_\lambda^{C+\mathcal{B}}(v_{m_i})$, we have $v_{m_i} \in [I + (C + \mathcal{B})]w_{m_i}$, that is,

$$\frac{1}{\lambda}(v_{m_i} - w_{m_i}) \in (C + \mathcal{B})w_{m_i}.$$

So, by maximal monotonicity of $(C + \mathcal{B})$, we have

$$\left\langle a - w_{m_i}, b - \frac{1}{\lambda}(v_{m_i} - w_{m_i}) \right\rangle \geq 0.$$

Hence we have

$$\langle a - w_{m_i}, b \rangle \geq \left\langle a - w_{m_i}, \frac{1}{\lambda}(v_{m_i} - w_{m_i}) \right\rangle. \tag{22}$$

Since $\|v_{m_i} - w_{m_i}\| \to 0$ and $w_{m_i} \rightharpoonup w$, we get

$$\lim_{m_i \to \infty} \langle a - w_{m_i}, b \rangle = \langle a - w, b \rangle \geq 0.$$

Because $C + \mathcal{B}$ is maximal monotone, this implies that $0 \in (C + \mathcal{B})w$, that is, $w \in (C + \mathcal{B})^{-1}0$. So $w \in P(A) \cap (C + \mathcal{B})^{-1}0$

Since $\|Aw_m - w_m\| \to 0$ and $w_{m_i} \rightharpoonup w \in P(A) \cap (C + \mathcal{B})^{-1}0$, from (20) and Remark 1.6, we get

$$\begin{aligned}
\limsup_{m \to \infty} \langle v - q, Aw_m - q \rangle &= \lim_{m_i \to \infty} \langle v - q, Aw_{m_i} - q \rangle \\
&= \lim_{m_i \to \infty} \langle v - q, Aw_{m_i} - w_{m_i} + w_{m_i} - q \rangle \\
&= \lim_{m_i \to \infty} \langle v - q, w - q \rangle \\
&\leq 0.
\end{aligned}$$

Hence (19) is proved.

Step 6. Finally we prove that

$$v_m \to q = proj_{P(A) \cap (C+\mathcal{B})^{-1}0}(v_0). \tag{23}$$

Using (8), (9) and Lemma 1.11, we obtain

$$\begin{aligned}
\|v_{m+1} - q\|^2 &= \|\beta_m(v - q) + (1 - \beta_m)(Aw_m - q)\|^2 \\
&\leq (1 - \beta_m)^2\|(Aw_m - q)\|^2 + 2\beta_m\langle v - q, v_{m+1} - q \rangle \\
&\leq (1 - \beta_m)^2\|w_m - q)\|^2 + 2\beta_m\langle v - q, v_{m+1} - q \rangle \\
&\leq (1 - \beta_m)^2\|v_m - q)\|^2 + 2\beta_m\langle v - q, v_{m+1} - q \rangle.
\end{aligned} \tag{24}$$

Let

$$\gamma_m = \max\{0, \langle v - q, v_{m+1} - q \rangle\}.$$

Then $\gamma_m \geq 0$.

Now we prove that $\gamma_m \to 0$ as $m \to \infty$.

It follows from (19) that for given $\delta > 0$, there exists m_0 such that

$$\langle v - q, v_{m+1} - q \rangle < \delta.$$

So, we have

$$0 \leq \gamma_m < \delta, \quad \forall m \geq m_0.$$

By the arbitrariness of $\delta > 0$, we get $\gamma_m \to 0$. So we can write (24) as follows:

$$|v_{m+1} - q\|^2 \leq (1 - \beta_m)^2\|v_m - q)\|^2 + 2\beta_m\gamma_m. \tag{25}$$

By taking $e_m = |v_{m+1} - q\|^2$, $f_m = 2\beta_m\gamma_m$ and $g_m = 0$, then all the conditions of the Lemma 1.10 are satisfied. Hence $v_m \to q$ as $m \to \infty$. This completes the proof. □

3. Numerical Examples

In this section we show through numerical examples that our algorithm (8) has merits over algorithm (6). All codes are written in MATLAB 2012.

Example 3.1. *Let $\mathcal{V} = \mathbb{R}$ be the set of all real numbers and let $C : \mathbb{R} \to \mathbb{R}$, $A : \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R} \to 2^{\mathbb{R}}$ be defined as $Cv = -6v$ for all $v \in \mathbb{R}$, $Av = v$ and $\mathcal{B}v = \{3v\}$ for all $v \in \mathbb{R}$. Then C is Lipschitz continuous but is not inverse strongly monotone. It is clear that Zhang et al.'s algorithm* (6) *is not applicable, see Fig. 1 while our algorithm* (8) *is applicable, see Fig. 2.*

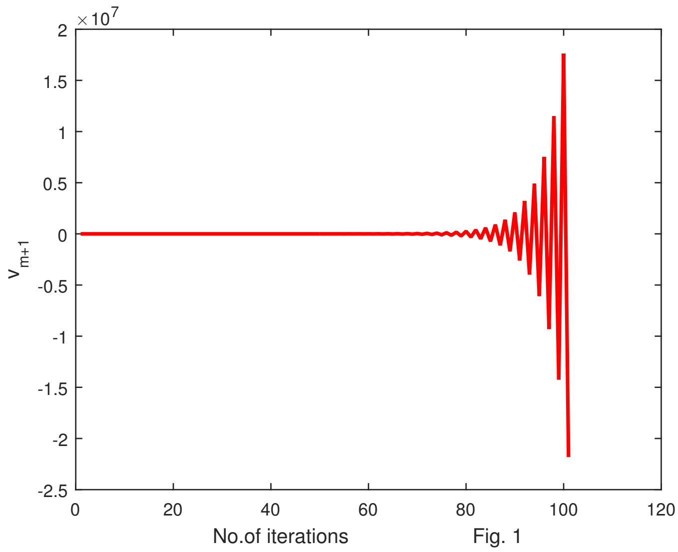

Figure 1: $\{v_m\}$ does not converge for using algorithm (6).

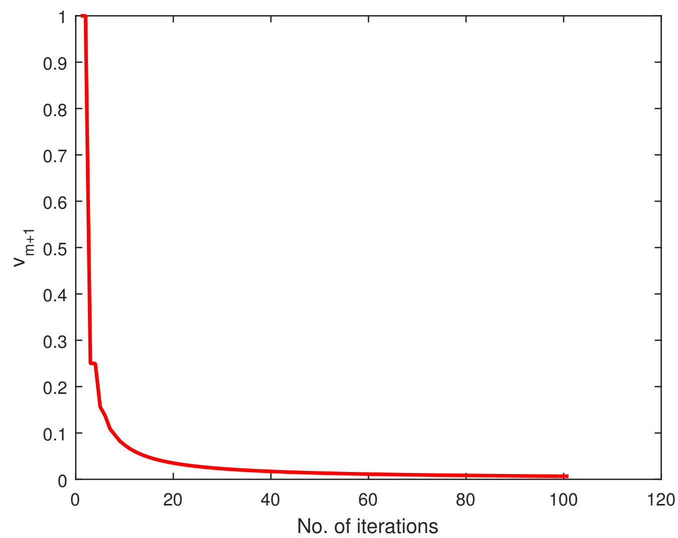

Figure 2: $\{v_m\}$ converges for using our algorithm (8).

Example 3.2. *Let $\mathcal{V} = \mathbb{R}$ be the set of all real numbers and let $C : \mathbb{R} \to \mathbb{R}$, $A : \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R} \to 2^{\mathbb{R}}$ be defined as $Cv = 2v$ for all $v \in \mathbb{R}$, $Av = v$ and $\mathcal{B}v = \{\frac{v}{2}\}$ for all $v \in \mathbb{R}$. Then C is $\frac{1}{2}$-inversely strongly monotone. It is clear that Zhang et al.'s algorithm* (6) *is not applicable when $\lambda = 2$, that is, $\lambda \notin (0, 2\alpha]$, see Fig. 3 while our algorithm* (8) *is applicable even though $\lambda \notin (0, 2\alpha]$, see Fig. 4.*

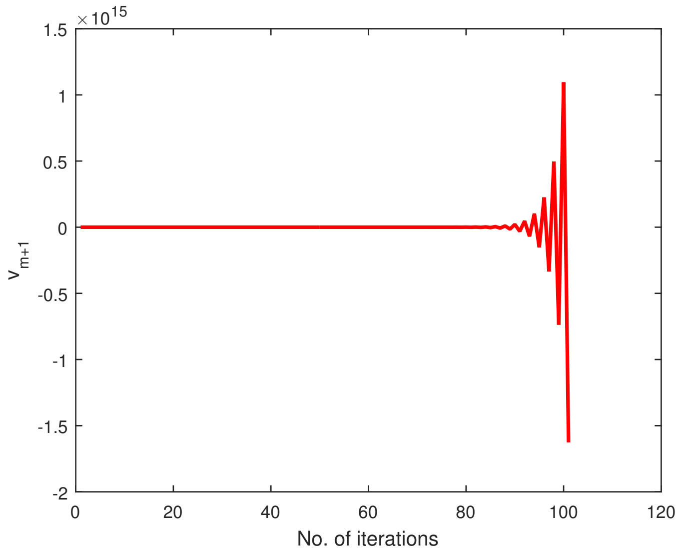

Figure 3: $\{v_m\}$ does not converge using algorithm (6) by taking $\lambda = 2$.

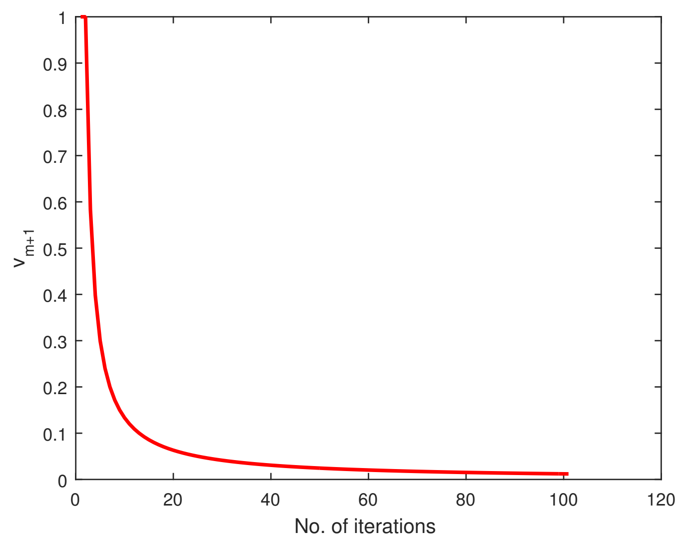

Figure 4: $\{v_m\}$ converges using our algorithm (8) by taking $\lambda = 2$

4. Application

We know that the minimization problem can be converted into an equivalent to the variational inclusion problem. So, our algorithm can be used to solve minimization problem over the fixed point set of a non-

expansive mapping.
Let $\mathcal{V} = \mathbb{R}$, a Hilbert space, $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(v) = \frac{v^2}{4}$, $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semi-continuous function defined as $g(v) = v^2$ and $A : \mathbb{R} \to \mathbb{R}$ by $Av = v$, a non-expansive mapping. Then our aim is to find $v^* \in P(A)$ such that

$$\{f(v^*) + g(v^*)\} = \min_{v \in P(A)} \{f(v) + g(v)\}, \tag{26}$$

which is equivalent (By Fermat's Rule) to find $v \in P(A)$ such that

$$0 \in \frac{v}{2} + 2v. \tag{27}$$

Then above problem is same as finding common solution of problems (1) and (5) by taking $Cv = \frac{v}{2}$, $\mathcal{B}v = 2v$, $Av = v$ and $K = [-1, 1]$.
Then solution set of Problem (26) or problem (27) is $\{0\}$ (see Fig.5).

Using our algorithm (8), we can easily see from Fig.6 that sequence converges to 0 (solution) for different initial values.

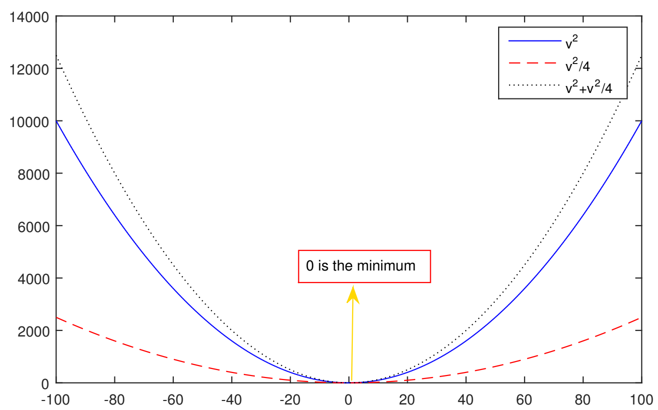

Figure 5: Solution of the problem (4.1) is 0.

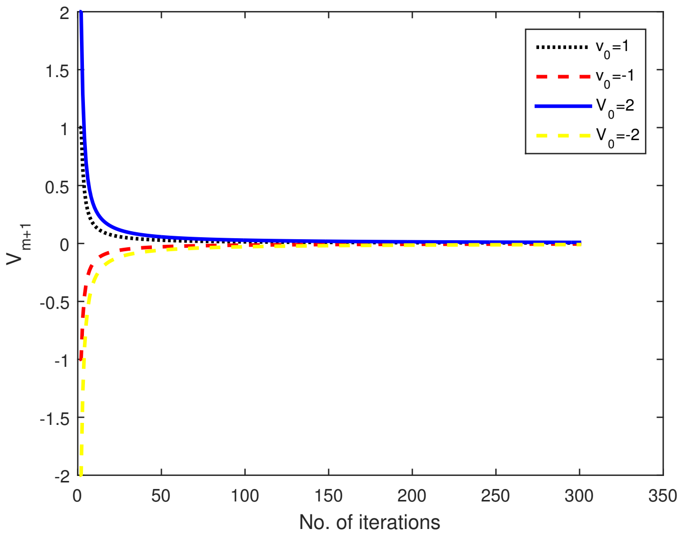

Figure 6: $\{v_m\}$ converges to 0(common solution) while starting with different initial values $v_0 = 1, -1, 2$ *and* -2

5. Conclusion

We presented a novel approach that corrects the weaknesses in the Zhang et al. [26] algorithm. We also defined a novel way to correct the weaknesses in the pronounced algorithm. We also use computer modeling to back up our claims with numerical evidence. In comparison to the results currently available in the current state-of-the-art, the results presented in this article are sharp.

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