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Finite sum of weighted composition operators with closed range

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Abstract. In this paper, first we characterize closedness of range of the finite sum of weighted composition operators between different L^p-spaces. Then we discuss polar decomposition and invertibility of these operators.

1. introduction

Weighted composition operators are a general class of operators and they appear naturally in the study of surjective isometries on most of the function spaces, semigroup theory, dynamical systems, Brennans conjecture, etc. This type of operators are a generalization of multiplication operators and composition operators.
There are many great papers on the investigation of weighted composition operators acting on the spaces of measurable functions. For instance, one can see [2–5, 7, 8, 10, 15, 17, 18]. Also, some basic properties of weighted composition operators on L^p-spaces were studied by Parrott [12], Nordgern [11], Singh and Manhas [16], Takagi [19] and some other mathematicians. As far as we know finite sum of weighted composition operators were studied on L^p-spaces by Jabbarzadeh and Estaremi in [6]. Also we investigated some basic properties of these operators in [15].

Let (X, Σ, μ) be a σ-finite measure space. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. For any σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ such that $(X, \mathcal{A}, \mu_{\mathcal{A}})$ is also σ-finite , the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma)$, $1 \le p \le \infty$, where $E^{\mathcal{A}} f$ is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad A \in \mathcal{A}.$$

As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. For more details on the properties of $E^{\mathcal{A}}$ see [9] and [13]. For a measurable function $u : X \to C$ and non-singular measurable transformation $\varphi : X \to X$, i.e, the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, we can define an operator $uC_\varphi : L^p(\Sigma) \to L^0(\Sigma)$ with $uC_\varphi(f) = u.f \circ \varphi$ and it is called a weighted composition operator. For non-singular measurable transformations $\{\varphi_i\}_{i=1}^n$, we put $W = \sum_{i=1}^n u_i C_{\varphi_i}$.
In this paper, we are going to give some sufficient and necessary condition for closedness of range of finite sum of weighted composition operators between different L^p-spaces. Moreover, we compute the polar decomposition of these operators on L^2. Finally we talk a bit about invertibility and injectivity.

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2. Main results

In this section first we give an equivalent condition for closedness of range on the Hilbert space L^2.

Theorem 2.1. *Let $W = \sum_{i=1}^{n} u_i C_{\varphi_i}$ be a bounded operator on $L^2(\mu)$ and $u_i(\varphi_j^{-1}) = 0, \quad i \neq j$. The following statements are equivalent.*

(a) W has closed range.

(b) There is a constant $c > 0$ such that $J = \sum_{i=1}^{n} h_i E_i(|u_i|^2) \circ \varphi_i^{-1} \geq c \quad \mu - a.e$ on $CozJ = \{x \in X : J(x) \neq 0\}$.

Proof. $(b) \Rightarrow (a)$ Suppose that there is some constant $c > 0$ such that $J \geq c, \quad \mu - a.e$ on $CozJ$. We know that $\ker W \subseteq L^2_{|X\setminus CozJ}(\mu)$. Since $W^*Wf = Jf$ for every $f \in L^2(\mu)$,

$$\begin{aligned} \|Wf\|_2^2 &= (Wf, Wf) \\ &= (W^*Wf, f) \\ &= \int_X J|f|^2 d\mu = \int_{CozJ} J|f|^2 d\mu + \int_{X\setminus CozJ} J|f|^2 d\mu \\ &\geq c\|f\|_2^2. \end{aligned}$$

Obviously $W_{|J}$ is injective and $W_{|J}(L^2_{|J}(\mu))$ is closed in $L^2(\mu)$, where $L^2_{|J}(\mu) = \{f \in L^2(\mu)\ ;\ f = 0 \ on \ X\setminus J\}$. Since $\ker W = L^2_{|X\setminus CozJ}(\mu)$, $W(L^2(\mu))$ must be closed in $L^2(\mu)$.

$(a) \Rightarrow (b)$ Assume W has closed range. Then $W_{|J}(L^2_{|J}(\mu))$ is closed in $L^2(\mu)$. Since $W_{|J}$ is injective so there exists a constant $d > 0$ such that $\|W_{|J}\|_2 \geq d\|f\|_2$, for any $f \in L^2(\mu)$. Take $c = \frac{d^2}{n}$, (b) follows immediately once we show that for any $E \in \Sigma$ with $E \subset CozJ$, $\int_E J d\mu \geq c\mu(E)$. PicK any $E \in \Sigma$ with $E \subset J$. We may assume $\mu(E) < \infty$. Then $\chi_E \in L^2_{|J}(\mu)$ and $n\int_E J d\mu = n\int_X J\chi_E d\mu \geq \|W_{|J}\chi_E\| \geq d^2\|\chi_E\|_2^2 = d^2\mu(E)$ so $\int_E J d\mu \geq c\mu(E)$. □

Now we find some necessary and sufficient conditions for closedness of range when the operator act on the L^p with $p > 1$. Recall that an atom of the measjure μ is an element $A \in \sigma$ with $\mu(A) > 0$ such that for each $F \in \sigma$, if $F \subset A$ then either $\mu(F) = 0$ or $\mu F = \mu(A)$. A measjure space (X, σ, μ) with no atoms is called non-atomic meajsure space.

Theorem 2.2. *Let $W = \sum_{i=1}^{m} u_i C_{\varphi_i}$ be a bounded operator on $L^p(\mu)$ with $p > 1$. Then the followings hold.*

(a) If $J(B) = 0, \quad \mu - a.e$ and $\sum_{i\in\mathbb{N}} J(A_i)\mu(A_i) < \infty$ then W has closed range.

(b) If W has closed range and is injective then $J(B) = 0, \quad \mu - a.e$.

(c) Let $\mu(X) < \infty$. If W has closed range and is injective then there exists a constant $\delta > 0$ such that $u = \sum_{i=1}^{n} u_i^p \geq \delta$ ox X.

Proof. (a) Take any sequence $(Wf_n)_{n\in\mathbb{N}}$ in $W(L^p(\mu))$ with $\|f_n\| < 1$ and $\|Wf_n - g\| \to 0$. For a fixed $i \in \mathbb{N}$ the sequence $(f_n(A_i))_{n\in\mathbb{N}}$ is bounded by $\frac{1}{\sqrt[p]{\mu(A_i)}}$. So we can find a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that with each fixed i, $f_{n_k}(A_i) \to \alpha_i$ for some $\alpha_i \in \mathbb{C}$. Define $f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}$. By Fatous lemma we have $\int_X |f|^p d\mu \leq \liminf_{k\to\infty} \int_X |f_{n_k}|^p d\mu \leq 1$, for $f \in L^p(\mu)$. Then for each $\epsilon > 0$, we have

$$\begin{aligned} \|g - Wf\|_p &\leq \|g - Wf_n\|_p + \|Wf_n - Wf_{n_k}\|_p + \|Wf_{n_k} - Wf\|_p \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \int_X |W(f_{n_k} - f)|^p d\mu \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + m^{p-1}\int_X J|f_{n_k} - f|^p d\mu \\ &= \frac{\epsilon}{3} + \frac{\epsilon}{3} + m^{p-1}\sum_{i=1}^{\infty} J(A_i)|f_{n_k}(A_i) - \alpha_i|^p \mu(A_i) \\ &\longrightarrow 0 \end{aligned}$$

Obviously $W(L^p(\mu))$ is closed in $L^p(\mu)$.
(b) Suppose on the contrary, $\mu(\{x \in B \ : \ J(x) > 0\}) > 0$. Then there exists $\delta > 0$ such that the set $G = \{x \in B \ : \ J(x) \geq \delta\}$ has positive measure. We assume $\mu(G) < \infty$. Moreover, as G is non atomic, we can further assume that $\mu(X\backslash G) > 0$. Consider the Banach space $L^p_{|_G}(\mu)$ and the operator $W_{|_G}$ defined on $L^p_{|_G}(\mu)$. We claim that $W_{|_G}(L^p_{|_G}(\mu))$ is closed in $L^p(\mu)$. To prove we take any convergent sequence $(W_{|_G}(f_n))_{n\in\mathbb{N}}$ in $W\ |_G\ (L^p_{|_G}(\mu))$. Let $g \in L^p(\mu)$ satisfy $\|W_{|_G}(f_n) - g\|_p \to 0$ as $n \to \infty$. Note that $(W_{|_G}(f_n))_{n\in\mathbb{N}}$ can be regarded as a sequence in $W(L^p(\mu))$. The closedness of range of W yields an $f \in L^p(\mu)$ with $g = Wf \ \ \mu - a.e$ On X. Then assume W has closed range and is injective so there exists a constant $d > 0$ such that $\|W_{|_G}(f_n) - Wf\|_p \geq d\|f_n - f\|_p$. As $\|W_{|_G}(f_n) - g\|_p = \|W_{|_G}(f_n) - Wf\|_p = 0$ and $\|f_n - f\|_p^p = \int_G |f_n - f|^p d\mu + \int_{X\backslash G} |f_n - f|^p d\mu$ we have that $\int_{X\backslash G} |f|^p d\mu = 0$ and so $f \in L^p_{|_G}(\mu)$. Then there exists some constant $c > 0$ such that $\|W_{|_G} f\|_p \geq c\|f\|_p$, for all $f \in L^p_{|_G}(\mu)$. We claim that this is impossible by showing that for any $\alpha > 0$, there is some $f_\alpha \in L^p_{|_G}(\mu)$ satisfying $\|W_{|_G} f\|_p < c\|f\|_p$. For any $n \in \mathbb{N}$, define $G_n = \{x \in G \ ; \ \left(\frac{(n-1)\alpha}{m^{\frac{p-1}{p}}}\right)^p \leq J(x) \leq \left(\frac{n\alpha}{m^{\frac{p-1}{p}}}\right)^p\}$. Then $G = (\cup_{n\in\mathbb{N}} G_n) \cup \{x \in G \ ; \ J(x) = \infty\}$. Since W is a bounded operator on $L^p(\mu)$ so J is finite valued μ-a.e on X, then we have $\mu(\{x \in G \ ; \ J(x) = \infty\}) = 0$. Now as $\mu(G) > 0$, $\mu(G_N) > 0$ for some $N \in \mathbb{N}$. Since G_N is non- atomic, for any $\alpha > 0$, we can choose some set $E_\alpha \in \Sigma$ such that $E_\alpha \subseteq G_N$ and $\mu(E_\alpha) \leq \mu(G_N)$. Take $f_\alpha = \chi_{E_\alpha}$. Obviously $f_\alpha \in L^p_{|_G}(\mu)$. Moreover $\|W_{|_G} f_\alpha\|_p \leq m^{\frac{p-1}{p}} \left(\int_X J|f_\alpha|^p d\mu\right)^{\frac{1}{p}} < m^{\frac{p-1}{p}} \left(\frac{N\alpha}{m^{\frac{p-1}{p}}}\right)\|f_\alpha\|_p = N\alpha\|f_\alpha\|_p$. This prove our claim and therefore we must have $J = 0$, $\ \mu - a.e$ on B.
(c) Assume W has closed range and is injective so there exists a constant $d > 0$ such that $\|Wf\|_p \geq d\|f\|_p$, for any $f \in L^p(\mu)$.

$$\begin{aligned}
n^{p-1}\int_X \sum_{i=1}^n u_i^p d\mu &\geq n^{p-1}\sum_{i=1}^n \int_{\varphi_i^{-1}(X)} u_i^p d\mu \\
&= n^{p-1}\sum_{i=1}^n \int_X u_i^p \chi_{\varphi_i^{-1}(X)} d\mu \\
&\geq \int_X |W\chi_X|^p d\mu \\
&= \|W\chi_X\|_p^p \\
&\geq d^p\|\chi_X\|_p^p = d^p\mu(X)
\end{aligned}$$

so $u \geq \delta$ on X . The proof is now complete. □

Here we give some necessary and sufficient conditions for closedness of range when the operator projects L^p into L^q when $1 \leq q < p < \infty$.

Theorem 2.3. *Suppose that $1 \leq q < p < \infty$ and let W be a bounded operator from $L^p(\mu)$ into $L^q(\mu)$. The followings hold.*

(a) If W has closed range and is injective then the set $\{i \in \mathbb{N} \ ; \ J(= \sum_{r=1}^n h_r E_r(|u_r|^q) \circ \varphi_r^{-1})(A_i) > 0\}$ is finite.
(b) If $J(B) = 0$, $\mu - a.e$ and the set $\{i \in \mathbb{N} \ ; \ J(= \sum_{r=1}^n h_r E_r(|u_r|^q) \circ \varphi_r^{-1})(A_i) > 0\}$ is finite then W has closed range.

Proof. (a) Suppose on the contrary, the set $\{i \in \mathbb{N} \ ; \ J(A_i) > 0\}$ is infinite. Since W is injective and has closed range there exists $d > 0$ such that $\|Wf\|_q \geq d\|f\|_p$, for all $f \in L^p(\mu)$. Thus for any $i \in \mathbb{N}$, $\|W\chi_{A_i}\|_q^q \geq d^q\mu(A_i)^{\frac{q}{p}}$

and so

$$\begin{aligned} d^q\mu(A_i)^{\frac{q}{p}} &\leq \|W\chi_{A_i}\|_q^q \\ &= \int_X |\sum_{r=1}^n u_r\chi_{A_i}\circ\varphi_r|^q d\mu \\ &\leq n^{q-1}\int_X J\chi_{A_i}d\mu \\ &= n^{q-1}J(A_i)\mu(A_i). \end{aligned}$$

It follows from the preceding inequality that

$$\frac{d^{\frac{pq}{p-q}}}{n^{\frac{p(q-1)}{p-q}}} \leq J(A_i)^{\frac{p}{p-q}}\mu(A_i).$$

Therefore,

$$\infty = \sum_{i\in\mathbb{N}} \frac{d^{\frac{pq}{p-q}}}{n^{\frac{p(q-1)}{p-q}}} \leq \sum_{i\in\mathbb{N}} J(A_i)^{\frac{p}{p-q}}\mu(A_i) < \infty.$$

This is a contradiction.
(b) Let $g \in \overline{W(L^p(\mu))}$ then there exists a sequence $(Wf_n)_{n\in\mathbb{N}} \subseteq W(L^p(\mu))$ such that $Wf_n \longrightarrow g$ and $\|f_n\| < 1$. If the set $\{i \in \mathbb{N}\ ;\ J(A_i) > 0\}$ is empty then W is the zero operator . Otherwise we may assume there exists some $k \in \mathbb{N}$ such that $J(A_i) > 0$ for $1 \leq i \leq k$ and $J(A_i) = 0$ for any $i > k$. As $f_n \in L^p(\mu)$ for all n, $|f_n(A_i)| \leq \frac{\|f_n\|_p}{\sqrt[p]{\mu(A_i)}} \leq \frac{1}{\sqrt[p]{\mu(A_i)}}$ for any $1 \leq i \leq k$ and any $n \in \mathbb{N}$. By Bolzano-Weierstrass there exists a subsequence of nutural number $(n_j)_{j\in\mathbb{N}}$ such that for each fixed $1 \leq i \leq k$ the sequence $(f_{n_j}(A_i))_{j\in\mathbb{N}}$ converges. Suppose $\lim_{j\to\infty} f_{n_j}(A_i) = \varsigma_j (\in \mathbb{C})$ and define $f = \sum_{i=1}^k \varsigma_j\chi_{A_i}$. Then $f \in L^p(\mu)$. For every $\epsilon > 0$, we have that

$$\begin{aligned} \|g - Wf\|_q &\leq \|g - Wf_n\|_q + \|Wf_n - Wf_{n_j}\|_q + \|Wf_{n_j} - Wf\|_q \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \int_X |W(f_{n_j} - f)|^q d\mu \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + n^{q-1}\int_X J|f_{n_j} - f|^q d\mu \\ &= \frac{\epsilon}{3} + \frac{\epsilon}{3} + n^{q-1}\sum_{i=1}^k J(A_i)|f_{n_j}(A_i) - \varsigma_j|^q\mu(A_i) \\ &\longrightarrow 0 \end{aligned}$$

□

In the next theorem we obtain some necessary and sufficient conditions for closedness of range when the operator projects L^p into L^q when $1 \leq p < q < \infty$.

Theorem 2.4. *Suppose that* $1 \leq p < q < \infty$ *and* $W = \sum_{i=1}^m u_iC_{\varphi_i}$ *be a bounded operator from* $L^p(\mu)$ *into* $L^q(\mu)$. *Then the followings hold.*

(a) If $J(B) = 0$, $\mu - a.e$ *and* $\sum_{i\in\mathbb{N}} J(A_i)\mu(A_i) < \infty$ *then* W *has closed range.*
(b) If W *has closed range and is injective then* $J(B) = 0$, $\mu - a.e$.
(c) Let $\mu(X) < \infty$. *If* W *has closed range and is injective then there exists a constant* $\delta > 0$ *such that* $u = \sum_{i=1}^n u_i^p \geq \delta$ *on* X.

Proof. (a) Take any sequence $(Wf_n)_{n\in\mathbb{N}}$ in $W(L^p(\mu))$ with $\|f_n\| < 1$. For fixed $i \in \mathbb{N}$ the sequence $(f_n(A_i))_{n\in\mathbb{N}}$ is bounded by $\frac{1}{\sqrt[p]{\mu(A_i)}}$. Applying contor's diagonalization procces, we extract a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that with each fixed i, $f_{n_k}(A_i) \to \alpha_i$ for each $\alpha_i \in \mathbb{C}$. Define $f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}$. By fatous lemma we have $\int_X |f|^p d\mu \le \liminf_{k\to\infty} \int_X |f_{n_k}|^p d\mu \le 1$, or $f \in L^p(\mu)$. Then for each $\epsilon > 0$, we have

$$
\begin{aligned}
\|g - Wf\|_q &\le \|g - Wf_n\|_q + \|Wf_n - Wf_{n_k}\|_q + \|Wf_{n_k} - Wf\|_q \\
&\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \int_X |W(f_{n_k} - f)|^q d\mu \\
&\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + m^{q-1} \int_X J|f_{n_k} - f|^q d\mu \\
&= \frac{\epsilon}{3} + \frac{\epsilon}{3} + m^{q-1} \sum_{i=1}^{\infty} J(A_i)|f_{n_k}(A_i) - \alpha_i|^q \mu(A_i) \\
&\longrightarrow 0
\end{aligned}
$$

Obviusly $W(L^p(\mu))$ is closed in $L^q(\mu)$.
(b) Suppose on the contrary, $\mu(\{x \in B \ : \ J(x) > 0\}) > 0$. Then there exists some $\delta > 0$ such that the set $G = \{x \in B \ : \ J(x) \ge \delta\}$ has positive μ- measure. We assume $\mu(G) < \infty$. Moreover, as G is non atomic, we can further assume that $\mu(X\backslash G) > 0$. Consider the Banach space $L^p_{|_G}(\mu)$ and the operator $W_{|_G}$ defined on $L^p_{|_G}(\mu)$. We claim that $W_{|_G}(L^p_{|_G}(\mu))$ is closed in $L^q(\mu)$. To prove we take any convergent sequence $(W_{|_G}(f_n))_{n\in\mathbb{N}}$ in $W_{|_G}(L^p_{|_G}(\mu))$. Let $g \in L^q(\mu)$ satisfy $\|W_{|_G}(f_n) - g\|_q \to 0$ as $n \to \infty$. Note that $(W_{|_G}(f_n))_{n\in\mathbb{N}}$ can be ragarded as a sequence in $W(L^p(\mu))$. The closedness of range of W yeilds an $f \in L^p(\mu)$ with $g = Wf \ \ \mu - a.e$ On X. Then assume W has closed range and is injective so there exists a constant $d > 0$ such that $\|W_{|_G}(f_n) - Wf\|_q \ge d\|f_n - f\|_p$. As $\|W_{|_G}(f_n) - g\|_q = \|W_{|_G}(f_n) - Wf\|_q = 0$ and $\|f_n - f\|_p^p = \int_G |f_n - f|^p d\mu + \int_{X\backslash G} |f_n - f|^p d\mu$ we have that $\int_{X\backslash G} |f|^p d\mu = 0$ and so $f \in L^p_{|_G}(\mu)$. Then there exists some conctant $c > 0$ such that $\|W_{|_G} f\|_q \ge c\|f\|_p$, for all $f \in L^p_{|_G}(\mu)$. We claim that this is impossible by showing that for any $\alpha > 0$, there is some $f_\alpha \in L^p_{|_G}(\mu)$ satisfying $\|W_{|_G} f\|_q < c\|f\|_p$. For any $n \in \mathbb{N}$, define $G_n = \{x \in G \ ; \ n - 1 \le J(x) \le n\}$. Then $G = (\cup_{n\in\mathbb{N}} G_n) \cup \{x \in G \ ; \ J(x) = \infty\}$. Since W is a bounded operator on $L^p(\mu)$ so J is finite valued μ-a.e on X, then we have $\mu(\{x \in G \ ; \ J(x) = \infty\}) = 0$. Now as $\mu(G) > 0$, $\mu(G_N) > 0$ for some $N \in \mathbb{N}$. Since G_N is non- atomic, for any $\alpha > 0$, we can choose some set $E_\alpha \in \Sigma$ such that $E_\alpha \subseteq G_N$ and $\mu(E_\alpha) = \frac{\mu(G_N)}{K}$, where $K < \frac{N^{\frac{q}{q-p}} \mu(G_N}{\alpha^{\frac{pq}{q-p}}}$. Take $f_\alpha = \chi_{E_\alpha}$. Obviously $f_\alpha \in L^p_{|_G}(\mu)$. Moreover

$$
\begin{aligned}
\|W_{|_G} f_\alpha\|_p &\le m^{\frac{q-1}{q}} \left(\int_X J|f_\alpha|^q d\mu \right)^{\frac{1}{q}} \\
&< m^{\frac{q-1}{q}} \left(\frac{N\mu(G_N}{K} \right)^{\frac{1}{q}} \\
&= m^{\frac{q-1}{q}} N^{\frac{1}{q}} \left(\frac{\mu(G_N}{K} \right)^{\frac{1}{p} + \frac{p-q}{pq}} \\
&= m^{\frac{q-1}{q}} N^{\frac{1}{q}} \|f_\alpha\|_p \left(\frac{K}{\mu(G_N} \right)^{\frac{q-p}{pq}} \\
&< \frac{N}{\alpha} \|f_\alpha\|_p.
\end{aligned}
$$

This prove our claim and therefore we must have $J = 0, \ \ \mu - a.e$ on B.
(c) Assume W has closed range and is injective so there exists a constant $d > 0$ such that $\|Wf\|_p \ge d\|f\|_p$ for

any $f \in L^p(\mu)$.

$$\begin{aligned} n^{p-1}\int_X \sum_{i=1}^{n} u_i^p d\mu &\geq n^{p-1}\sum_{i=1}^{n}\int_{\varphi_i^{-1}(X)} u_i^p d\mu \\ &= n^{p-1}\sum_{i=1}^{n}\int_X u_i^p \chi_{\varphi_i^{-1}(X)} d\mu \\ &\geq \int_X |W\chi_X|^p d\mu \\ &= \|W\chi_X\|_p^p \\ &\geq d^p\|\chi_X\|_p^p = d^p\mu(X) \end{aligned}$$

so $u \geq \delta$ on X . The proof is now complete. □

In the sequel we investigate the closedness of range the operator in from L^∞ into L^q and the converse. First we find some necessary and sufficient conditions for the case that W is a bounded operator from L^∞ into L^q with $1 < q < \infty$.

Theorem 2.5. *Suppose that* $1 \leq q < \infty$. *Let* $J = \sum_{r=1}^{n} h_r E_r(|u_r|^q) \circ \varphi_r^{-1}$ *and* W *be a operator from* $L^\infty(\mu)$ *into* $L^q(\mu)$. *The followings hold.*

(a) If
   (1) W has closed range.
   (2) W is injective.
   (3) $\sum_{i\in\mathbb{N}} J(A_i)\mu(A_i) < \infty$.

   Then the set $\{i \in \mathbb{N} \; ; \; J(A_i) > 0\}$ *is finite.*

(b) If $J(B) = 0$, $\mu - a.e$ *and the set* $\{i \in \mathbb{N} \; ; \; J(A_i) > 0\}$ *is finite then W has closed range.*

Proof. (a) Suppose on the contray, the set $\{i \in \mathbb{N} \; ; \; J(A_i) > 0\}$ is infinite. Since W has closed range and is injective we can find some constant$d > 0$ such that $\|Wf\|_q \geq d\|f\|_\infty$, for all $f \in L^\infty(\mu)$. Thus for any $i \in \mathbb{N}$, $\|W\chi_{A_i}\|_q^q \geq d^q$ and so we have,

$$\begin{aligned} d^q &\leq \|W\chi_{A_i}\|_q^q \\ &= \int_X |\sum_{r=1}^{n} u_r\chi_{A_i} \circ \varphi_r|^q d\mu \\ &\leq n^{q-1}\int_X J\chi_{A_i} d\mu \\ &= n^{q-1}J(A_i)\mu(A_i) \end{aligned}$$

It follows from the preceding inequality that

$$\frac{d^q}{n^{q-1}} \leq J(A_i)\mu(A_i)$$

Therefore,

$$\infty = \sum_{i\in\mathbb{N}} \frac{d^q}{n^{q-1}} \leq \sum_{i\in\mathbb{N}} J(A_i)\mu(A_i) < \infty$$

contradiction arises.
(b) Let $g \in \overline{W(L^\infty(\mu))}$ then there exists a sequence $(Wf_n)_{n\in\mathbb{N}} \subseteq W(L^\infty(\mu))$ such that$Wf_n \longrightarrow g$ with $\|f_n\| < 1$. If the set $\{i \in \mathbb{N} \,;\, J(A_i) > 0\}$ is empty then W is the zero operator . Otherwise we may assume there exists some $k \in \mathbb{N}$ such that $J(A_i) > 0$ for $1 \le i \le k$ and $J(A_i) = 0$ for any $i > k$. As $f_n \in L^\infty(\mu)$ for all n, $|f_n(A_i)| \le \|f_n\|_\infty$ for any $1 \le i \le k$ and any $n \in \mathbb{N}$. By Bolzano-Weierstrass there exists a subsequence of nutural number $(n_j)_{j\in\mathbb{N}}$ such thst for each fixed $1 \le i \le k$ the sequence $(f_{n_j}(A_i))_{j\in\mathbb{N}}$ converjes. Suppose $\lim_{j\to\infty} f_{n_j}(A_i) = \varsigma_j (\in \mathbb{C})$ and define $f = \sum_{i=1}^{k} \varsigma_j \chi_{A_i}$. Then $f \in L^\infty(\mu)$. For every $\epsilon > 0$, we have that

$$\begin{aligned}
\|g - Wf\|_q &\le \|g - Wf_n\|_q + \|Wf_n - Wf_{n_j}\|_q + \|Wf_{n_j} - Wf\|_q \\
&\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \int_X |W(f_{n_j} - f)|^q d\mu \\
&\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + n^{q-1} \int_X J|f_{n_j} - f|^q d\mu \\
&= \frac{\epsilon}{3} + \frac{\epsilon}{3} + n^{q-1} \sum_{i=1}^{k} J(A_i)|f_{n_j}(A_i) - \varsigma_j|^q \mu(A_i) \\
&\longrightarrow 0
\end{aligned}$$

□

Now we find some necessary and sufficient conditions for the case that W is a bounded operator from L^p into L^∞ with $1 < p < \infty$.

Theorem 2.6. *Let u_i's are nonnegative and $\mu(X) < \infty$. Suppose that $1 \le p < \infty$ and let W be a operator from $L^p(\mu)$ into $L^\infty(\mu)$. The followings hold.*

(a) If (X, Σ, μ) be a purely atomic space and W is bounded operator then W has closed range.
(b) If W has closed range and is injective then there exists a constant $\delta > 0$ such that $u = \sum_{i=1}^{n} u_i^p \ge \delta$ on X.

Proof. (a) Take any sequence $(Wf_n)_{n\in\mathbb{N}}$ in $W(L^p(\mu))$ with $\|f_n\| < 1$. For fixed $i \in \mathbb{N}$ the sequence $(f_n(A_i))_{n\in\mathbb{N}}$ is bounded by $\frac{1}{\sqrt[p]{\mu(A_i)}}$. Applying contor's diagonalization procces, we extract a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that with each fixed i, $f_{n_k}(A_i) \to \alpha_i$ for each $\alpha_i \in \mathbb{C}$. Define $f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}$. By fatous lemma we have $\int_X |f|^p d\mu \le \liminf_{k\to\infty} \int_X |f_{n_k}|^p d\mu \le 1$, or $f \in L^p(\mu)$. Then for each $\epsilon > 0$, we have

$$\begin{aligned}
\|g - Wf\|_\infty &\le \|g - Wf_n\|_\infty + \|Wf_n - Wf_{n_k}\|_\infty + \|Wf_{n_k} - Wf\|_\infty \\
&\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|W\| \int_X |f_{n_k} - f|^q d\mu \\
&\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|W\| \int_{\cup_{i\in\mathbb{N}} A_i} |f_{n_k} - f|^q d\mu \\
&= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|W\| \sum_{i=1}^{\infty} |f_{n_k}(A_i) - \alpha_i|^q \mu(A_i) \\
&\longrightarrow 0
\end{aligned}$$

Obviusly $W(L^p(\mu))$ is closed in $L^\infty(\mu)$.
(b) Assume W has closed range and is injective so there exists a constant $d > 0$ such that $\|Wf\|_\infty \ge d\|f\|_p$ for any $f \in L^p(\mu)$. Take $\delta = \frac{d^p \mu(X)}{n^{p-1}}$, Then,

$$|\sum_{i=1}^{n} u_i \chi_X \circ \varphi_i|^p \quad \le \quad n^{p-1} \sum_{i=1}^{n} u_i^p$$

Therefore,

$$\begin{aligned} n^{p-1}\sum_{i=1}^{n} u_i^p &\geq (\sum_{i=1}^{n} u_i)^p \\ &\geq \|W\chi_X\|_\infty^p \\ &\geq d^p\|\chi_X\|_p^p = d^p\mu(X) \end{aligned}$$

so $u \geq \delta$ on X . The proof is now complete. □

Here we consider W as a bounded operator on L^∞.

Theorem 2.7. *Let u_i's are nonnegative and $\mu(X) < \infty$. Suppose that W be a bounded operator from $L^\infty(\mu)$ into $L^\infty(\mu)$. The followings hold.*

(a) If If (X, Σ, μ) be a purely atomic space then W has closed range.
(b) If W has closed range and is injective then there exists a constant $\delta > 0$ such that $u = \sum_{i=1}^{n} u_i \geq \delta$ on X.

Proof. (a) Take any sequence $(Wf_n)_{n\in\mathbb{N}}$ in $W(L^\infty(\mu))$ with $\|f_n\| < 1$. For fixed $i \in \mathbb{N}$ the sequence $(f_n(A_i))_{n\in\mathbb{N}}$ is bounded by $|f_n(A_i)| \leq \|f_n\| < 1$. Applying contor's diagonalization procces, we extract a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that with each fixed i, $f_{n_k}(A_i) \to \alpha_i$ for each $\alpha_i \in \mathbb{C}$. Define $f = \sum_{i=1}^{\infty} \alpha_i\chi_{A_i}$. Then for each $\epsilon > 0$, we have

$$\begin{aligned} \|g - Wf\|_\infty &\leq \|g - Wf_n\|_\infty + \|Wf_n - Wf_{n_k}\|_\infty + \|Wf_{n_k} - Wf\|_\infty \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|W\|\|f_{n_k} - f\|_\infty \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|W\| \sup_{i\in\mathbb{N}} |f_{n_k}(A_i) - \alpha_i| \\ &\longrightarrow 0 \end{aligned}$$

Obviously $W(L^\infty(\mu))$ is closed in $L^\infty(\mu)$.

(b) Assume W has closed range and is injective so there exists a constant $d > 0$ such that $\|Wf\|_\infty \geq d\|f\|_\infty$, for any $f \in L^\infty(\mu)$. Take $\delta = d$. Then ,

$$|\sum_{i=1}^{n} u_i\chi_X \circ \varphi_i| \leq \sum_{i=1}^{n} u_i$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{n} u_i &\geq \sum_{i=1}^{n} u_i\chi_X \circ \varphi_i \\ &\geq d \end{aligned}$$

so $u \geq \delta$ on X . The proof is now complete. □

In the next theorem we obtain the polar decomposition of W as a bounded operator on the Hilbert space L^2. The polar decomposition of every linear map $A : X \longrightarrow U$ can be written as $A = UP$ where $P \geq 0$ an U is unitary.

Theorem 2.8. *Suppose $u_i(\varphi_j^{-1}) = 0, \quad i \neq j$. The unique polar decomposition of $W = \sum_{i=1}^{n} u_iC_{\varphi_i}$ is $V|W|$ where $|W|(f) = M_Jf$, $V(g) = \sum_{i=1}^{n} u_i\frac{\chi_B g}{\sqrt{J}} \circ \varphi_i$ and $B = Coz(J = \sum_{i=1}^{n} h_iE_i(u_i^2) \circ \varphi_i^{-1})$.*

Proof. We have that

$$\begin{aligned} \|Wf\|_2^2 &= (Wf, Wf) \\ &= (W^*Wf, f) \\ &= \int_X J|f|^2 d\mu = \int_B J|f|^2 d\mu + \int_{X\backslash B} J|f|^2 d\mu \\ &= \int_B J|f|^2 d\mu \end{aligned}$$

where $J = \sum_{i=1}^n h_i E_i(u_i)^2 \circ \varphi_i^{-1}$. Then $\ker W = L^2(X\backslash B) = (L^2(B))^{\perp}$. For each $f \in L^2(\mu)$ write $f = \chi_B f + \chi_{X\backslash B} f$ so that $Wf = W\chi_B f$. We may define partial isometry V with initial space $(\ker W)^{\perp} = L^2(B)$ and final space $RanW$ by $V(g) = \sum_{i=1}^n u_i \frac{\chi_B g}{\sqrt{J}} \circ \varphi_i, \quad g \in L^2(\mu)$. Then the unique polar for W is given $W = VM_{\sqrt{J}}$. □

Finally, the next two assertions we investigate invertibility of W.

Theorem 2.9. *Let (X, Σ, μ) be apurely atomic measure space, $0 \neq u_i \in L^{\infty}(\mu)$ and W be a sum finite of weighted composition operators on $L^p(\mu)$. If there is a positive integer N_i such that $\varphi_i^{N_i}(A_n) = A_n$ up to a null set for all $n \geq 1$ and $u_i(\varphi_j) = 0, \quad i \neq j$ then*

- *(a) W is invertible.*
- *(b) The set function E that is defined as $E(B) = M_{\chi_B \circ v}$ for all borel sets B of $\mathbb{C}$ is a spectral measure where $v = \sum_{i=1}^n u_i u_i \circ \varphi_i \cdots u_i \circ \varphi_i^{N-1}$, $N = [N_1, \cdots, N_n]$.*

Proof. (a) Not that $\ker W^r \subseteq \ker W^{r+1}$ and $W^{r+1}(L^p(\mu)) \supseteq W^r(L^p(\mu))$. If there is a positive integer N_i such that $\varphi_i^{N_i}(A_n) = A_n$ up to a null set for all $n \geq 1$ then W^N is a multiplication operator induced by function $v = \sum_{i=1}^n u_i u_i \circ \varphi_i \cdots u_i \circ \varphi_i^{N-1}$,where $N = [N_1, \cdots, N_n]$.
If $f \in \ker W^N$ then $W^N f(A_n) = 0$ for all $n \geq 1$. We have that $vf(A_n) = 0$ therefor $f = 0$, $\mu - a.e$ on X. So W is injective.
Let $g \in L^p(\mu)$ then $W^N(\frac{g}{v})(A_n) = g(A_n)$ for all $n \geq 1$. So W is surjective.
(b) As observed by Rho and Yoo ([14], example 1), the multiplication operator M_{χ_v} is spectral. In fact the spectral measure E is given by $E(B) = M_{\chi_B \circ v}$ for all Borel set B of $\mathbb{C}$. □

Theorem 2.10. *Let $W = \sum_{i=1}^n u_i C_{\varphi_i}$ be a bounded operator on $L^2(\mu)$ and $u_i(\varphi_j^{-1}) = 0, \quad i \neq j$. The following statements are equivalent.*

- *(a) W is injective.*
- *(b) $J = \sum_{i=1}^n h_i E_i(|u_i|^2) \circ \varphi_i^{-1} > 0, \quad \mu - a.e$ on X.*
- *(c) whenever $J(E) = 0$ for $E \in \Sigma$, $\mu(E) = 0$.*

Proof. $(b) \Rightarrow (a)$ Take any $f \in \ker W$, then we have

$$\begin{aligned} 0 = \|Wf\|_2^2 &= (Wf, Wf) \\ &= (W^*Wf, f) \\ &= \int_X J|f|^2 d\mu = \int_{Cozf} J|f|^2 d\mu + \int_{X\backslash Cozf} J|f|^2 d\mu \\ &= \int_{Cozf} J|f|^2 d\mu \end{aligned}$$

Since $J > 0, \quad \mu - a.e$ on $Cozf$, it follows that $\mu(Cozf) = 0$ or $f = 0 \quad \mu - a.e$ on X.
$(a) \Rightarrow (c)$ Let $E \in \Sigma$ satisfy $J(E) = 0$ we may also assume $\mu(E) < \infty$. Then $\chi_E \in L^2(\mu)$ and $\|W\chi_E\|_2^2 = \int_X J\chi_E d\mu = \int_E J d\mu = 0$. Now the injectivity of W implies that $\chi_E = 0, \quad \mu - a.e$ on X. Hence $\mu(E) = 0$.
$(c) \Rightarrow (b)$ Put $B = CozJ$. Clearly, $X\backslash B \in \Sigma$. Moreover, since $J(X\backslash B) = 0$ We must have $\mu(X\backslash B) = 0$. This shows that $J > 0, \quad \mu - a.e$ on X. □

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