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On the characterization of matrix domains using Cesàro and backward difference operators

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Abstract. In this manuscript, the new sequence space Z_Φ for $Z \in \{c, c_0, \ell_\infty\}$ and $\Phi = C^m\Delta$ have been introduced using the Cesàro matrix and backward difference operator. Some of the topological properties of these spaces have been studied and the existence of the Schauder basis for new spaces have been verified. Also, the α, β and γ-duals have been computed along with the characterization of the matrix transformation between new spaces.

1. Introduction

The analysis of the sequence space theory has always been of greatest interest in the numerous branches of analysis such as the theory of summability, structural theory of topological linear spaces, Schauder basis theory, etc. Moreover, sequence space theory contains a useful tool for acquiring the geometrical and topological results through the Schauder basis.

Cesàro [6] propounded his work in the discipline of differential geometry. He also worked on the averaging method of the divergent series for Cesàro summation, called Cesàro-mean. He defined an infinite Cesàro matrix $C = C_{rv}$ of order one, denoted as C^1, has the entries as follows

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{1}$$

Several authors concluded their research on the Cesàro sequence and the Cesàro function spaces, however, they all have been around the Cesàro matrix of order one. Ng and Lee [17] have defined X_p and X_∞, the non-absolute type of Cesàro sequence spaces as the domains of matrix C^1, in ℓ_p and ℓ_∞ for $1 \le p < \infty$. Later on, Başar and Sengonul [22] introduced the non-absolute type of Cesàro spaces $\tilde{c}$ and $\tilde{c_0}$ as the domains of matrix C^1, in c and c_0 respectively. Moreover, Altay and Başar [2, 3] have studied and investigated the

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space of bounded variation bv_p as the domain of backward difference ∇ in ℓ_p for $0 < p < 1$ and $1 \leq p < \infty$, respectively.

In [20], Roopaei et al. introduced and studied the Cesàro sequence spaces of order m as the domain of matrix C^m in ℓ_p and ℓ_∞ and discussed their duals for $1 < p < \infty$.

Let s indicate the space of all complex sequences. For any sequence space μ and infinite matrix A, the matrix domain of A is defined as $\mu_A = \{x \in s : Ax \in \mu\}$. In the last few years, there is a procedure of obtaining new spaces via the matrix domain of a convenient matrix and characterizing the classes of matrix transformation between the sequence spaces. Many authors studied and investigated new Banach spaces by means of matrix domains of the special triangle matrices, in the classical sequence spaces. For more details, one can refer to [4, 5, 8, 9, 11, 16] and references therein.

The study of difference sequence space was initiated by Kizmaz [12]. Subsequently, Başar and Altay [2], Et and Colak [7], Ahmad and Mursaleen [1], Altay and Polat [18] studied and introduced new sequence spaces by means of difference operator.

Besides this, Cesàro sequence spaces are defined through the domains of Cesàro matrix C^m of order m. Some of them can be viewed in Roopaei [20] and, Roopaie and Başar [19] which incorporates the earlier known Cesàro Banach spaces. Recently, Roopaei and Başar [19] have investigated the Cesàro spaces $\ell_p(C^m)$, $0 < p < 1$, $c(C^m)$ and $c_0(C^m)$ as the domains of matrix C^m of order m in ℓ_p, c and c_0, respectively.

In this paper, we defined new sequence spaces through the backward difference operator and Cesàro matrix. Besides, we determine some topological properties of new spaces along with α-, β- and γ-duals, and constructed the bases of these spaces. Finally, we discuss the characterization of some related matrix classes between these sequence spaces.

Motivation. In [21] Roopaei and Hazarika have investigated the sequence space $\ell_p(S^{m,n})$, where $S^{m,n} = C^m\Delta^n$. Here the authors have introduced the matrix domain Z_Φ for $Z \in \{c, c_0, \ell_\infty\}$ based on Cesàro matrix of order m and backward difference operator. Through this research, the authors have found the topological properties, basis, duals, and matrix transformations that have not been known before.

2. Preliminaries

By c, c_0, and ℓ_∞, we indicate the spaces of all convergent, null convergent, and bounded sequences $x = (x_v)$, endowed with norm $\|x\|_\infty = \sup_r |x_r|$. We also indicate the spaces of all convergent and bounded series by cs, and bs respectively. Throughout the text, $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$, and $e = (1, 1, 1, ...)$ and $e_r = (0, 0, \cdots, 1, 0, \cdots)$ where 1 is in the r^{th} place, and 0 everywhere.

If a normed linear space U contains a sequence (b_r), then for every $x \in U$, there is a unique sequence of scalars (α_r) such that

$$\|x - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_r b_r)\| \to 0 \; as \; r \to \infty,$$

then (b_r) is known as the Schauder basis for U. The series $\sum_{r=0}^{\infty} \alpha_r b_r$ has the sum x, known as the expansion of x about the basis (b_r), and we write $x = \sum_{r=0}^{\infty} \alpha_r b_r$, [14].

Let U and V be any two sequence spaces. Then, the multiplier space $\mathcal{M}(U, V)$ is given as

$$\mathcal{M}(U, V) = \{(a_r) \in s : ay = (a_r y_r) \in V, \text{ for every } y \in U\}$$

Thus, the α-dual, the β-dual and the γ-dual of U are denoted as

$$U^\alpha = \mathcal{M}(U, \ell_1), \; U^\beta = \mathcal{M}(U, cs), \; U^\gamma = \mathcal{M}(U, bs).$$

An infinite matrix can be observed as the linear operator from a sequence space into another sequence space. For this, let U and V be any arbitrary subsets of s. Let $A = (a_{rv})$ is an infinite matrix with complex entries (a_{rv}). By $A(x) = (A_r(x)) = (Ax)_r$, we write the $A-$transform of a sequence $x = (x_v)$, if the series $A_r(x) = \sum_v a_{rv} x_v$ is convergent for $r \geq 0$.

If $Ax \in V$ with $x \in U$, then A defines a matrix mapping from U into V. Further, (U, V) indicates the family of all infinite matrices that maps U into V. Thus, A is in (U, V) if and only if $Ax = ((Ax)_r) \in V$, $\forall$ $x \in U$, that is, $A \in (U, V)$ if and only if $A_r \in U^\beta$, $\forall$ r (see [24]).

A Banach sequence space ν is known as BK-space, if the projection mappings $q_r : \nu \to \mathbb{C}$ such that $q_r(x) = x_r,\ r \geq 1$ are continuous. For the natural number r and the sequence $x = (x_1, x_2, \cdots, x_r, \cdots)$, the r^{th} section of x is denoted as $x^{(r)} = (x_1, x_2, \cdots, x_r, 0, 0, \cdots)$. If for each $x \in \nu$, $x^{(r)}$ tends to x, then ν is known as AK-space.

The infinite matrix $A = (a_{rv})$ is said to be a summability matrix if it is a lower triangular matrix, i.e., $a_{rv} = 0$ for $r < v$ and $\sum_{v=0}^{r} a_{rv} = 1$, for every r.

Let us consider the Hausdorff matrix with generating sequence $\mu = (\mu_r)$, which is a lower triangular matrix, denoted as $H_\mu = (h_{rv})_{r,v=0}^{\infty}$ and their entries are as follows:

$$h_{rv} = \binom{r}{v} \int_0^1 \tau^v (1-\tau)^{r-v} d\mu(\tau), \text{ for } 0 \leq v \leq r$$

for every $v,\ r \in \mathbb{N}^0$, where μ be the probability measure on $[0, 1]$. For the probability measure μ, the Hausdorff matrix H_μ is called as totally regular.

For $m > 0$, the Hausdorff matrix consist of the following matrices classes:

(i) if $d\mu(\tau) = m(1-\tau)^{m-1}$, then the Hausdorff matrix introduces the Cesàro matrix of order m,
(ii) If $d\mu(\tau) = m\tau^{m-1} d\tau$, then the Hausdorff matrix introduces the Gamma matrix of order m,
(iii) if $d\mu(\tau) = \dfrac{\left|\log \tau\right|^{m-1}}{\Gamma(m)} d\tau$, then the Hausdorff matrix introduces the Hölder matrix of order m.

Such matrices have always been studied for a long time in connection with the summability of series, and subsequently as operators on the sequence spaces.

In [13], Hardy's formula follows that if the measure μ satisfying

$$\int_0^1 \tau^{-1/p} d\mu(\tau) < \infty,$$

then the Hausdorff matrix H_μ is a bounded linear operator on ℓ_p furnished with the norm

$$\left\|H_\mu\right\|_{\ell_p} = \int_0^1 \tau^{-1/p} d\mu(\tau), \ \ (1 \leq p < \infty).$$

For $d\mu(\tau) = m(1-\tau)^{m-1} d\mu$ in the Hausdorff matrix H_μ, the Cesàro matrix of order m, $C^m = (C^m_{rv})$ is given as

$$C^m_{rv} = \begin{cases} \dfrac{\binom{m+r-v-1}{r-v}}{\binom{m+r}{r}}, & 0 \leq v \leq r \\ 0, & v > r \end{cases}$$

for $r, v \in \mathbb{N}^0$.

In accordance with Hardy's formula, C^m is endowed with the norm

$$\|C^m\|_{\ell_p} = \frac{\Gamma(m+1)\Gamma(1/q)}{\Gamma(m+1/q)},$$

where q is a conjugate of p, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

For instance, the ℓ_p norm of the Cesàro matrix C^1 is $\|C\|_{\ell_p} = q$.

3. Matrix Domain Z_Φ

We embark with the concept of convergent, null convergent, and bounded sequences through the composition of Cesàro operator of order m and the backward difference operator Δ, where $\Delta x_v = x_v - x_{v-1}$ and $x_{-1} = 0$.

The infinite Cesàro matrix of order m, $C^m = (C^m_{rv})$, is invertible and its inverse is defined by

$$C^{-m}_{rv} = \begin{cases} (-1)^{r-v}\binom{m}{r-v}\binom{m+v}{v}, & v \le r \le m+v \\ 0, & \text{otherwise} \end{cases},$$

$\forall\ r, v \in \mathbb{N}^0$.

We now define the sequence space, Z_Φ for $Z \in \{c, c_0, \ell_\infty\}$ as follows:

$$Z_\Phi = \left\{ x = (x_v) \in s : \left(\frac{1}{\binom{m+r}{r}} \sum_{v=0}^{r} \binom{m+r-v-1}{r-v} \Delta x_v \right)_{r=0}^{\infty} \in Z \right\}.$$

With the definition of matrix domain, we can write

$$Z_\Phi = \{x \in s : \Phi x \in Z\}.$$

As $y = (\Delta y_v)$ is the C^m-transform of a sequence $x = \Delta x_v$, i.e.,

$$y_r = (C^m x)_r = \frac{1}{\binom{m+r}{r}} \sum_{v=0}^{r} \binom{m+r-v-1}{r-v} \Delta x_v.$$

Theorem 3.1. *The sequence space Z_Φ for $Z \in \{c,\ c_0, \ell_\infty\}$ is a complete normed linear space furnished with the norm*

$$\|x\|_{Z_\Phi} = \sup_r \left| \frac{1}{\binom{m+r}{r}} \sum_{v=0}^{r} \binom{m+r-v-1}{r-v} \Delta x_v \right|.$$

Proof. Let $x,\ y \in Z_\Phi$ and a, and b be any two scalars. Then

$$\sup_r \left| \frac{1}{\binom{m+r}{r}} \sum_{v=0}^{r} \binom{m+r-v-1}{r-v} \Delta(a x_v + b y_v) \right| \le |a| \sup_r \left| \frac{1}{\binom{m+r}{r}} \sum_{v=0}^{r} \binom{m+r-v-1}{r-v} \Delta x_v \right| + |b| \sup_r \left| \frac{1}{\binom{m+r}{r}} \sum_{v=0}^{r} \binom{m+r-v-1}{r-v} \Delta y_v \right|$$

and so $a x_v + b y_v \in Z_\Phi$. Hence, Z_Φ is a linear space.

Clearly, the functional $\|\cdot\|_{Z_\Phi}$ defined above introduce a norm on the space Z_Φ.

For completeness, let (x^n) is a Cauchy sequence in Z_Φ, where $x^n = (x^n_v) = (x^n_0, x^n_1, x^n_2, \cdots) \in Z_\Phi$, for every $n \in \mathbb{N}^0$. Then, for every $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ with $\left\|x^n - x^j\right\|_{Z_\Phi} < \epsilon$ for $n,\ j \ge n_0$.

Thus, for each $v \in \mathbb{N}^0$,

$$\left|(Z_\Phi x^n)_v - (Z_\Phi x^j)_v\right| < \epsilon,\ \ \forall\ n, j \ge n_0. \tag{2}$$

So that $((Z_\Phi x^n)_v)_n$ is a Cauchy sequence of scalars for $v \in \mathbb{N}^0$. Therefore, $((Z_\Phi x^n)_v)_n$ converges for each v, and

$$\lim_{n\to\infty} (Z_\Phi x^n)_v = (Z_\Phi x)_v,\ v \in \mathbb{N}^0.$$

Letting, $j \to \infty$ in (2), we have

$$|(Z_\Phi x^n)_v - (Z_\Phi x)_v| \leq \epsilon \ \ \forall\, n \geq n_0, \forall\, v \in \mathbb{N}^0.$$

Thus, by definition $\|x^n - x\|_{Z_\Phi} \leq \epsilon$ for all $n \geq n_0$. Further, assume that

$$\|x\|_{Z_\Phi} \leq \|x^n\|_{Z_\Phi} + \|x^n - x\|_{Z_\Phi},$$

which is finite for $n \geq n_0$ and so $x \in Z_\Phi$. □

Theorem 3.2. *The spaces $c(\Phi), c_0(\Phi)$ and $\ell_\infty(\Phi)$ are linearly isomorphic to c, c_0 and ℓ_∞, respectively.*

Proof. Here, we consider the case for ℓ_∞. For this, it suffices to show the existence of a linear bijection from $\ell_\infty(\Phi)$ to ℓ_∞. Now, define a map $Q : \ell_\infty(\Phi) \to \ell_\infty$, as $x \to \Phi(x)$, where $\phi = C^m\Delta$. Since, C^m and Δ are both linear and invertible also is the matrix Φ, which completes the proof. □

Since, $Z_\Phi \cong Z$ for $Z \in \{c,\ c_0\}$, the basis for the spaces Z_Φ are the inverse images of basis for Z. Therefore we state the following result.

Theorem 3.3. *Let $\lambda_v = (\Phi x)_v$ and the sequences $b^{(i)} = (b_r^{(i)}),\ \ (i \in \mathbb{N}^0)$, and (b_r^{-1}) be defined as*

$$(b_i^{(v)}) = \begin{cases} \sum_{j=0}^{v-i} (-1)^j \binom{m+i}{i}\binom{m}{j}, & if\ 0 \leq i \leq v \\ 0, & if\ i > v \end{cases} \quad and\ (b_r^{-1}) = \sum_{i=0}^{v} \sum_{j=0}^{v-i} (-1)^j \binom{m+i}{i}\binom{m}{i+j}$$

Then,

(i) *The sequence $(b^{(i)})_{i=0}^{\infty}$ is a basis for space $c_0(\Phi)$ and every $x \in c_0(\Phi)$ is expressed uniquely as $x = \sum_{i=0}^{\infty} \lambda_i b^{(i)}$.*

(ii) *The sequence $(b^{(i)})_{i=-1}^{\infty}$ be a basis for space $c(\Phi)$ and every $x \in c(\Phi)$ is expressed uniquely as $x = \ell b_r^{-1} + \sum_{i=0}^{\infty} (\lambda_i - \ell) b^{(i)}$, where $\ell = \lim_{r\to\infty} (\Phi x)_r$.*

4. The $\alpha-, \beta-$ and $\gamma-$duals of Z_Φ

Let G indicates the collection of all non-empty finite subsets of $\mathbb{N}$, and $T = (t_{rv})$ be an infinite matrix which satisfy the following conditions:

$$\sup_{r\in\mathbb{N}_0} \sum_{v\in\mathbb{N}_0} |t_{rv}| < \infty \tag{3}$$

$$\sup_{K\in G} \sum_{r=0}^{\infty} \left| \sum_{v\in K} t_{rv} \right| < \infty \tag{4}$$

$$\lim_{r\to\infty} t_{rv} = 0,\ \ (v \in \mathbb{N}^0) \tag{5}$$

$$\lim_{r\to\infty} t_{rv} \text{ exists for all } v \tag{6}$$

$$\lim_{r\to\infty} \sum_{v=0}^{\infty} t_{rv} = 0 \tag{7}$$

$$\lim_{r\to\infty}\sum_{v=0}^{\infty}|t_{rv}|=0 \tag{8}$$

$$\lim_{r\to\infty}\sum_{v=0}^{\infty}t_{rv} \text{ exists} \tag{9}$$

$$\lim_{r\to\infty}\sum_{v=0}^{\infty}\left|t_{rv}-\lim_{r}t_{rv}\right|=0 \tag{10}$$

We now state the following results given by Stieglitz et.al [23] which are useful to compute the duals.

Lemma 4.1. *[23] (i) $T=(t_{rv})\in(c_0,\ c_0)$ if and only if (3) and (5) hold.*
(ii) $T=(t_{rv})\in(c_0,\ c)$ if and only if (3) and (6) hold.
(iii) $T=(t_{rv})\in(c,\ c_0)$ if and only if (3), (5) and (7) hold.
(iv) $T=(t_{rv})\in(c,\ c)$ if and only if (3), (6) and (9) hold.
(v) $T=(t_{rv})\in(c_0,\ \ell_\infty)$ (or $(c,\ \ell_\infty)$, or $(\ell_\infty,\ \ell_\infty)$) if and only if (3) holds.
(vi) $T=(t_{rv})\in(c_0,\ \ell_1)$ (or $(c,\ \ell_1)$, or $(\ell_\infty,\ \ell_1)$) if and only if (4) holds.
(vii) $T=(t_{rv})\in(\ell_\infty,\ c_0)$ if and only if (8) holds.
(viii) $T=(t_{rv})\in(\ell_\infty,\ c)$ if and only if (3), (6) and (10) hold.

Theorem 4.2. *The α-dual of space Z_Φ for $Z\in\{c,\ c_0,\ \ell_\infty\}$ is ψ, where*

$$\psi=\left\{a=(a_v)\in s:\sup_{K\in G}\sum_{r}\left|\sum_{i\in K}\sum_{j=0}^{v-i}(-1)^j\binom{m+i}{i}\binom{m}{j}a_v\right|<\infty\right\}.$$

Proof. Let $a=(a_v)\in s$. Given that $x\in Z_\Phi$, $y\in Z$ for $Z\in\{c,\ c_0,\ \ell_\infty\}$. Then, for every $v\in\mathbb{N}^0$,

$$a_vx_v=\sum_{i=0}^{v}\sum_{j=0}^{v-i}(-1)^j\binom{m+i}{i}\binom{m}{j}a_vy_v=(Ay)_v,$$

where $A=(a_{vi})$, is defined as

$$a_{vi}=\begin{cases}\sum\limits_{j=0}^{v-i}(-1)^j\binom{m+i}{i}\binom{m}{j}a_v, & 0\le i\le v\\ 0, & i>v\end{cases},$$

for all $i,\ v\in\mathbb{N}^0$.

Hence, for each $x\in Z_\Phi$, $a_vx_v\in\ell_1$ if and only if $Ay\in\ell_1$ with $y\in Z$ for $Z\in\{c,\ c_0,\ \ell_\infty\}$. Thus, we may conclude that $a\in[Z_\Phi]^\alpha$ if and only if $A\in(Z,\ell_1)$. Applying Lemma (4.1) part (vi), we obtain $[Z_\Phi]^\alpha=\psi$. □

Theorem 4.3. *The γ-dual of space Z_Φ for $Z\in\{c,\ c_0,\ \ell_\infty\}$ is*

$$\kappa=\left\{a=(a_v)\in s:\sup_{r\in\mathbb{N}_0}\sum_{v=0}^{\infty}|c_{rv}|<\infty\right\},$$

where $C=(c_{rv})$ the matrix defined as

$$C_{rv}=\begin{cases}\binom{m+v}{v}\left[a_v+\binom{m}{m-2}-\binom{m}{m-1}\sum\limits_{i=v+1}^{r}a_i+\sum\limits_{i=v+2}^{r}(-1)^{v-i}\binom{m}{v-i}\left(\sum\limits_{v=i}^{r}a_v\right)\right], & 0\le v\le r\\ 0, & v>r\end{cases} \tag{11}$$

Proof. Let $a = (a_v) \in s,\ x \in Z_\Phi$ and $y \in Z$ for $Z \in \{c, c_0, \ell_\infty\}$. Consider the equality

$$\begin{aligned}
\sum_{v=0}^{r} a_v x_v &= \sum_{v=0}^{r} a_v \left[\sum_{i=0}^{v}\sum_{j=0}^{v-i}(-1)^j\binom{m+i}{i}\binom{m}{j}y_i\right] \\
&= \sum_{v=0}^{r-1}\sum_{i=0}^{v}\sum_{j=0}^{v-i}(-1)^j\binom{m+i}{i}\binom{m}{j}y_r a_v + \sum_{i=0}^{v}\sum_{j=0}^{v-i}(-1)^j\binom{m+i}{i}\binom{m}{j}y_j a_j \\
&= \binom{m+v}{v}\left[a_v + \binom{m}{m-2} - \binom{m}{m-1}\sum_{i=v+1}^{r} a_i + \sum_{i=v+2}^{r}(-1)^{v-i}\binom{m}{v-i}\left(\sum_{v=i}^{r} a_v\right)\right] \\
&= (Cy)_r,
\end{aligned}$$

where $C = (C_{rv})$ is defined in (11).

Thus, $a \in [Z_\Phi]^\gamma$ if and only if $ax \in bs$ for $x \in Z_\Phi$ if and only if $(\sum_{v=0}^{r} a_v x_v) \in \ell_\infty$. So, $Cy \in \ell_\infty$ for $y \in Z$. Hence, by using Lemma (4.1) Part (v) we obtain, $[Z_\Phi]^\gamma = \kappa$. □

We now state the following results to compute the β-dual of the sequence space Z_Φ.

Let T be a triangle matrix with matrix domain Z_T.

Lemma 4.4. *[10] Let Z be a BK-space with AK, and $P = Q^t$, the transpose of the matrix Q, where $Q = (q_{iv})$ be the inverse of matrix T. Then, $a \in [Z_T]^\beta$ if and only if $Pa \in [Z]^\beta$ and the matrix $E \in (Z, c_0)$, where $E = (e_{nv})$ is defined as*

$$e_{nv} = \begin{cases} \sum\limits_{i=n}^{\infty} a_i q_{iv}, & 0 \le v \le n \\ 0, & v > n \end{cases},$$

for every $v,\ n \in \mathbb{N}^0$. Also , if $a \in [Z_T]^\beta$, then $\sum_{v=0}^{\infty} a_v y_v = \sum_{v=0}^{\infty} P_v(a)T_v(x),\ \forall\ x = (x_v) \in Z_T$.

Remark 4.5. *(i) [10] For $Z = \ell_\infty$, the result holds by above lemma.*
(ii) [15] $a \in [c_T]^\beta$ whenever $Pa \in \ell_1$, and $E \in (c, c)$. Also, if $a \in [c_T]^\beta$, then for every $x \in c_T$,

$$\sum_{v=0}^{\infty} a_v x_v = \sum_{v=0}^{\infty} P_v(a)T_v(x) - \nu\rho,$$

where $\nu = \lim_{v\to\infty} T_v(x)$ and $\rho = \lim_{n\to\infty}\sum_{v=0}^{n} e_{nv}$.

Theorem 4.6. *Define the following sets:*

$$d_1 = \left\{a = (a_v) \in s : \sum_{v=0}^{\infty} |P_v(a)| < \infty\right\}$$

$$d_2 = \left\{a = (a_v) \in s : \lim_{n\to\infty} e_{nv} = 0 \quad \textit{for all } v\right\}$$

$$d_3 = \left\{a = (a_v) \in s : \sup_n \sum_{v=0}^{\infty} |e_{nv}| < \infty\right\}$$

$$d_4 = \left\{a = (a_v) \in s : \lim_{n\to\infty} \sum_{v=0}^{n} (e_{nv}) \ \textit{ exists}\right\}$$

$$d_5 = \left\{a = (a_v) \in s : \lim_{n\to\infty} \sum_{v=0}^{n} |e_{nv}| = 0\right\}$$

$$d_6 = \left\{a = (a_v) \in s : \lim_{n\to\infty} e_{nv} \ \textit{ exists for all } v\right\}$$

where

$$P(a) = P_v(a) = \binom{m+v}{v}\left[a_v + \left(\binom{m}{m-2} - \binom{m}{m-1}\right)\sum_{i=v+1}^{\infty} a_i + \sum_{k=2}^{\infty}(-1)^k\binom{m}{k}\left(\sum_{i=v+k}^{\infty} a_i\right)\right]$$

and

$$e_{nv} = \binom{m+v}{v}\left[\sum_{j=0}^{n-v}(-1)^j\binom{m}{j}\sum_{i=n}^{\infty} a_i + \sum_{j=n-v+1}^{\infty}(-1)^j\binom{m}{j}\sum_{i=v+j}^{\infty} a_i\right].$$

Then, $[c_0(\Phi)]^\beta = d_1 \cap d_2 \cap d_3$, $[c(\Phi)]^\beta = d_1 \cap d_3 \cap d_4 \cap d_6$ *and* $[\ell_\infty(\Phi)]^\beta = d_1 \cap d_5$.

Proof. The matrix $T = (t_{rv})$ is defined as

$$t_{rv} = \begin{cases} \frac{1}{\binom{m+r}{r}}\left[\binom{m+r-v-1}{r-v} - \binom{m+r-v-2}{r-v-1}\right], & 0 \le v < r \\ \frac{1}{\binom{m+r}{r}}, & v = r \\ 0, & v > r \end{cases}$$

Let $Q = (q_{iv})$ is the inverse of T. Then,

$$q_{iv} = \begin{cases} \sum_{j=0}^{i-v}(-1)^j\binom{m+v}{v}\binom{m}{j}, & 0 \le v \le i \\ 0, & v > i. \end{cases}$$

Here, we first compute $E = (e_{nv})$ and $P_v(a)$ to get the β-dual. Consider the equality

$$\begin{aligned} P_v(a) &= \sum_{i=v}^{\infty} a_i q_{iv} \\ &= \binom{m+v}{v}a_v + \sum_{i=v+1}^{\infty}\sum_{j=0}^{i-v}(-1)^j\binom{m+v}{v}\binom{m}{j}a_j \\ &= \binom{m+v}{v}\left[a_v + \left\{\binom{m}{m-2} - \binom{m}{m-1}\right\}\sum_{i=v+1}^{\infty} a_i + \sum_{k=2}^{\infty}(-1)^k\binom{m}{k}\sum_{i=v+k}^{\infty} a_i\right] \end{aligned}$$

and

$$\begin{aligned} e_{nv} &= \sum_{i=n}^{\infty} a_i q_{iv} \\ &= \sum_{i=n}^{\infty}\sum_{j=0}^{i-v}(-1)^j\binom{m+v}{v}\binom{m}{j}a_i \\ &= \binom{m+v}{v}\left[\sum_{j=0}^{n-v}(-1)^j\binom{m}{j}\sum_{i=n}^{\infty} a_i + \sum_{j=n-v+1}^{\infty}(-1)^j\binom{m}{j}\sum_{i=v+j}^{\infty} a_i\right]. \end{aligned}$$

By Lemma (4.4) and Remark (4.5), conclude that
$[c_0(\Phi)]^\beta = d_1 \cap d_2 \cap d_3$, $[c(\Phi)]^\beta = d_1 \cap d_3 \cap d_4 \cap d_6$ and $[\ell_\infty(\Phi)]^\beta = d_1 \cap d_5$. $\square$

5. Matrix Mappings on Z_Φ

Here, the necessary and sufficient conditions for matrix transformation from the space Z_Φ to Z, for $Z \in \{c,\ c_0, \ell_\infty\}$ have been discussed in detail.

We begin with the results which are useful in the characterization of matrix classes about the spaces of Cesàro and backward difference operators.

Lemma 5.1. *[10] Let Z is a BK-space having AK, and Y is a sequence space of s and $P = Q^t$. Then, $A \in (Z_T, Y)$ if and only if $B^A \in (Z, Y)$ and $E^{A_r} \in (Z, c_0)$ for $r \in \mathbb{N}_0$, where B^A with rows $B^A_r = P(A_r)$, A_r are the rows of A, and the triangles E^{A_r} are given as*

$$e^{A_r}_{nv} = \begin{cases} \sum\limits_{i=n}^{\infty} a_{ri} q_{iv}, & 0 \le v \le n \\ 0, & v > n \end{cases}$$

Lemma 5.2. *[10] Let Y be any subset of s. Then, $A \in (c_T, Y)$ if and only if $P_v(A_r) \in (c_0, Y)$ and $E^{A_r} \in (c, c)$ for every r and $P_v(A_r)e - (\rho_r) \in Y$, where $\rho_r = \lim\limits_{n\to\infty} \sum_{v=0}^{n} e^{A_r}_{nv}$ for $r \in \mathbb{N}^0$ and, $e = (1, 1, 1, \cdots)$. Also, if $A \in (c_T, Y)$ then, $Az = P_v(A_r)(T(z)) - \nu(\rho_r)$, for every $z \in c_T$ and $\nu = \lim\limits_{v\to\infty} T_v(z)$.*

We now characterize some matrix classes concerning the space $A \in Z_\Phi$ for $Z \in (c,\ c_0, \ell_\infty)$ by considering the following conditions:

$$\sup_{r\in\mathbb{N}_0} \sum_{v=0}^{\infty} |P_v(A_r)| < \infty \tag{12}$$

$$\lim_{r\to\infty} P_v(A_r) = 0, \text{ for all } v \tag{13}$$

$$\lim_{r\to\infty} P_v(A_r) \text{ exists for all } v \tag{14}$$

$$\lim_{r\to\infty} \sum_{v=0}^{\infty} |P_v(A_r)| = 0 \tag{15}$$

$$\sup_{n\in\mathbb{N}_0} \sum_{v=0}^{n} \left|e^{A_r}_{nv}\right| < \infty, \ \forall\ r \tag{16}$$

$$\lim_{n\to\infty} e^{A_r}_{nv} = 0, \text{ for all } r \text{ and } v \tag{17}$$

$$\lim_{n\to\infty} \sum_{v=0}^{n} \left|e^{A_r}_{nv}\right| = 0, \ \forall\ r \tag{18}$$

$$\lim_{r\to\infty} \sum_{v=0}^{\infty} \left|P_v(A_r) - \lim_{r\to\infty} P_v(A_r)\right| = 0 \tag{19}$$

$$\lim_{n\to\infty} e_{nv}^{A_r} \text{ exists for all } r,\ v \tag{20}$$

$$\lim_{n\to\infty} \sum_{v=0}^{n} e_{nv}^{A_r} \text{ exists for all } r \tag{21}$$

$$P_v(A_r)e - (\rho_r) \in Z, \text{ for } Z \in \{c, c_0, \ell_\infty\},\ \forall\ \rho_r \text{ and } r \in \mathbb{N}^0, \tag{22}$$

where $\rho_r = \lim\limits_{n\to\infty} \sum_{v=0}^{n} e_{nv}^{A_r}$,
with

$$P_v(A_r) = \binom{m+v}{v}\left[a_{rv} + \left(\binom{m}{m-2} - \binom{m}{m-1}\right)\sum_{i=v+1}^{\infty} a_{ri} + \sum_{k=2}^{\infty}(-1)^k\binom{m}{k}\left(\sum_{i=v+k}^{\infty} a_{ri}\right)\right],$$

and

$$e_{nv}^{A_r} = \binom{m+v}{v}\left[\sum_{j=0}^{n-v}(-1)^j\binom{m}{j}\sum_{i=n}^{\infty} a_{ri} + \sum_{j=n-v+1}^{\infty}(-1)^j\binom{m}{j}\sum_{i=v+j}^{\infty} a_{ri}\right].$$

Theorem 5.3. *(i) $A = (a_{rv}) \in (c_0(\Phi), \ell_\infty)$ if and only if (12), (16) and, (17) hold.*
(ii) $A = (a_{rv}) \in (c_0(\Phi), c)$ if and only if (12), (14), (16) and, (17) hold.
(iii) $A = (a_{rv}) \in (c_0(\Phi), c_0)$ if and only if (12), (13), (16) and (17) hold.

Proof. Here, part (iii) is considered for verification. One may similarly prove the other parts. The proof is on similar lines to the Theorem (4.6). For this, to prove that $B^A \in (c_0, c_0)$, and $E^{A_r} \in (c_0, c_0)$, it suffices to prove the matrices $B^A = P_v(A_r)$ and $E^{A_r} = (e_{nv}^{A_r})$ for $r \in \mathbb{N}^0$ of Lemma (5.1).

$$\begin{aligned}
P_v(A_r) &= \sum_{i=v}^{\infty} a_{ri} q_{iv} \\
&= \binom{m+v}{v} a_{rv} + \sum_{i=v+1}^{\infty}\sum_{j=0}^{i-v}(-1)^j\binom{m+v}{v}\binom{m}{j} a_{rj} \\
&= \binom{m+v}{v}\left[a_{rv} + \left(\binom{m}{m-2} - \binom{m}{m-1}\right)\sum_{i=v+1}^{\infty} a_{ri} + \sum_{k=2}^{\infty}(-1)^k\binom{m}{k}\left(\sum_{i=v+k}^{\infty} a_{ri}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
e_{nv} &= \sum_{i=n}^{\infty} a_{ri} q_{iv} \\
&= \sum_{i=n}^{\infty}\sum_{j=0}^{i-v}(-1)^j\binom{m+v}{v}\binom{m}{j} a_{ri} \\
&= \binom{m+v}{v}\left[\sum_{j=0}^{n-v}(-1)^j\binom{m}{j}\sum_{i=n}^{\infty} a_{ri} + \sum_{j=n-v+1}^{\infty}(-1)^j\binom{m}{j}\sum_{i=v+j}^{\infty} a_{ri}\right].
\end{aligned}$$

Hence by Lemma (5.2), we conclude that $A \in (c_0(\Phi), c_0)$ if and only if the conditions (12), (13), (16) and (17) hold. $\square$

From Theorem (5.3), we can state the following Corollaries.

Corollary 5.4. *(i)* $A = (a_{rv}) \in (\ell_\infty(\Phi), \ell_\infty)$ *if and only if (12), and (18) hold.*
(ii) $A = (a_{rv}) \in (\ell_\infty(\Phi), c)$ *if and only if (12), (14), (18), and (19) hold.*
(iii) $A = (a_{rv}) \in (\ell_\infty(\Phi), c_0)$ *if and only if (15) and (18) hold.*

Corollary 5.5. *(i)* $A = (a_{rv}) \in (c(\Phi), \ell_\infty)$ *if and only if (12)-(14) and (20)-(22) hold.*
(ii) $A = (a_{rv}) \in (c(\Phi), c)$ *if and only if (12), (14), (16) and (20)-(22) hold.*
(iii) $A = (a_{rv}) \in (c(\Phi), c_0)$ *if and only if (12), (14) and (20)-(22) hold.*

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