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On the set of all generalized Drazin invertible elements in a ring

Fei Peng[a], **Xiaoxiang Zhang**[a,*]

[a] *School of Mathematics, Southeast University, Nanjing 210096, China*

Abstract. Berkani and Sarihr [Studia Math. (2001) 148: 251–257] showed that the set of all Drazin invertible elements in an algebra over a filed is a regularity in the sense of Kordula and Müller [Studia Math. (1996) 119: 109–128]. In this paper, the above result is extended to the case of a ring. Counterexamples are provided to show that the set of all generalized Drazin invertible elements in a ring need not be a regularity in general. We determine when the set of all generalized Drazin invertible matrices in the 2×2 full matrix ring over a commutative local ring is a regularity. We also give a sufficient condition for the set of all generalized Drazin invertible elements in a ring to be a regularity.

1. Introduction

To develop the axiomatic theory of spectrum, Kordula and Müller [14] introduced the notion of a regularity in a complex Banach algebra using a purely algebraic method. Here we restate the definition of a regularity in the setting of rings. Thus, a non-empty subset S in a ring R is called a *regularity* if the following two conditions are satisfied:

(1) for any $a \in R$ and positive integer n, $a \in S \Leftrightarrow a^n \in S$, and

(2) for any mutually commutative elements $a, b, c, d \in R$ such that $ac + bd = 1$, $ab \in S \Leftrightarrow a, b \in S$.

In 2001, Berkani and Sarihr [1] proved that the set of all Drazin invertible elements in an algebra over a filed is a regularity. For the case of generalized Drazin inverse, Lubansky [16] obtained a similar result in a complex Banach algebra.

In this note, the Berkani-Sarihr's result mentioned above is extended to the case of a ring. Counterexamples are provided to show that the set of all generalized Drazin invertible elements in a ring need not be a regularity in general. We determine when the set of all generalized Drazin invertible matrices in the 2×2 full matrix ring over a commutative local ring is a regularity. We also give a sufficient condition for the set of all generalized Drazin invertible elements in a ring to be a regularity.

Throughout this paper, all rings R are associative with unity 1. The symbol $\mathrm{U}(R)$ stands for the set of all invertible elements of R. Write $\mathrm{J}(R)$ to denote the Jacobson radical of R. The commutant of $a \in R$ is denoted by $\mathrm{comm}(a)$, i.e.,

$$\mathrm{comm}(a) = \{x \in R \colon xa = ax\}.$$

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* Corresponding author: Xiaoxiang Zhang
Email addresses: pfmath@163.com (Fei Peng), 101009915@seu.edu.cn (Xiaoxiang Zhang)

Similarly, the double commutant $\text{comm}^2(a) = \{y \in R : yx = xy \text{ for all } x \in \text{comm}(a)\}$. Following Harte [11], an element $a \in R$ is said to be *quasi-nilpotent* if $1 - ax \in U(R)$ for each $x \in \text{comm}(a)$, which is equivalent to $\|a^n\|^{\frac{1}{n}} \to 0$ as $n \to +\infty$ in case R is a complex Banach algebra. Nilpotent elements and elements in the Jacobson radical are well-known examples of quasi-nilpotent elements. We denote by R^{qnil} the set of all quasi-nilpotent elements of R.

Recall that the *Drazin inverse* of $a \in R$, whenever it exists, is the unique element $y \in R$ (denoted by a^{D}) such that $yay = y \in \text{comm}(a)$ and $ya^{k+1} = a^k$ for some non-negative integer k [9]. It is known that $y = a^{\text{D}}$ if and only if $yay = y \in \text{comm}^2(a)$ and $a - aya$ is nilpotent. Based on this fact, Koliha and Patrício [15] introduced the notion of generalized Drazin inverses in a ring. They called $b \in R$ a *generalized Drazin inverse* of a if $bab = b \in \text{comm}^2(a)$ and $a - aba \in R^{\text{qnil}}$. The generalized Drazin inverse of a is unique if it exists, and will be denoted by a^{gD}. It is worth mentioning that if R is a complex Banach algebra, then $b = a^{\text{gD}}$ if and only if $bab = b \in \text{comm}(a)$ and $a - aba \in R^{\text{qnil}}$ (see [13] for the proof and much more, including topological and spectral properties of the generalized Drazin inverse). By R^{D} and R^{gD} we mean the set of all elements which have Drazin inverses and generalized Drazin inverses in R, respectively. An element $a \in R$ is called *quasipolar* [15] if there exists $p \in R$ such that $p^2 = p \in \text{comm}^2(a)$, $ap \in R^{\text{qnil}}$ and $a + p \in U(R)$. Following [18], a ring R is said to be *quasipolar* if each element in R is quasipolar. It is shown [15] that $a \in R^{\text{gD}}$ if and only if it is quasipolar. This fact will be used below repeatedly.

2. Main results

Proposition 2.1. *The set R^{D} of all Drazin invertible elements in any ring R is a regularity.*

Proof. First of all, R^{D} is nonempty since $0, \pm 1 \in R^{\text{D}}$. According to [9, Theorem 4], $a \in R^{\text{D}}$ if and only if there is a positive integer m such that

$$a^m R = a^{m+1} R = a^{m+2} R = \cdots \quad \text{and} \quad Ra^m = Ra^{m+1} = Ra^{m+2} = \cdots$$

From this fact, it is easy to see that, for each integer $n \geq 1$, $a \in R^{\text{D}}$ if and only if $a^n \in R^{\text{D}}$ (see [2, Theorem 11.5], [9, Theorem 2] and [12, Theorem 2.1] for different proofs).

Let $a, b, c, d \in R$ be mutually commuting elements such that $ac + bd = 1$. If $a, b \in R^{\text{D}}$, then $a^k \in a^{k+1}R \cap Ra^{k+1}$ and $b^k \in b^{k+1}R \cap Rb^{k+1}$ for some positive integer k. One easily shows that $(ab)^k \in (ab)^{k+1}R \cap R(ab)^{k+1}$. Thus $ab \in R^{\text{D}}$ in view of [9, Theorem 4].

Conversely, suppose $ab \in R^{\text{D}}$ with $(ab)^m = (ab)^{m+1}(ab)^{\text{D}}$, we shall prove $a, b \in R^{\text{D}}$. From the binomial expansion of $(ac + bd)^{2m+1} = 1$ one can obtain $c', d' \in \text{comm}(a) \cap \text{comm}(b)$ such that $a^{m+1}c' + b^{m+1}d' = 1$. Let $y = a^m - a^{m+1}b(ab)^{\text{D}}$, then $a^m = y + a^{m+1}(ab)^{\text{D}}b$ and

$$\begin{aligned} y &= (a^{m+1}c' + b^{m+1}d')y \\ &= a^{m+1}c'y + d'b^{m+1}[a^m - a^{m+1}b(ab)^{\text{D}}] \\ &= a^{m+1}c'y + d'b[(ab)^m - (ab)^{m+1}(ab)^{\text{D}}] \\ &= a^{m+1}c'y \in a^{m+1}R. \end{aligned}$$

So we have $a^m \in a^{m+1}R$. Similarly, $a^m \in Ra^{m+1}$ and $b^m \in b^{m+1}R \cap Rb^{m+1}$. Therefore $a, b \in R^{\text{D}}$. □

The following lemma will be repeatedly used in the sequel.

Lemma 2.2. *Let $a \in R$. If $a^n \in R^{\text{gD}}$ for some integer $n > 1$, then $a \in R^{\text{gD}}$ with $a^{\text{gD}} = a^{n-1}(a^n)^{\text{gD}} = (a^n)^{\text{gD}}a^{n-1}$ and $(a^n)^{\text{gD}} = (a^{\text{gD}})^n$. In particular, $a^n \in R^{\text{qnil}}$ implies $a \in R^{\text{qnil}}$.*

Proof. Suppose $a^n \in R^{\text{gD}}$. Then $a \in R^{\text{gD}}$ and $a^{\text{gD}} = (a^n)^{\text{gD}}a^{n-1}$ (see, for instance, [12, Theorem 2.7 (i)]). From $(a^n)^{\text{gD}} \in \text{comm}^2(a^n)$ and $a^{n-1} \in \text{comm}(a^n)$, we derive $(a^n)^{\text{gD}}a^{n-1} = a^{n-1}(a^n)^{\text{gD}}$, and hence $(a^{\text{gD}})^n = a^{\text{gD}}a(a^{\text{gD}})^n = [(a^n)^{\text{gD}}a^{n-1}]a(a^{\text{gD}})^n = (a^n)^{\text{gD}}a^n(a^{\text{gD}})^n = (a^n)^{\text{gD}}aa^{\text{gD}} = (a^n)^{\text{gD}}a[a^{n-1}(a^n)^{\text{gD}}] = (a^n)^{\text{gD}}$.

The last statement follows from the fact that $a \in R^{\text{qnil}}$ if and only if $a^{\text{gD}} = 0$. □

Next, we provide two examples of rings in which the set of all generalized Drazin invertible elements is not a regularity.

Example 2.3. Let $S = \mathbb{Z}_2[t_1, t_2, \ldots]$ be the ring of all polynomials in countably many indeterminates over the field $\mathbb{Z}_2$ of integers modulo 2 and $S_{(t_1)}$ denote the localization of S at the prime ideal (t_1). Consider the ring endomorphism $\sigma : S_{(t_1)} \to S_{(t_1)}$ induced by $\sigma(t_i) = t_{i+1}$ for all $i \geq 1$. Let $S_{(t_1)}[[x;\sigma]]$ be the skew formal power series ring over $S_{(t_1)}$ subject to $xa = \sigma(a)x$ for all $a \in S_{(t_1)}$ and $R = \mathrm{T}_2(S_{(t_1)}[[x;\sigma]])$ be the ring of all 2×2 upper triangular matrices over $S_{(t_1)}[[x;\sigma]]$, then

(i) $A = \begin{pmatrix} t_2 & x \\ 0 & -t_1 \end{pmatrix} \in R^{\mathrm{gD}}$ but $A^2 = \begin{pmatrix} t_2^2 & 0 \\ 0 & t_1^2 \end{pmatrix} \notin R^{\mathrm{gD}}$;

(ii) $B = \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}$, $C = \begin{pmatrix} t_2^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ commute with each other, $BC + D^2 = I$, where I denotes the identity of R, and $BD \in R^{\mathrm{gD}}$ but $B \notin R^{\mathrm{gD}}$.

Proof. (i) We claim that $A \in R^{\mathrm{qnil}}$. Indeed, suppose that

$$X = \begin{pmatrix} \sum\limits_{i=0}^{\infty} \mu_i x^i & \sum\limits_{i=0}^{\infty} \nu_i x^i \\ 0 & \sum\limits_{i=0}^{\infty} \rho_i x^i \end{pmatrix} \in R$$

commutes with A. Write $\mu_{-1} = \nu_{-1} = \rho_{-1} = 0$. Then, we have

$$AX = \begin{pmatrix} \sum\limits_{i=0}^{\infty} t_2 \mu_i x^i & \sum\limits_{i=0}^{\infty} [t_2 \nu_i + \sigma(\rho_{i-1})] x^i \\ 0 & -\sum\limits_{i=0}^{\infty} t_1 \rho_i x^i \end{pmatrix}$$

and

$$XA = \begin{pmatrix} \sum\limits_{i=0}^{\infty} \mu_i t_{2+i} x^i & \sum\limits_{i=0}^{\infty} [\mu_{i-1} - \nu_i t_{1+i}] x^i \\ 0 & \sum\limits_{i=0}^{\infty} -\rho_i t_{1+i} x^i \end{pmatrix}.$$

Now $AX = XA$ implies

$$(t_2 - t_{2+i})\mu_i = 0, \tag{1}$$

$$(t_1 - t_{1+i})\rho_i = 0, \tag{2}$$

$$t_2 \nu_i + \sigma(\rho_{i-1}) = \mu_{i-1} - \nu_i t_{1+i}, \tag{3}$$

for all $i \in \mathbb{N}$.

From the above equalities (1) and (2) one can see that $\mu_j = \rho_j = 0$ for $j \geq 1$ since $t_2 - t_{2+j}$ and $t_1 - t_{1+j}$ are invertible in $S_{(t_1)}[[x;\sigma]]$. Combining this fact with the above equality (3), we obtain

$$(t_2 + t_1)\nu_0 = 0,\ \sigma(\rho_0) = \mu_0 \text{ and } (t_2 + t_{1+j})\nu_j = 0 \text{ for } j > 1.$$

Consequently, it follows that $\nu_0 = \nu_2 = \nu_3 = \cdots = 0$ and $t_2 \mu_0 \neq 1$. We thus conclude

$$I - AX = \begin{pmatrix} 1 - t_2 \mu_0 & (\mu_0 - \nu_1 t_2) x \\ 0 & 1 + t_1 \rho_0 \end{pmatrix} \in \mathrm{U}(R).$$

This shows $A \in R^{\mathrm{qnil}}$ and hence $A \in R^{\mathrm{gD}}$ with $A^{\mathrm{gD}} = 0$.

Assume that $A^2 \in R^{\mathrm{gD}}$. By Lemma 2.2, we have $(A^2)^{\mathrm{gD}} = (A^{\mathrm{gD}})^2 = 0$, which means $A^2 \in R^{\mathrm{qnil}}$. However, there is a matrix $C = \begin{pmatrix} t_2^{-2} & 0 \\ 0 & 0 \end{pmatrix} \in R$ such that $A^2 C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = CA^2$ and $I - A^2 C \notin \mathrm{U}(R)$, a contradiction.

(ii) It is clear that B, C and D commute with each other, $BC + D^2 = I$ and $BD \in \mathrm{J}(R) \subseteq R^{\mathrm{gD}}$. From [5, Example 2.11], we know that B is not quasipolar, i.e., not generalized Drazin invertible. □

Example 2.4. Let $\mathbb{Z}_{(3)}$ be the localization of the ring $\mathbb{Z}$ of integers at the prime ideal $3\mathbb{Z}$ and $R = \mathrm{M}_2(\mathbb{Z}_{(3)})$ be the ring of all 2×2 matrices over $\mathbb{Z}_{(3)}$. Consider the following matrices

$$A = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 6 \\ 2 & 3 \end{pmatrix}, C = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } D = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R.$$

An easy computation shows that A, B, C and D are mutually commutative, $AC+BD = I_2$ and $AB \in \mathrm{J}(R) \subseteq R^{\mathrm{gD}}$. However, in view of [6, Corollary 2.14], $B \notin R^{\mathrm{gD}}$ because $\det B \in \mathrm{J}(\mathbb{Z}_{(3)})$, $\mathrm{tr}B \in \mathrm{U}(\mathbb{Z}_{(3)})$ and the equation $x^2 = (\mathrm{tr}B)^2 - 4\det B = 52$ has no solution in $\mathrm{U}(\mathbb{Z}_{(3)})$, where $\det B$ and $\mathrm{tr}B$ denote, respectively, the determinant and trace of B.

Remark 2.5. Let $\mathrm{M}_2(R)$ be the 2×2 full matrix ring over an arbitrary commutative local ring R. We remark that $(\mathrm{M}_2(R))^{\mathrm{gD}}$ is "almost" a regularity, i.e., (1) for any integer $n > 1$, $X \in (\mathrm{M}_2(R))^{\mathrm{gD}}$ if and only if $X^n \in (\mathrm{M}_2(R))^{\mathrm{gD}}$; (2) if $A, B, C, D \in \mathrm{M}_2(R)$ are mutually commutative, $AC + BD = I_2$ and $A, B \in (\mathrm{M}_2(R))^{\mathrm{gD}}$, then $AB \in (\mathrm{M}_2(R))^{\mathrm{gD}}$. Indeed, according to Lemma 2.2, it suffices to show $AB \in (\mathrm{M}_2(R))^{\mathrm{gD}}$ under the hypothesis of $A, B \in (\mathrm{M}_2(R))^{\mathrm{gD}}$ and $AB = BA$. First of all, using [7, Proposition 4.1], we have $(A - AA^{\mathrm{gD}}A)^2, (B - BB^{\mathrm{gD}}B)^2 \in \mathrm{J}(\mathrm{M}_2(R))$. Then, by $A^{\mathrm{gD}} \in \mathrm{comm}^2(A)$ and $B^{\mathrm{gD}} \in \mathrm{comm}^2(B)$, it follows that $B^{\mathrm{gD}}A^{\mathrm{gD}}ABB^{\mathrm{gD}}A^{\mathrm{gD}} = B^{\mathrm{gD}}A^{\mathrm{gD}} \in \mathrm{comm}(AB)$ and

$$\begin{aligned}(AB - ABB^{\mathrm{gD}}A^{\mathrm{gD}}AB)^2 &= [(A - AA^{\mathrm{gD}}A)B + AA^{\mathrm{gD}}A(B - BB^{\mathrm{gD}}B)]^2 \\ &= (A - AA^{\mathrm{gD}}A)^2B^2 + (AA^{\mathrm{gD}}A)^2(B - BB^{\mathrm{gD}}B)^2 \\ &\in \mathrm{J}(\mathrm{M}_2(R)) \subseteq (\mathrm{M}_2(R))^{\mathrm{qnil}}.\end{aligned}$$

Consequently, $AB - ABB^{\mathrm{gD}}A^{\mathrm{gD}}AB \in (\mathrm{M}_2(R))^{\mathrm{qnil}}$ by Lemma 2.2. Let $P = I_2 - ABB^{\mathrm{gD}}A^{\mathrm{gD}}$, from the proof of [15, Theorem 4.2], one can see that $P^2 = P \in \mathrm{comm}(AB)$, $ABP \in (\mathrm{M}_2(R))^{\mathrm{qnil}}$ and $AB + P \in \mathrm{U}(\mathrm{M}_2(R))$. Therefore, we obtain $AB \in (\mathrm{M}_2(R))^{\mathrm{gD}}$ by [7, Proposition 3.5].

For the $n \times n$ full matrix ring over a commutative local ring without zero divisor, we have the following result.

Proposition 2.6. *Let n be any integer greater than 1 and $\mathrm{M}_n(R)$ be the ring of all $n \times n$ matrices over a commutative local ring R without zero divisor. If $A, B \in (\mathrm{M}_n(R))^{\mathrm{gD}}$ and $AB = BA$, then $AB \in (\mathrm{M}_n(R))^{\mathrm{gD}}$.*

Proof. Let $C = AB$ for convenience. Similarly to Remark 2.5, by virtue of [4, Theorem 2.5], we conclude that there exists $P \in \mathrm{M}_n(R)$ such that $P^2 = P \in \mathrm{comm}(C)$, $CP \in (\mathrm{M}_n(R))^{\mathrm{qnil}}$ and $C + P \in \mathrm{U}(\mathrm{M}_n(R))$. Note that R is projective free (see, e.g., [10, Charpter VIII, Proposition 4.8]). From [3, Charpter 0, Proposition 4.5], there is $V \in \mathrm{U}(\mathrm{M}_n(R))$ such that $P = V\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}V^{-1}$, where $0 \leq r \leq n$. If $r = 0$ then C is invertible and hence the result is clear. If $r = n$ then $C \in (\mathrm{M}_n(R))^{\mathrm{qnil}} \subseteq (\mathrm{M}_n(R))^{\mathrm{gD}}$.

Now suppose $0 < r < n$ and write $V^{-1}CV = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$, where C_1 is an $r \times r$ matrix over R. Then $CP = PC$ implies $C_2 = C_3 = 0$. Moreover,

$$V\begin{pmatrix} C_1 + I_r & 0 \\ 0 & C_4 \end{pmatrix}V^{-1} = V\begin{pmatrix} C_1 & 0 \\ 0 & C_4 \end{pmatrix}V^{-1} + V\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}V^{-1} = C + P \in \mathrm{U}(\mathrm{M}_n(R))$$

implies $C_4 \in \mathrm{U}(\mathrm{M}_{n-r}(R))$. Furthermore,

$$V\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}V^{-1} = V\begin{pmatrix} C_1 & 0 \\ 0 & C_4 \end{pmatrix}V^{-1}V\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}V^{-1} = CP \in (\mathrm{M}_n(R))^{\mathrm{qnil}}$$

gives rise to $\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \in (\mathrm{M}_n(R))^{\mathrm{qnil}}$ by [6, Lemma 2.3]. For any $D \in \mathrm{comm}(C_1)$, we have $\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, and hence $\begin{pmatrix} I_r - C_1D & 0 \\ 0 & I_{n-r} \end{pmatrix} \in \mathrm{U}(\mathrm{M}_n(R))$. This means $I_r - C_1D \in \mathrm{U}(\mathrm{M}_r(R))$, i.e., $C_1 \in (\mathrm{M}_r(R))^{\mathrm{qnil}}$. From [4, Theorem 2.5] it follows that $C_1^k \in \mathrm{J}(\mathrm{M}_r(R))$ for some $k \geq 1$. Write $W = V^{-1}C^kV = \begin{pmatrix} C_1^k & 0 \\ 0 & C_4^k \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 0 \\ 0 & C_4^{-k} \end{pmatrix}$. We proceed to show that $W \in (\mathrm{M}_n(R))^{\mathrm{gD}}$ with $W^{\mathrm{gD}} = X$. A trivial verification gives that $XWX = X$ and $W - WXW \in \mathrm{J}(\mathrm{M}_n(R)) \subseteq (\mathrm{M}_n(R))^{\mathrm{qnil}}$. We next prove $X \in \mathrm{comm}^2(W)$. For any $Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \in \mathrm{comm}(W)$ with $Y_1 \in \mathrm{M}_r(R)$, one has

$$\begin{pmatrix} C_1^k Y_1 & C_1^k Y_2 \\ C_4^k Y_3 & C_4^k Y_4 \end{pmatrix} = WY = YW = \begin{pmatrix} Y_1 C_1^k & Y_2 C_4^k \\ Y_3 C_1^k & Y_4 C_4^k \end{pmatrix}.$$

Write $f(x) = a_0 + a_1 x + \cdots + x^{n-r}$ for the characteristic polynomial of C_4^k. Clearly $a_0 \in \mathrm{U}(R)$ since $C_4 \in \mathrm{U}(\mathrm{M}_{n-r}(R))$. By the Hamilton-Cayley theorem, $f(C_4^k) = 0$ and so $(C_4^k)^{\mathrm{D}} = (C_4^k)^{-1} = g(C_4^k)$, where $g(x) = -a_0^{-1}a_1 - \cdots - a_0^{-1}x^{n-r-1}$. Let F be the quotient field of R. Note that C_1^k is Drazin invertible in $\mathrm{M}_r(F)$ (see [9, Corollary 5]). Then we have

$$(C_1^k)^{\mathrm{D}} Y_2 = Y_2 (C_4^k)^{\mathrm{D}} = Y_2 g(C_4^k) = g(C_1^k) Y_2,$$

where the first equality can be obtained by a similar argument to the proof of [8, Theorem 2.2], and the last equality follows from $Y_2 C_4^k = C_1^k Y_2$. Hence

$$[I_r - C_1^k (C_1^k)^{\mathrm{D}}] Y_2 = Y_2 - C_1^k Y_2 (C_4^k)^{\mathrm{D}} = Y_2 [I_r - C_4^k (C_4^k)^{\mathrm{D}}] = 0,$$

and so $Y_2 = C_1^k (C_1^k)^{\mathrm{D}} Y_2$. Consequently,

$$\begin{aligned} [I_r - g(C_1^k) C_1^k] Y_2 &= Y_2 - g(C_1^k) C_1^k Y_2 \\ &= C_1^k (C_1^k)^{\mathrm{D}} Y_2 - C_1^k g(C_1^k) Y_2 \\ &= C_1^k [(C_1^k)^{\mathrm{D}} Y_2 - Y_2 g(C_4^k)] \\ &= C_1^k [(C_1^k)^{\mathrm{D}} Y_2 - Y_2 (C_4^k)^{\mathrm{D}}] \\ &= 0. \end{aligned}$$

Since $C_1^k \in \mathrm{J}(\mathrm{M}_r(R)) = \mathrm{M}_r(\mathrm{J}(R))$ and all the coefficients of $g(x)$ are in R, we conclude that $g(C_1^k)C_1^k \in \mathrm{M}_r(\mathrm{J}(R)) = \mathrm{J}(\mathrm{M}_r(R))$. This forces $I_r - g(C_1^k)C_1^k \in \mathrm{U}(\mathrm{M}_r(R))$ and hence $Y_2 = 0$ as we have seen that $[I_r - g(C_1^k)C_1^k]Y_2 = 0$. In the same manner one can show that $Y_3 = 0$. In addition, the equation $C_4^k Y_4 = Y_4 C_4^k$ implies $C_4^{-k} Y_4 = Y_4 C_4^{-k}$. Whence it follows that $YX = XY$, showing $X \in \mathrm{comm}^2(W)$. Thus, $W^{\mathrm{gD}} = X$ as desired. In view of [6, Lemma 2.3], $C^k = VWV^{-1} \in (\mathrm{M}_n(R))^{\mathrm{gD}}$. Finally, we obtain that $C = AB \in (\mathrm{M}_n(R))^{\mathrm{gD}}$ by Lemma 2.2. □

The above Example 2.4 and Remark 2.5 motivate us to consider under what condition $(\mathrm{M}_2(R))^{\mathrm{gD}}$ is a regularity.

Theorem 2.7. *Let $\mathrm{M}_2(R)$ be the 2×2 matrix ring over a commutative local ring R. Then $(\mathrm{M}_2(R))^{\mathrm{gD}}$ is a regularity if and only if $\mathrm{M}_2(R)$ is quasipolar.*

Proof. Suppose that $(\mathrm{M}_2(R))^{\mathrm{gD}}$ is a regularity. By [7, Theorem 3.7], it suffices to prove that for any $u \in \mathrm{U}(R)$ and $j \in \mathrm{J}(R)$, $\begin{pmatrix} 0 & j \\ 1 & u \end{pmatrix}$ is quasipolar. Let

$$A = \begin{pmatrix} 0 & j \\ 1 & u \end{pmatrix}, C = \begin{pmatrix} u^{-1} - u - 2u^{-1}j & j \\ 1 & u^{-1} - 2u^{-1}j \end{pmatrix},$$

$$B = \begin{pmatrix} -u & j \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 2u^{-1}j - u^{-1} & j \\ 1 & 2u^{-1}j - u^{-1} + u \end{pmatrix}.$$

One can check that A, B, C and D are mutually commutative, $AC + BD = I_2$ and $AB \in \mathrm{J}(\mathrm{M}_2(R)) \subseteq (\mathrm{M}_2(R))^{\mathrm{gD}}$. Therefore $A \in (\mathrm{M}_2(R))^{\mathrm{gD}}$, i.e., A is quasipolar.

The converse is obvious. □

We refer the readers to [6, 7] for more sufficient and necessary conditions under which the 2×2 matrix ring $M_2(R)$ over a commutative local ring R is quasipolar.

Recall that a ring R is said to be *abelian* if every idempotent in R is central. Following Nicholson and Zhou [17], we say that idempotents in a ring R *lift strongly modulo an ideal* I if, whenever $a^2 - a \in I$, there exists $e^2 = e \in aR$ (equivalently $e^2 = e \in Ra$) such that $e - a \in I$. As usual, we write $\sqrt{J(R)} = \{x \in R : x^n \in J(R)$ for some positive integer $n\}$.

Theorem 2.8. *Let R be an abelian ring such that $R^{\text{qnil}} \subseteq \sqrt{J(R)}$ and idempotents in R lift strongly modulo $J(R)$, then R^{gD} is a regularity.*

Proof. We will use the following fact repeatedly in the sequel: if $a, x \in R$ satisfy $xax = x$, then $x \in \text{comm}^2(a)$. Indeed, for any $y \in \text{comm}(a)$, $yx = y(xa)x = (xa)yx = x(ya)x = xy(ax) = x(ax)y = (xax)y = xy$ since R is abelian.

Given an integer $n \geq 1$, if $a^n \in R^{\text{gD}}$, then by Lemma 2.2, we have $a \in R^{\text{gD}}$. Conversely, suppose $a \in R^{\text{gD}}$. Then $(a^{\text{gD}})^n a^n (a^{\text{gD}})^n = (a^{\text{gD}})^n$ and hence $(a^{\text{gD}})^n \in \text{comm}^2(a^n)$. Note that $a - aa^{\text{gD}}a \in R^{\text{qnil}} \subseteq \sqrt{J(R)}$. We have $(a - aa^{\text{gD}}a)^{kn} \in J(R) \subseteq R^{\text{qnil}}$ for some integer $k \geq 1$. Consequently, it follows that $a^n - a^n(a^{\text{gD}})^n a^n = (a - aa^{\text{gD}}a)^n \in R^{\text{qnil}}$ by Lemma 2.2. Thus, $a^n \in R^{\text{gD}}$.

Now let $a, b, c, d \in R$ be mutually commutative elements such that $ac + bd = 1$. If $a, b \in R^{\text{gD}}$ then it follows that $b^{\text{gD}}a^{\text{gD}}abb^{\text{gD}}a^{\text{gD}} = b^{\text{gD}}bb^{\text{gD}}(a^{\text{gD}}a)a^{\text{gD}} = b^{\text{gD}}a^{\text{gD}} \in \text{comm}^2(ab)$ and

$$(ab - abb^{\text{gD}}a^{\text{gD}}ab)^l = (a - aa^{\text{gD}}a)^l b^l + (aa^{\text{gD}}a)^l (b - bb^{\text{gD}}b)^l \in J(R) \subseteq R^{\text{qnil}}$$

for some positive integer l. Using Lemma 2.2 we get $ab - abb^{\text{gD}}a^{\text{gD}}ab \in R^{\text{qnil}}$. This shows $b^{\text{gD}}a^{\text{gD}} = (ab)^{\text{gD}}$. Conversely, if $ab \in R^{\text{gD}}$, write $p = 1 - ab(ab)^{\text{gD}}$. Since $abp \in R^{\text{qnil}} \subseteq \sqrt{J(R)}$, it follows that $(abp)^t \in J(R)$ for some positive integer t. Note that $a, b, c, d, p, (ab)^{\text{gD}}$ commute with each other as $(ab)^{\text{gD}} \in \text{comm}^2(ab)$ and $a, b, c, d \in \text{comm}(ab)$. Let $g = b(ab)^{\text{gD}} + pc$ and $h = 1 - (1 - ga)^t$, then $h \in aR \cap Ra$ and

$$\begin{aligned} a^t - a^t h &= (a - aga)^t = (a - ab(ab)^{\text{gD}}a - apca)^t \\ &= (pa - apca)^t = [ap(1 - ac)]^t \\ &= (apbd)^t = (abp)^t d^t \in J(R). \end{aligned}$$

Hence

$$a^t + J(R) = a^t h + J(R) = ha^t + J(R) \in [a^{t+1}R + J(R)] \cap [Ra^{t+1} + J(R)].$$

This implies that $a + J(R) \in (R/J(R))^{\text{D}}$ by [9, Theorem 4]. Let $x \in R$ with $x + J(R) = (a + J(R))^{\text{D}}$, one has that $ax - (ax)^2 \in J(R)$ and $(a - axa)^m \in J(R)$ for some positive integer m. As idempotents in R lift strongly modulo $J(R)$, there is an idempotent $e \in R$ such that $ax - e \in J(R)$ and $e = axw$ for some $w \in R$. It is easily seen that $(xwe)a(xwe) = xwe \in \text{comm}^2(a)$ and

$$[a - a(xwe)a]^m = [(1 - e)a]^m = [(a - axa) + (ax - e)a]^m \in J(R) \subseteq R^{\text{qnil}}.$$

By Lemma 2.2, $a - a(xwe)a \in R^{\text{qnil}}$. Therefore $a \in R^{\text{gD}}$ with $a^{\text{gD}} = xwe$. Similarly one gets $b \in R^{\text{gD}}$. □

Remark 2.9. (1) Note that the ring $\mathbb{Z}$ of integers, the polynomial ring $\mathbb{Z}[x]$, the formal power series ring $\mathbb{Z}[[x]]$ and all local rings satisfy the hypothesis of Theorem 2.8.

(2) Let $R = R_1 \times R_2$ be the direct product of two rings R_1 and R_2 such that R_1^{gD} and R_2^{gD} are regularities, then $R^{\text{gD}} = R_1^{\text{gD}} \times R_2^{\text{gD}}$ is a regularity. Thus, one can construct more examples of rings R in which R^{gD} is a regularity.

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