On the set of all generalized Drazin invertible elements in a ring

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Abstract. Berkani and Sarihr [Studia Math. (2001) 148: 251–257] showed that the set of all Drazin invertible elements in an algebra over a field is a regularity in the sense of Kordula and Müller [Studia Math. (1996) 119: 109–128]. In this paper, the above result is extended to the case of a ring. Counterexamples are provided to show that the set of all generalized Drazin invertible elements in a ring need not be a regularity in general. We determine when the set of all generalized Drazin invertible matrices in the $2 \times 2$ full matrix ring over a commutative local ring is a regularity. We also give a sufficient condition for the set of all generalized Drazin invertible elements in a ring to be a regularity.

1. Introduction

To develop the axiomatic theory of spectrum, Kordula and Müller [14] introduced the notion of a regularity in a complex Banach algebra using a purely algebraic method. Here we restate the definition of a regularity in the setting of rings. Thus, a non-empty subset $S$ in a ring $R$ is called a regularity if the following two conditions are satisfied:

1. For any $a \in R$ and positive integer $n$, $a \in S \iff a^n \in S$, and
2. For any mutually commutative elements $a, b, c, d \in R$ such that $ac + bd = 1$, $ab \in S \iff a, b \in S$.

In 2001, Berkani and Sarihr [1] proved that the set of all Drazin invertible elements in an algebra over a field is a regularity. For the case of generalized Drazin inverse, Lubansky [16] obtained a similar result in a complex Banach algebra.

In this note, the Berkani-Sarihr’s result mentioned above is extended to the case of a ring. Counterexamples are provided to show that the set of all generalized Drazin invertible elements in a ring need not be a regularity in general. We determine when the set of all generalized Drazin invertible matrices in the $2 \times 2$ full matrix ring over a commutative local ring is a regularity. We also give a sufficient condition for the set of all generalized Drazin invertible elements in a ring to be a regularity.

Throughout this paper, all rings $R$ are associative with unity $1$. The symbol $U(R)$ stands for the set of all invertible elements of $R$. Write $J(R)$ to denote the Jacobson radical of $R$. The commutant of $a \in R$ is denoted by $\text{comm}(a)$, i.e.,

$$\text{comm}(a) = \{x \in R: xa = ax\}.$$
Similarly, the double commutant comm²(a) = {y ∈ R : xy = yx for all x ∈ comm(a)}. Following Harte [11], an element a ∈ R is said to be quasi-nilpotent if 1 − ax ∈ U(R) for each x ∈ comm(a), which is equivalent to ||aⁿ|| → 0 as n → +∞ in case R is a complex Banach algebra. Nilpotent elements and elements in the Jacobson radical are well-known examples of quasi-nilpotent elements. We denote by Rqnil the set of all quasi-nilpotent elements of R.

Recall that the Drazin inverse of a ∈ R, whenever it exists, is the unique element y ∈ R (denoted by a⁻D) such that yay = y ∈ comm(a) and yay⁺₁ = a for some non-negative integer k [9]. It is known that y = a⁻D if and only if yay = y ∈ comm²(a) and a − aya is nilpotent. Based on this fact, Koliha and Patricio [15] introduced the notion of generalized Drazin inverses in a ring. They called b ∈ R a generalized Drazin inverse of a if bab = b ∈ comm²(a) and a − aba ∈ Rqnil. The generalized Drazin inverse of a is unique if it exists, and will be denoted by a⁻Dg. It is worth mentioning that if R is a complex Banach algebra, then b = a⁻D if and only if bab = b ∈ comm(a) and a − aba ∈ Rqnil (see [13] for the proof and much more, including topological and spectral properties of the generalized Drazin inverse). By R¹D and RqD we mean the set of all elements which have Drazin inverses and generalized Drazin inverses in R, respectively. An element a ∈ R is called quasipolar [15] if there exists p ∈ R such that p² = p ∈ comm²(a), ap ∈ Rqnil and a + p ∈ U(R). Following [18], a ring R is said to be quasipolar if each element in R is quasipolar. It is shown [15] that a ∈ RqD if and only if it is quasipolar. This fact will be used below repeatedly.

2. Main results

Proposition 2.1. The set R¹D of all Drazin invertible elements in any ring R is a regularity.

Proof. First of all, R¹D is nonempty since 0, ±1 ∈ R¹D. According to [9, Theorem 4], a ∈ R¹D if and only if there is a positive integer m such that

\[ aⁿR = aⁿ⁺１R = aⁿ⁺２R = \cdots \text{ and } Raⁿ = Raⁿ⁺１ = Raⁿ⁺₂ = \cdots \]

From this fact, it is easy to see that, for each integer n ≥ 1, a ∈ R¹D if and only if aⁿ ∈ R¹D (see [2, Theorem 11.5], [9, Theorem 2] and [12, Theorem 2.1] for different proofs).

Let a, b, c, d ∈ R be mutually commuting elements such that ac + bd = 1. If a, b ∈ R¹D, then aᵏ ∈ aᵏ⁺₁R ∩ R⁺ and bᵏ ∈ bᵏ⁺₁R ∩ Rbᵏ⁺₁ for some positive integer k. One easily shows that (ab)ᵏ ∈ (ab)ᵏ⁺₁R ∩ R(ab)ᵏ⁺₁. Thus ab ∈ R¹D in view of [9, Theorem 4].

Conversely, suppose ab ∈ R¹D with (ab)ᵐ = (ab)ᵐ⁺₁(ab)D, we shall prove a, b ∈ R¹D. From the binomial expansion of (ac + bd)₂ᵐ⁺₁ = 1 one can obtain c', d' ∈ comm(a) ∩ comm(b) such that aᵐ⁺¹c' + bᵐ⁺¹d' = 1. Let y = aᵐ − aᵐ⁺¹b(ab)D, then aᵐ⁺¹c'y + bᵐ⁺¹d'y = 1.

\[ y = (aᵐ⁺¹c'y + bᵐ⁺¹d'y) = aᵐ⁺¹c'y + d'b⁺¹[aᵐ − aᵐ⁺¹b(ab)D] = aᵐ⁺¹c'y + d'b[aᵐ − (ab)ᵐ⁺¹(ab)D] = aᵐ⁺¹c'y + bᵐ⁺¹R. \]

So we have aᵐ⁺¹ ∈ aᵐ⁺¹R. Similarly, aᵐ ∈ Raᵐ⁺¹ and bᵐ ∈ bᵐ⁺¹R ∩ Rbᵐ⁺¹. Therefore a, b ∈ R¹D. □

The following lemma will be repeatedly used in the sequel.

Lemma 2.2. Let a ∈ R. If aⁿ ∈ Rq¹D for some integer n > 1, then a ∈ Rq¹D with a⁻¹Dg = a⁻¹Dg⁻¹ = (a⁻¹D)⁻¹ and (aⁿ)⁻¹Dg = (aⁿ⁻¹)⁻¹Dg = (aⁿ⁻¹)⁻¹Dg⁻¹. In particular, aⁿ ∈ Rqnil implies a ∈ Rqnil.

Proof. Suppose aⁿ ∈ Rq¹D. Then a ∈ Rq¹D and a⁻¹Dg = (a⁻¹D)⁻¹ (see, for instance, [12, Theorem 2.7 (i)]). From (aⁿ)⁻¹Dg ∈ comm²(aⁿ) and a⁻¹Dg ∈ comm(a⁻¹), we derive (a⁻¹D)⁻¹ = a⁻¹Dg⁻¹ and hence (a⁻¹D)⁻¹ = a⁻¹Dg⁻¹ = (a⁻¹D)⁻¹ = a⁻¹Dg⁻¹ = (a⁻¹D)⁻¹. The last statement follows from the fact that a ∈ Rqnil if and only if a⁻¹D = 0. □
Next, we provide two examples of rings in which the set of all generalized Drazin invertible elements is not a regularity.

**Example 2.3.** Let \( S = \mathbb{Z}_2[t_1, t_2, \ldots] \) be the ring of all polynomials in countably many indeterminates over the field \( \mathbb{Z}_2 \) of integers modulo 2 and \( S_{(t_i)} \) denote the localization of \( S \) at the prime ideal \((t_i)\). Consider the ring endomorphism \( \sigma : S_{(t_i)} \rightarrow S_{(t_i)} \) induced by \( \sigma(t_i) = t_{i+1} \) for all \( i \geq 1 \). Let \( S_{(t_i)}[[x; \sigma]] \) be the skew formal power series ring over \( S_{(t_i)} \) subject to \( xa = \sigma(a)x \) for all \( a \in S_{(t_i)} \) and \( R = T_2(S_{(t_i)}[[x; \sigma]]) \) be the ring of all \( 2 \times 2 \) upper triangular matrices over \( S_{(t_i)}[[x; \sigma]] \), then

(i) \( A = \begin{pmatrix} t_2 & x \\ 0 & -t_i \end{pmatrix} \in R^{\text{qnil}} \) but \( A^2 = \begin{pmatrix} t_2^2 & 0 \\ 0 & t_i^2 \end{pmatrix} \notin R^{\text{qnil}} \);

(ii) \( B = \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}, \ C = \begin{pmatrix} t_2^2 & 0 \\ 0 & 0 \end{pmatrix} \) and \( D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) commute with each other, \( BC + D^2 = I \), where \( I \) denotes the identity of \( R \), and \( BD \in R^{\text{qnil}} \) but \( B \notin R^{\text{qnil}} \).

**Proof.** (i) We claim that \( A \in R^{\text{qnil}} \). Indeed, suppose that

\[
X = \begin{pmatrix} \sum_{i=0}^{\infty} \mu_i x^i \\ \sum_{i=0}^{\infty} \nu_i x^i \\ \sum_{i=0}^{\infty} \rho_i x^i \end{pmatrix} \in R
\]

commutes with \( A \). Write \( \mu_{-1} = \nu_{-1} = \rho_{-1} = 0 \). Then, we have

\[
AX = \begin{pmatrix} \sum_{i=0}^{\infty} t_2 \mu_i x^i \\ \sum_{i=0}^{\infty} \sum_{j=0}^{i} \mu_j x^i + \sigma(\rho_{i-1})x^i \\ 0 \end{pmatrix}
\]

and

\[
XA = \begin{pmatrix} \sum_{i=0}^{\infty} \mu_i t_2 x^i \\ \sum_{i=0}^{\infty} \sum_{j=0}^{i} \mu_{i-1} x^i - \nu i t_{1+i} x^i \\ 0 \end{pmatrix}
\]

Now \( AX =XA \) implies

\[
(t_2 - t_{2+j}) \mu_i = 0, \quad (t_2 - t_{2+j}) \nu_i = 0, \quad t_2 \nu_i + \sigma(\rho_{i-1}) = \mu_{i-1} - \nu t_{1+i},
\]

for all \( i \in \mathbb{N} \).

From the above equalities (1) and (2) one can see that \( \mu_j = \rho_j = 0 \) for \( j \geq 1 \) since \( t_2 - t_{2+j} \) and \( t_2 - t_{1+j} \) are invertible in \( S_{(t_i)}[[x; \sigma]] \). Combining this fact with the above equality (3), we obtain

\[
(t_2 + t_i) \nu_0 = 0, \quad \sigma(\rho_0) = \mu_0 \text{ and } (t_2 + t_{1+i}) \nu_j = 0 \text{ for } j > 1.
\]

Consequently, it follows that \( \nu_0 = \nu_2 = \nu_3 = \cdots = 0 \) and \( t_2 \mu_0 \neq 1 \). We thus conclude

\[
I - AX = \begin{pmatrix} 1 - t_2 \mu_0 & (\mu_0 - \nu_1 t_2)X \\ 0 & 1 + t_1 \rho_0 \end{pmatrix} \in U(R).
\]

This shows \( A \in R^{\text{qnil}} \) and hence \( A \in R^{\text{qnil}} \) with \( A^{\text{qnil}} = 0 \).

Assume that \( A^2 \in R^{\text{qnil}} \). By Lemma 2.2, we have \( (A^2)^{\text{qnil}} = (A^{\text{qnil}})^2 = 0 \), which means \( A^2 \in R^{\text{qnil}} \). However, there is a matrix \( C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R \) such that \( A^2 C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = CA^2 \) and \( I - A^2 C \notin U(R) \), a contradiction.

(ii) It is clear that \( B, C \) and \( D \) commute with each other, \( BC + D^2 = I \) and \( BD \in J(R) \subseteq R^{\text{qnil}} \). From [5, Example 2.11], we know that \( B \) is not quasispolar, i.e., not generalized Drazin invertible. \( \Box \)
Example 2.4. Let $\mathbb{Z}_{(3)}$ be the localization of the ring $\mathbb{Z}$ of integers at the prime ideal $3\mathbb{Z}$ and $R = M_2(\mathbb{Z}_{(3)})$ be the ring of all $2 \times 2$ matrices over $\mathbb{Z}_{(3)}$. Consider the following matrices

$$A = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } D = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R.$$

An easy computation shows that $A, B, C$ and $D$ are mutually commutative, $AC + BD = I_2$ and $AB \in J(R) \subseteq R^{\text{qd}}$. However, in view of [6, Corollary 2.14], $B \not\in R^{\text{qd}}$ because $det B \in J(\mathbb{Z}_{(3)})$, $tr B \in U(\mathbb{Z}_{(3)})$ and the equation $x^2 = (tr B)^2 - 4det B = 52$ has no solution in $U(\mathbb{Z}_{(3)})$, where $det B$ and $tr B$ denote, respectively, the determinant and trace of $B$.

Remark 2.5. Let $M_2(R)$ be the $2 \times 2$ full matrix ring over an arbitrary commutative local ring $R$. We remark that $(M_2(R))^{\text{qd}}$ is “almost” a regularity, i.e., (1) for any integer $n > 1$, $X \in (M_2(R))^{\text{qd}}$ if and only if $X^n \in (M_2(R))^{\text{qd}}$; (2) if $A, B, C, D \in M_2(R)$ are mutually commutative, $AC + BD = I_2$ and $A, B \in (M_2(R))^{\text{qd}}$, then $AB \in (M_2(R))^{\text{qd}}$. Indeed, according to Lemma 2.2, it suffices to show $AB \in (M_2(R))^{\text{qd}}$ under the hypothesis of $A, B \in (M_2(R))^{\text{qd}}$ and $AB = BA$. First of all, using [7, Proposition 4.1], we have $(A - AA^{\text{qd}}A)^2, (B - BB^{\text{qd}}B)^2 \in J(M_2(R))$. Then, by $A^{\text{qd}} \in \text{comm}(A)$ and $B^{\text{qd}} \in \text{comm}(B)$, it follows that $B^{\text{qd}}A^{\text{qd}}AB^{\text{qd}}A^{\text{qd}} = B^{\text{qd}}A^{\text{qd}} \in \text{comm}(AB)$ and

$$\begin{align*}
(AB - ABB^{\text{qd}}A^{\text{qd}}AB)^2 &= [(A - AA^{\text{qd}}A)B + AA^{\text{qd}}A(B - BB^{\text{qd}}B)]^2 \\
&= (A - AA^{\text{qd}}A)^2B^2 + (AA^{\text{qd}}A)^2(B - BB^{\text{qd}}B)^2 \in J(M_2(R)) \subseteq (M_2(R))^{\text{nil}}.
\end{align*}$$

Consequently, $AB - ABB^{\text{qd}}A^{\text{qd}}AB \in (M_2(R))^{\text{nil}}$ by Lemma 2.2. Let $P = I_2 - ABB^{\text{qd}}A^{\text{qd}}$, from the proof of [15, Theorem 4.2], one can see that $P^2 = P \in \text{comm}(AB), ABB \in (M_2(R))^{\text{nil}}$ and $AB + P \in U(M_2(R))$. Therefore, we obtain $AB \in (M_2(R))^{\text{qd}}$ by [7, Proposition 3.5].

For the $n \times n$ full matrix ring over a commutative local ring without zero divisor, we have the following result.

Proposition 2.6. Let $n$ be any integer greater than 1 and $M_n(R)$ be the ring of all $n \times n$ matrices over a commutative local ring $R$ without zero divisor. If $A, B \in (M_n(R))^{\text{qd}}$ and $AB = BA$, then $AB \in (M_n(R))^{\text{qd}}$.

Proof. Let $C = AB$ for convenience. Similarly to Remark 2.5, by virtue of [4, Theorem 2.5], we conclude that there exists $P \in M_n(R)$ such that $P^2 = P \in \text{comm}(C), CP \in (M_n(R))^{\text{nil}}$ and $C + P \in U(M_n(R))$. Note that $R$ is projective free (see, e.g., [10, Chapter VIII, Proposition 4.8]). From [3, Chapter 0, Proposition 4.5], there is $V \in U(M_n(R))$ such that $P = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V^{-1}$, where $0 \leq r \leq n$. If $r = 0$ then $C$ is invertible and hence the result is clear. If $r = n$ then $C \in (M_n(R))^{\text{nil}} \subseteq (M_n(R))^{\text{qd}}$.

Now suppose $0 < r < n$ and write $V^{-1}CV = \begin{pmatrix} C_1 & C_2 \\ 0 & C_3 \end{pmatrix}$, where $C_1$ is an $r \times r$ matrix over $R$. Then $CP = PC$ implies $C_2 = C_3 = 0$. Moreover,

$$V \begin{pmatrix} C_1 & 0 \\ 0 & C_3 \end{pmatrix} V^{-1} = V \begin{pmatrix} 0 & 0 \\ C_3 & C_1 \end{pmatrix} V^{-1} + V \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} V^{-1} = C + P \in U(M_n(R))$$

implies $C_4 \in U(M_{n-r}(R))$. Furthermore,

$$V \begin{pmatrix} C_1 & 0 \\ 0 & C_3 \end{pmatrix} V^{-1} = V \begin{pmatrix} C_1 & 0 \\ 0 & C_3 \end{pmatrix} V^{-1} V \begin{pmatrix} C_1 & 0 \\ 0 & C_3 \end{pmatrix} V^{-1} = CP \in (M_n(R))^{\text{nil}}$$

gives rise to $\begin{pmatrix} C_1 & 0 \\ 0 & C_3 \end{pmatrix} \in (M_n(R))^{\text{nil}}$ by [6, Lemma 2.3]. For any $D \in \text{comm}(C_1)$, we have $\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_3 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_3 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, and hence $\begin{pmatrix} I_r - C_1D \\ 0 \end{pmatrix} \in U(M_n(R))$. This means $I_r - C_1D \in U(M_n(R))$, i.e., $C_1 \in (M_n(R))^{\text{nil}}$.

From [4, Theorem 2.5] it follows that $C_1^{\ast} \in J(M_n(R))$ for some $k \geq 1$. Write $W = V^{-1}C_1^{\ast}V = \begin{pmatrix} C_1^{\ast} & 0 \\ 0 & C_3 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & C_3 \\ 0 & C_1^{\ast} \end{pmatrix}$. We proceed to show that $W \in (M_n(R))^{\text{qd}}$ with $W^{\text{qd}} = X$. A trivial verification gives
that $XWX = X$ and $W - WXW \in J(M_{\nu}(R)) \subseteq (M_{\nu}(R))^{gD}$. We next prove $X \in \text{comm}^{2}(W)$. For any $Y = \left( \begin{array}{cc} Y_{1} & Y_{2} \\ Y_{3} & Y_{4} \end{array} \right) \in \text{comm}(W)$ with $Y_{1} \in M_{\nu}(R)$, one has

$$
\begin{pmatrix}
Y_{1}C_{1}^{k} & Y_{2}C_{2}^{k} \\
Y_{3}C_{1}^{k} & Y_{4}C_{2}^{k}
\end{pmatrix} = WY = YW = \begin{pmatrix}
Y_{1}C_{1}^{k} & Y_{2}C_{1}^{k} \\
Y_{3}C_{1}^{k} & Y_{4}C_{2}^{k}
\end{pmatrix}.
$$

Write $f(x) = a_{0} + a_{1}x + \cdots + x^{n-\tau}$ for the characteristic polynomial of $C_{1}^{k}$. Clearly $a_{0} \in U(R)$ since $C_{4} \in U(M_{\nu-1}(R))$. By the Hamilton-Cayley theorem, $f(C_{1}^{k}) = 0$ and so $(C_{1}^{k})^{D} = (C_{1}^{k})^{-1} = g(C_{1}^{k})$, where $g(x) = -a_{0}^{-1}a_{1} - \cdots - a_{0}^{-1}x^{n-\tau-1}$. Let $F$ be the quotient field of $R$. Note that $C_{1}^{k}$ is Drazin invertible in $M_{\nu}(F)$ (see [9, Corollary 5]). Then we have

$$(C_{1}^{k})^{D}Y_{2} = Y_{2}(C_{1}^{k})^{D} = Y_{2}g(C_{1}^{k}) = g(C_{1}^{k})Y_{2},$$

where the first equality can be obtained by a similar argument to the proof of [8, Theorem 2.2], and the last equality follows from $Y_{2}C_{1}^{k} = C_{1}^{k}Y_{2}$. Hence

$$[I_{\nu} - C_{1}^{k}(C_{1}^{k})^{D}]Y_{2} = Y_{2} - C_{1}^{k}Y_{2}(C_{1}^{k})^{D} = Y_{2}[I_{\nu} - C_{1}^{k}(C_{1}^{k})^{D}] = 0,$$

and so $Y_{2} = C_{1}^{k}(C_{1}^{k})^{D}Y_{2}$. Consequently,

$$[I_{\nu} - g(C_{1}^{k})C_{1}^{k}]Y_{2} = Y_{2} - g(C_{1}^{k})C_{1}^{k}Y_{2} = C_{1}^{k}(C_{1}^{k})^{D}Y_{2} - C_{1}^{k}g(C_{1}^{k})Y_{2} = C_{1}^{k}[(C_{1}^{k})^{D}Y_{2} - Y_{2}g(C_{1}^{k})] = C_{1}^{k}[(C_{1}^{k})^{D}Y_{2} - Y_{2}(C_{1}^{k})^{D}] = 0.$$

Since $C_{1}^{k} \in J(M_{\nu}(R)) = M_{\nu}(J(R))$ and all the coefficients of $g(x)$ are in $R$, we conclude that $g(C_{1}^{k})C_{1}^{k} \in M_{\nu}(J(R)) = J(M_{\nu}(R))$. This forces $I_{\nu} - g(C_{1}^{k})C_{1}^{k} \in U(M_{\nu}(R))$ and hence $Y_{2} = 0$ as we have seen that $[I_{\nu} - g(C_{1}^{k})C_{1}^{k}]Y_{2} = 0$. In the same manner one can show that $Y_{3} = 0$. In addition, the equation $C_{1}^{k}Y_{4} = Y_{4}C_{1}^{k}$ implies $C_{1}^{k}Y_{4} = Y_{4}C_{2}^{k}$.

Whence it follows that $XY = XY$, showing $X \in \text{comm}^{2}(W)$. Thus, $W^{gD} = X$ as desired. In view of [6, Lemma 2.3], $C^{k} = VWV^{-1} \in (M_{\nu}(R))^{gD}$. Finally, we obtain that $C = AB \in (M_{\nu}(R))^{gD}$ by Lemma 2.2.

The above Example 2.4 and Remark 2.5 motivate us to consider under what condition $(M_{2}(R))^{gD}$ is a regularity.

**Theorem 2.7.** Let $M_{2}(R)$ be the $2 \times 2$ matrix ring over a commutative local ring $R$. Then $(M_{2}(R))^{gD}$ is a regularity if and only if $M_{2}(R)$ is quasipolar.

**Proof.** Suppose that $(M_{2}(R))^{gD}$ is a regularity. By [7, Theorem 3.7], it suffices to prove that for any $u \in U(R)$ and $j \in J(R)$, \( \begin{pmatrix} 0 & j \\ 1 & u \end{pmatrix} \) is quasipolar. Let

$$A = \begin{pmatrix} 0 & j \\ 1 & u \end{pmatrix}, \quad C = \begin{pmatrix} u^{-1}-u-2u^{-1}j & j \\ 1 & u^{-1}-2u^{-1}j \end{pmatrix},$$

$$B = \begin{pmatrix} -u & j \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2u^{-1}j-u^{-1} & j \\ 1 & 2u^{-1}j-u^{-1}+u \end{pmatrix}.$$

One can check that $A, B, C$ and $D$ are mutually commutative, $AC + BD = I_{2}$ and $AB \in J(M_{2}(R)) \subseteq (M_{2}(R))^{gD}$. Therefore $A \in (M_{2}(R))^{gD}$, i.e., $A$ is quasipolar.

The converse is obvious. \( \square \)
We refer the readers to [6, 7] for more sufficient and necessary conditions under which the $2 \times 2$ matrix ring $M_2(R)$ over a commutative local ring $R$ is quasipolar.

Recall that a ring $R$ is said to be abelian if every idempotent in $R$ is central. Following Nicholson and Zhou [17], we say that idempotents in a ring $R$ lift strongly modulo an ideal $I$ if, whenever $a^2 - a \in I$, there exists $e^2 = e \in aR$ (equivalently $e^2 = e \in Ra$) such that $e - a \in I$. As usual, we write $\sqrt{I(R)} = \{x \in R : x^n \in I(R)\}$ for some positive integer $n$.

**Theorem 2.8.** Let $R$ be an abelian ring such that $R^{\text{qnil}} \subseteq \sqrt{I(R)}$ and idempotents in $R$ lift strongly modulo $I(R)$, then $R^{gD}$ is a regularity.

**Proof.** We will use the following fact repeatedly in the sequel: if $a, x \in R$ satisfy $xax = x$, then $x \in \text{comm}^2(a)$. Indeed, for any $y \in \text{comm}(a)$, $yx = y(ax)x = (xa)y = x(ya)x = xy(ax) = x(ay)y = (xa)y = xy$ since $R$ is abelian.

Given an integer $n \geq 1$, if $a^n \in R^{gD}$, then by Lemma 2.2, we have $a \in R^{gD}$. Conversely, suppose $a \in R^{gD}$. Then $(a^{gD})^n = (a^{gD})^n$ and hence $(a^{gD})^n \in \text{comm}^2(a^n)$. Note that $a - a^{gD}a \in R^{\text{qnil}} \subseteq \sqrt{I(R)}$. We have $(a - a^{gD}a)^n \in \sqrt{I(R)} \subseteq R^{\text{qnil}}$ for some integer $k \geq 1$. Consequently, it follows that $a^n - a^n(a^{gD})^n = (a - a^{gD}a)^n \in R^{gD}$ by Lemma 2.2. Thus, $a^n \in R^{gD}$.

Now let $a, b, c, d \in R$ be mutually commutative elements such that $ac + bd = 1$. If $a, b \in R^{gD}$ then it follows that $b^{gD}a^{gD}c^{gD}d^{gD} = b^{gD}b^{gD}(a^{gD}a)d^{gD} = b^{gD}a^{gD} \in \text{comm}^2(ab)$ and

$$(ab - ab^{gD}a^{gD}ab)^l = (a - a^{gD}a)b^l + (a^{gD}a)(b - b^{gD}b)^l \in \sqrt{I(R)} \subseteq R^{\text{qnil}}$$

for some positive integer $l$. Using Lemma 2.2 we get $ab - ab^{gD}a^{gD}ab \in R^{\text{qnil}}$. This shows $b^{gD}a^{gD} = (ab)^{gD}$. Conversely, if $ab \in R^{gD}$, write $p = 1 - ab(ab)^{gD}$. Since $abp \in R^{\text{qnil}} \subseteq \sqrt{I(R)}$, it follows that $(ab)p^l \in I(R)$ for some positive integer $t$. Note that $a, b, c, d, p, (ab)^{gD}$ commute with each other as $(ab)^{gD} \in \text{comm}^2(ab)$ and $a, b, c, d \in \text{comm}(ab)$. Let $g = b(ab)^{gD} + pc$ and $h = 1 - (1 - ga)^t$, then $h \in aR \cap Ra$ and

$$a^l - a^l h = (a - aga)^t = (a - ab(ab)^{gD}a - apca)^t = (pa - apca)^t = (ap(1 - ac))^t$$

Hence $a^l + I(R) = a^l h + I(R) = ha^l + I(R) \in \{a^{l+1}R + I(R)\} \cap [Ra^{l+1} + I(R)]$.

This implies that $a + I(R) \in (R/J(R))^{gD}$ by [9, Theorem 4]. Let $x \in R$ with $x + I(R) = (a + I(R))^l$, one has that $ax - (ax)^2 \in I(R)$ and $(a - ax)^m \in I(R)$ for some positive integer $m$. As idempotents in $R$ lift strongly modulo $I(R)$, there is an idempotent $e \in R$ such that $ax - e \in I(R)$ and $e = axw$ for some $w \in R$. It is easily seen that $(xwe)ax(xwe) = xwe \in \text{comm}^2(a)$ and

$$[a - a(xwe)ax]^m = [(1 - e)x]^m = [(a - axa) + (ax - e)a]^m \in I(R) \subseteq R^{\text{qnil}}.$$  

By Lemma 2.2, $a - a(xwe)ax \in R^{\text{qnil}}$. Therefore $a \in R^{gD}$ with $a^{gD} = xwe$. Similarly one gets $b \in R^{gD}$. 

**Remark 2.9.** (1) Note that the ring $\mathbb{Z}$ of integers, the polynomial ring $\mathbb{Z}[x]$, the formal power series ring $\mathbb{Z}[x]$ and all local rings satisfy the hypothesis of Theorem 2.8.

(2) Let $R = R_1 \times R_2$ be the direct product of two rings $R_1$ and $R_2$ such that $R_1^{gD}$ and $R_2^{gD}$ are regularities, then $R^{gD} = R_1^{gD} \times R_2^{gD}$ is a regularity. Thus, one can construct more examples of rings in which $R^{gD}$ is a regularity.

**References**
