Filomat 37:20 (2023), 6727–6740
https://doi.org/10.2298/FIL2320727K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Liftings from a Para-Sasakian manifold to its tangent bundles

Mohammad Nazrul Islam Khan[a]

[a]*Department of Computer Engineering, College of Computer, Qassim University, Buraydah 51452, Saudi Arabia*

Abstract. The purpose of the present paper is to study the liftings of a quarter symmetric non-metric connection from a para-Sasakian manifold to its tangent bundles. By liftings, some results of the curvature tensor, projective curvature tensor, concircular curvature tensor and conformal curvature tensor wrt a quarter symmetric non-metric connection in a P-Sasakian manifold to its tangent bundles are obtained.

1. Introduction

Differential geometry places a significant emphasis on the theory of linear connections. Semi-symmetric connections were first discussed by Friedmann and Schouten in 1924 [14] and Hayden defined metric connections with torsion tensor in 1932 [17]. However, quarter-symmetric linear connections in a differential manifold were first presented by Golab in [15]. A linear connection is said to be a quarter-symmetric connection on M if its torsion tensor $\bar{T}$ is of the form

$$\bar{T}(\beta_1, \beta_2) = u(\beta_2)\phi\beta_1 - u(\beta_1)\phi\beta_2, \quad \forall \beta_1, \beta_2 \in \mathfrak{I}_0^1(M), \tag{1}$$

where u is a 1-form and ϕ is a tensor of type (1, 1). If there is a Riemannian metric g in M such that $\bar{\nabla} g = 0$, then the connection $\bar{\nabla}$ is said to be metric; otherwise, it is non-metric. Yano established a relationship between the Levi-Civita connection and the quarter-symmetric metric connection in [37].

Recently, Maksimović [30] studied a new quarter-symmetric non-metric connection on a generalized Riemannian manifold and some basic results are obtained. Moreover, certain properties of curvature tensors are developed. Mondal and De [28] investigated a P-Sasakian manifold with a quarter-symmetric non-metric connection. They examined the circular and conformal curvature tensors wrt the quarter-symmetric non-metric connection on such a manifold. Various types of quarter-symmetric connections (metric and non-metric) on the different manifolds were investigated by Ali and Nivas [4], Barman and Ghosh [5], De et. al. [10]), Biswas and De [7], Khan [23], Mishra and Pandey [29], Poyraz and Yolda [31], Sular et. al. [34] and others.

One of the fundamental goals of differential geometry to explaining how a surface is curved, either intrinsically or extrinsically. The space of tangent vectors to the curved space is one of the most vital tools

2020 *Mathematics Subject Classification*. Primary 53C15 ; Secondary 53C25, 53C40, 58A30.
Keywords. Complete lift; Tangent bundle; Differential equations; Para-Sasakian manifold; Quarter-symmetric connection; Nonlinear equations; Partial differential equations.
Received: 26 November; Accepted: 05 March 2023
Communicated by Ljubica Velimirović
The researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project.
Email address: m.nazrul@qu.eu.sa, mnazrul@rediffmail.com (Mohammad Nazrul Islam Khan)

for analyzing curved space. The employment of mathematical operators in the context of partial differential equations is the primary method for studying nonlinear equations. Numerous researchers have examined various connections and geometric structres on the tangent bundle and provided their findings in ([11], [12], [13], [3], [19] ,[21], [22], [20], [26], [27], [36]).

Investigating liftings from the P-Sasakian manifold with regard to a quarter-symmetric non-metric connection to tangent bundles is the goal of the proposed paper. The key conclusions of the paper are summarized as follows:

- Lifts are established for the para-Sasakian manifold endowed with a quarter-symmetric non-metric connection $\bar{\nabla}^C$ to the tangent bundle.
- It is shown that if the Ricci tensor $\bar{S}^C$ of $\bar{\nabla}^C$ on TM, then TM is an η^C Einstein manifold wrt $\bar{\nabla}^C$.
- The results of the concircular curvature tensor on the tangent bundle are obtained successfully.
- Some calculations for the projective curvature tensor on the tangent bundle have been done.
- A theorem on the conformal curvature tensor on the tangent bundle is proved.

2. Preliminaries

2.1. Para-Sasakian manifold

A differentiable manifold M (dim= n) is said to be an almost paracontact Riemannian structure (ϕ, ξ, η, g) ([1], [16]) where ϕ is a tensor field of type (1,1), ξ a vector field , a η 1-form and if it fufills

$$\eta(\xi) = 0, \tag{2}$$
$$\phi\xi = 0, \quad \phi\xi = 0, \quad \eta(\phi\beta_1) = 0, \tag{3}$$
$$\phi^2(\beta_1) = \beta_1 - \eta(\beta_1)\xi, \tag{4}$$
$$g(\phi\beta_1, \phi\beta_2) = g(\beta_1, \beta_2) - \eta(\beta_1)\eta(\beta_2) \tag{5}$$
$$g(\beta_1, \xi) = \eta(\beta_1), \tag{6}$$
$$(\nabla_{\beta_1}\eta)\beta_2 = g(\beta_1\phi), \forall \beta_1, \beta_2 \in \mathfrak{I}_0^1(M), \tag{7}$$

where ∇ is Levi-Civita connection wrt the Riemannian metric g. If $g(\beta_1, \phi\beta_2) = \Phi(\beta_1, \beta_2)$, then $\Phi \in \mathfrak{I}_0^2(M)$ is a symmetric.

In addition, the following relations hold in M

$$\nabla_{\beta_1}\xi = \phi\beta_1, \tag{8}$$
$$(\nabla_{\beta_1}\phi)(\beta_2) = -g(\beta_1, \beta_2)\xi - \eta(\beta_2)\beta_1 + 2\eta(\beta_1)\eta(\beta_2)\xi, \tag{9}$$
$$d\eta = 0, \tag{10}$$

then M is said to be a para-Sasakian manifold (P-Sasakian).

Let R and S be the curvature tensor and the Ricci tensor of M.Then we infer ([2], [16], [35])

$$g(R(\beta_1, \beta_2)\beta_3, \xi) = \eta(R(\beta_1, \beta_2)\beta_3) = g(\beta_1, \beta_3)\eta(\beta_2) - g(\beta_2, \beta_3)\eta(\beta_1), \tag{11}$$
$$R(\xi, \beta_1)\beta_2 = \eta(\beta_2)\beta_1 - g(\beta_1, \beta_2)\xi, \tag{12}$$
$$R(\beta_1, \beta_2)\xi = \eta(\beta_1)\beta_2 - \eta(\beta_2)\beta_1, \tag{13}$$
$$S(\beta_1, \xi) = -(n-1)\eta(\beta_1), \tag{14}$$
$$S(\phi\beta_1, \phi\beta_2) = S(\beta_1, \beta_2) + (n-1)\eta(\beta_1)\eta(\beta_2), \tag{15}$$

where $\forall \beta_1, \beta_2, \beta_3 \in \mathfrak{I}_0^1(M)$.

Further, M is said to be η Einstein if

$$S(\beta_1, \beta_2) = ag(\beta_1, \beta_2) + b\eta(\beta_1)\eta(\beta_2), \forall \beta_1, \beta_2 \in \mathfrak{I}_0^1(M) \tag{16}$$

where $a, b(b \neq 0)$ are scalar functions, M is called Einstein manifold for $b = 0$.

2.2. Lifts of a para-Sasakian manifold on TM

Suppose TM be the tangent bundle and $\beta_1 = {\beta_1}^i \frac{\partial}{\partial x^i}$ be a local vector field on M, then its vertical and complete lifts in the term of partial differential equations are

$$ {\beta_1}^V = {\beta_1}^i \frac{\partial}{\partial y^i}, \tag{17} $$

$$ {\beta_1}^C = {\beta_1}^i \frac{\partial}{\partial x^i} + \frac{\partial {\beta_1}^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}. \tag{18} $$

Let $\eta^C, \eta^V, {\beta_1}^C, {\beta_1}^V, \phi^C, \phi^V, \nabla^C, \nabla^V$ be the complete and vertical lifts of $\eta, \beta_1, \phi, \nabla$. By mathematical operators, we infer ([9], [32], [36])

$$ \eta^V({\beta_1}^C) = \eta^C({\beta_1}^V) = \eta(\beta_1)^V, \eta^C({\beta_1}^C) = \eta(\beta_1)^C, \tag{19} $$

$$ \phi^V X^C = (\phi\beta_1)^V, \phi^C X^C = (\phi\beta_1)^C, \tag{20} $$

$$ [\beta_1, \beta_2]^V = [{\beta_1}^C, {\beta_2}^V] = [{\beta_1}^V, {\beta_2}^C], [\beta_1, \beta_2]^C = [{\beta_1}^C, {\beta_2}^C], \tag{21} $$

$$ \nabla^C_{{\beta_1}^C} {\beta_2}^C = (\nabla_{\beta_1}\beta_2)^C, \quad \nabla^C_{{\beta_1}^C} {\beta_2}^V = (\nabla_{\beta_1}\beta_2)^V. \tag{22} $$

Employing the complete lift on (2)-(16), we acquire

$$ \eta^C(\xi^C) = \eta^V(\xi^V) = 0, \ \eta^C(\xi^V) = \eta^V(\xi^C) = 1, \tag{23} $$

$$ \phi^C\xi^C = \phi^V\xi^V = \phi^C\xi^V = \phi^V\xi^C = 0 $$

$$ \phi^C\xi^C = \phi^V\xi^V = \phi^C\xi^V = \phi^V\xi^C = 0, $$

$$ \eta^C(\phi\beta_1)^C = \eta^V(\phi\beta_1)^V = \eta^C(\phi\beta_1)^V = \eta^V(\phi\beta_1)^C = 0, \tag{24} $$

$$ ((\phi(\beta_1))^2)^C = {\beta_1}^C - \eta^C({\beta_1}^C)\xi^V - \eta^V({\beta_1}^C)\xi^C, \tag{25} $$

$$ \begin{aligned} g^C((\phi\beta_1)^C, (\phi\beta_2)^C) &= g^C({\beta_1}^C, {\beta_2}^C) - \eta^C({\beta_1}^C)\eta^V({\beta_2}^C) \\ &- \eta^V({\beta_1}^C)\eta^C({\beta_2}^C), \end{aligned} \tag{26} $$

$$ g^C({\beta_1}^C, \xi^C) = \eta^C({\beta_1}^C), \tag{27} $$

$$ (\nabla^C_{{\beta_1}^C}\eta^C){\beta_2}^C = g^C({\beta_1}^C, (\phi\beta_2)^C). \tag{28} $$

and

$$ \nabla^C_{{\beta_1}^C}\xi^C = (\phi\beta_1)^C, \tag{29} $$

$$ \begin{aligned} (\nabla^C_{{\beta_1}^C}\phi^C)({\beta_2}^C) &= -g^C({\beta_1}^C, {\beta_2}^C)\xi^V - g^C({\beta_1}^V, {\beta_2}^C)\xi^C - \eta^C({\beta_2}^C){\beta_1}^V \\ &- \eta^V({\beta_2}^C){\beta_1}^C + 2\eta^C({\beta_1}^C)\eta^C({\beta_2}^C)\xi^V \\ &+ 2\eta^C({\beta_1}^C)\eta^V({\beta_2}^C)\xi^C + 2\eta^V({\beta_1}^C)\eta^C({\beta_2}^C)\xi^C, \end{aligned} \tag{30} $$

$$ d\eta^C = 0. \tag{31} $$

In addition,

$$ \begin{aligned} g^C(R^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C, \xi^C) &= \eta^C(R^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C) = g^C({\beta_1}^C, {\beta_3}^C)\eta^V({\beta_2}^C) \\ &+ g^C({\beta_1}^V, {\beta_3}^C)\eta^C({\beta_2}^C) - g^C({\beta_2}^C, {\beta_3}^C)\eta^V({\beta_1}^C) \\ &- g^C({\beta_2}^V, {\beta_3}^C)\eta^C({\beta_1}^C) \end{aligned} \tag{32} $$

$$ \begin{aligned} R^C(\xi^C, {\beta_1}^C){\beta_2}^C &= \eta^C({\beta_2}^C){\beta_1}^V + \eta^V({\beta_2}^C){\beta_1}^V \\ &- g^C({\beta_1}^C, {\beta_2}^C)\xi^V - g^C({\beta_1}^V, {\beta_2}^C)\xi^C, \end{aligned} \tag{33} $$

$$ \begin{aligned} R^C({\beta_1}^C, {\beta_2}^C)\xi^C &= \eta^C({\beta_1}^C){\beta_2}^V + \eta^V({\beta_1}^C){\beta_2}^C \\ &- \eta^C({\beta_2}^C){\beta_1}^V + \eta^V({\beta_2}^C){\beta_1}^C, \end{aligned} \tag{34} $$

$$ S^C({\beta_1}^C, \xi^C) = -(n-1)\eta^C({\beta_1}^C), \tag{35} $$

$$ \begin{aligned} S^C((\phi\beta_1)^C, (\phi\beta_2)^C) &= S^C({\beta_1}^C, {\beta_2}^C) + (n-1)\{\eta^C({\beta_1}^C)\eta^V({\beta_2}^C) \\ &+ \eta^V({\beta_1}^C)\eta^C({\beta_2}^C)\}. \end{aligned} \tag{36} $$

$$\begin{aligned} S^C({\beta_1}^C,{\beta_2}^C) &= ag^C({\beta_1}^C,{\beta_2}^C)+b\{\eta^C({\beta_1}^C)\eta^V({\beta_2}^C) \\ &+ \eta^V({\beta_1}^C)\eta^C({\beta_2}^C)\}, \end{aligned} \tag{37}$$

where $\forall {\beta_1}^C,{\beta_2}^C,\xi^C \in \mathfrak{I}^1_0(TM)$, $\eta^C \in \mathfrak{I}^0_1(TM), R^C, g^C$ and S^C stand for the complete lifts of R, g and S, respectively.

3. Lifts of a quarter-symmetric non-metric connection on *TM*

Let TM be the tangent bundle of P-Sasakain manifold M of dim n and the mathematical operators ∇ and $\bar{\nabla}$ be a Levi-Civita connection and a linear connection on M, respectively such that

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(\beta_2)\phi\beta_1, \tag{38}$$

where $\forall \beta_1,\beta_2 \in \mathfrak{I}^1_0(M), \phi \in \mathfrak{I}^1_1(M)$.

The torsion tensor $\bar{T}$ wrt connection $\bar{\nabla}$ is defined as

$$\bar{T}(\beta_1,\beta_2) = \bar{\nabla}_X Y - \bar{\nabla}_{\beta_2}\beta_1 - [\beta_1,\beta_2]. \tag{39}$$

Applying the complete lift on (38) and (39), we infer

$$\bar{\nabla}^C_{{\beta_1}^C}{\beta_2}^C = \nabla^C_{{\beta_1}^C}{\beta_2}^C + \eta^C({\beta_2}^C)(\phi\beta_1)^V + \eta^V({\beta_2}^C)(\phi\beta_1)^C, \tag{40}$$

$$\bar{T}^C({\beta_1}^C,{\beta_2}^C) = \bar{\nabla}^C_{{\beta_1}^C}{\beta_2}^C - \bar{\nabla}^C_{{\beta_2}^C}{\beta_1}^C - [{\beta_1}^C,{\beta_2}^C]. \tag{41}$$

From (40), (42) and (43) are obtained.

$$\begin{aligned} \bar{T}^C({\beta_1}^C,{\beta_2}^C) &= \eta^C({\beta_2}^C)(\phi\beta_1)^V + \eta^V({\beta_2}^C)(\phi\beta_1)^C \\ &- \eta^C({\beta_2}^C)(\phi\beta_1)^V - \eta^V({\beta_2}^C)(\phi\beta_1)^C. \end{aligned} \tag{42}$$

and

$$\begin{aligned} (\bar{\nabla}^C_{{\beta_1}^C}g^C)({\beta_2}^C,{\beta_3}^C) &= -\eta^C({\beta_2}^C)\Phi^V({\beta_1}^C,{\beta_3}^C) - \eta^V({\beta_2}^C)\Phi^C({\beta_1}^C,{\beta_3}^C) \\ &= -\eta^C({\beta_3}^C)\Phi^V({\beta_1}^C,{\beta_2}^C) - \eta^V({\beta_1}^C)\Phi^C({\beta_2}^C,{\beta_3}^C), \end{aligned} \tag{43}$$

$\forall {\beta_1}^C,{\beta_2}^C \in \mathfrak{I}^1_0(TM)$, which indicates that $\bar{\nabla}^C$ is a quarter symmetric non-metric connection on TM.

In view of (24) and (43), we infer

$$(\bar{\nabla}^C_{\xi^C}g^C)({\beta_2}^C,{\beta_3}^C) = 0,$$

which indicating that g^C is ξ^C-parallel wrt $\bar{\nabla}^C$ on TM.

From (24), (26), (29), (30) and (40), we conclude the following theorem:

Theorem 3.1. *Let TM be the tangent bundle of P-Sasakain manifold M. Then*

$$\bar{\nabla}^C_{{\beta_1}^C}\xi^C = 2(\phi\beta_1)^C, \tag{44}$$

$$\begin{aligned} (\bar{\nabla}^C_{{\beta_1}^C}\phi^C){\beta_2}^C &= -g^C({\beta_1}^C,{\beta_2}^C)\xi^V - g^C({\beta_1}^V,{\beta_2}^C)\xi^C \\ &- 2\{\eta^C({\beta_2}^C){\beta_1}^V + \eta^V({\beta_2}^C){\beta_1}^C\} \\ &+ 3\{\eta^C({\beta_1}^C)\eta^C({\beta_2}^C)\xi^V + \eta^C({\beta_1}^C)\eta^V({\beta_2}^C)\xi^C \\ &+ \eta^V({\beta_1}^C)\eta^C({\beta_2}^C)\xi^C\}, \end{aligned} \tag{45}$$

$\forall {\beta_1}^C,{\beta_2}^C \in \mathfrak{I}^1_0(TM)$.

The curvature tensor $\bar{R}$ of $\bar{\nabla}$ on M is given as

$$\bar{R}(\beta_1,\beta_2)\beta_3 = \bar{\nabla}_{\beta_1}\bar{\nabla}_{\beta_2}\beta_3 - \bar{\nabla}_{\beta_2}\bar{\nabla}_{\beta_1}\beta_3 - \bar{\nabla}_{[\beta_1,\beta_2]}\beta_3. \tag{46}$$

Applying the complete lift on (46), we infer

$$\bar{R}^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C = \bar{\nabla}^C_{{\beta_1}^C}\bar{\nabla}^C_{{\beta_2}^C}{\beta_3}^C - \bar{\nabla}^C_{{\beta_2}^C}\bar{\nabla}^C_{{\beta_1}^C}{\beta_3}^C - \bar{\nabla}^C_{[{\beta_1}^C,{\beta_2}^C]}{\beta_3}^C. \tag{47}$$

From differential equations (28), (40) and (47) we infer

$$\begin{aligned}\bar{R}^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C &= R^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C\\ &+ \Phi^C({\beta_1}^C,{\beta_3}^C)(\phi\beta_2)^V + \Phi^V({\beta_1}^C,{\beta_3}^C)(\phi\beta_2)^C\\ &- \Phi^C({\beta_2}^C,{\beta_3}^C)(\phi\beta_1)^V - \Phi^V({\beta_2}^C,{\beta_3}^C)(\phi\beta_1)^C\\ &+ \eta^C({\beta_3}^C)\eta^C({\beta_1}^C){\beta_2}^V + \eta^C({\beta_3}^C)\eta^V({\beta_1}^C){\beta_2}^C\\ &+ +\eta^V({\beta_3}^C)\eta^C({\beta_1}^C){\beta_2}^C - \eta^C({\beta_3}^C)\eta^C({\beta_2}^C){\beta_1}^V\\ &+ \eta^C({\beta_3}^C)\eta^V({\beta_2}^C){\beta_1}^C + \eta^V({\beta_3}^C)\eta^C({\beta_2}^C){\beta_1}^C,\end{aligned} \tag{48}$$

where

$$R^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C = \nabla^C_{{\beta_1}^C}\nabla^C_{{\beta_2}^C}{\beta_3}^C - \nabla^C_{{\beta_2}^C}\nabla^C_{{\beta_1}^C}{\beta_3}^C - \nabla^C_{[{\beta_1}^C,{\beta_2}^C]}{\beta_3}^C \tag{49}$$

is the curvature tensor wrt ∇^C on TM. Thus, we conclude:

Theorem 3.2. *Let TM be the tangent bundle of M with $\bar{\nabla}^C$. Then the first Bianchi identity of $\bar{\nabla}^C$ on TM is provided.*

From (34) and (48) we deduce

$$\begin{aligned}\bar{K}^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) &= K^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) + \Phi^C({\beta_1}^C,{\beta_3}^C)\Phi^V({\beta_2}^C,U^C)\\ &+ \Phi^V({\beta_1}^C,{\beta_3}^C)\Phi^C({\beta_2}^C,U^C) - \Phi^C({\beta_2}^C,{\beta_3}^C)\Phi^V({\beta_1}^C,U^C)\\ &- \Phi^V({\beta_2}^C,{\beta_3}^C)\Phi^C({\beta_1}^C,U^C) + \eta^C({\beta_3}^C)K^V({\beta_1}^C,{\beta_2}^C,\xi^C,U^C)\\ &+ \eta^V({\beta_3}^C)K^C({\beta_1}^C,{\beta_2}^C,\xi^C,U^C),\end{aligned} \tag{50}$$

where K^C and $\bar{K}^C$ are given by

$$K^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) = g^C(R^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C,U^C)$$

and

$$\bar{K}^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) = g^C(\bar{R}^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C,U^C).$$

Theorem 3.3. *Let $\forall {\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C \in \mathfrak{I}^1_0(TM)$. Then*

$$\bar{K}^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) + \bar{K}^C({\beta_2}^C,{\beta_1}^C,{\beta_3}^C,U^C) = 0, \tag{51}$$

$$\begin{aligned}\bar{K}^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) &+ \bar{K}^C({\beta_2}^C,{\beta_1}^C,U^C,{\beta_3}^C)\\ &= \eta^C({\beta_3}^C)K^V({\beta_1}^C,{\beta_2}^C,\xi^C,U^C)\\ &+ \eta^V({\beta_3}^C)K^C({\beta_1}^C,{\beta_2}^C,\xi^C,U^C)\\ &+ \eta^C(U^C)K^V({\beta_1}^C,{\beta_2}^C,\xi^C,{\beta_3}^C)\\ &+ \eta^V(U^C)K^C({\beta_1}^C,{\beta_2}^C,\xi^C,{\beta_3}^C),\\ \bar{K}^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) &- \bar{K}^C({\beta_3}^C,U^C,{\beta_1}^C,{\beta_2}^C)\\ &= \eta^C({\beta_1}^C)\eta^C(U^C)g^C({\beta_2}^V,{\beta_3}^C)\\ &+ \eta^C({\beta_1}^C)\eta^V(U^C)g^C({\beta_2}^C,{\beta_3}^C)\\ &+ \eta^V({\beta_1}^C)\eta^C(U^C)g^C({\beta_2}^C,{\beta_3}^C)\\ &- \eta^C({\beta_2}^C)\eta^C({\beta_3}^C)g^C({\beta_1}^V,U^C)\\ &- \eta^C({\beta_2}^C)\eta^V({\beta_3}^C)g^C({\beta_1}^C,U^C)\\ &- \eta^V({\beta_2}^C)\eta^C({\beta_3}^C)g^C({\beta_1}^C,U^C).\end{aligned} \tag{52}$$

Proof. In view of (50), we get (53). Also in view of (34) and (50) we infer (52). Furthermore, in view of (50), we deduce (53).

Theorem 3.4. *Let* $\forall {\beta_1}^C, {\beta_2}^C, {\beta_3}^C \in \Im^1_0(TM)$. *Then*

$$\begin{aligned} g^C(\bar{R}^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C, \xi^C) &= \eta^C(\bar{R}^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C) \\ &= g^C(R^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C, \xi^C), \quad (53) \\ \bar{R}^C({\beta_1}^C{\beta_2}^C)\xi^C &= 2R^C({\beta_1}^C, {\beta_2}^C)\xi^C, \quad (54) \\ \bar{R}^C(\xi^C, {\beta_1}^C){\beta_2}^C &= -g^C((\phi\beta_1)^C, (\phi\beta_2)^C Y)\xi^V \\ &- g^V((\phi\beta_1)^C, (\phi\beta_2)^C Y)\xi^C \quad (55) \end{aligned}$$

$\forall {\beta_1}^C, {\beta_2}^C, {\beta_3}^C \in \Im^1_0(TM)$.

Proof. In view of (27), (24) and (47) we deduce (53). Also, in view of (23), (34) and (47) we deduce (54). Furthermore, in view of (23), (36) and (47) we deduce (55).

In a *P*-Sasakian manifold *M*, the Ricci tensor *S* wrt a quarter symmetric non-metric connection $\bar{\nabla}$ is defined as ([33], [8])

$$\bar{S}(\beta_1, \beta_2) = \sum_{i=1}^{n} g(\bar{R}(e_i, \beta_1)\beta_2, e_i),$$

where $\{e_i, i = 1, 2, ..., n\}$ is an orthonormal frame on *M*.

Applying the complete lift on the above equation, we infer

$$\bar{S}^C({\beta_1}^C, {\beta_2}^C) = \sum_{i=1}^{n} g^C(\bar{R}^C(e_i^C, {\beta_1}^C){\beta_2}^C, e_i^C).$$

The scalar curvature $\bar{r}^C$ wrt $\bar{\nabla}^C$ on *TM* is defined by

$$\bar{r}^C = \sum_{i=1}^{n} \bar{S}^C(e_i^C, e_i^C).$$

where ${\beta_1}^C, {\beta_2}^C \in \Im^1_0(TM)$ and $\{e_i^C, i = 1, 2, ..., n\}$ is an orthonormal frame on *TM* of *M*.

From (50) we have

$$\begin{aligned} \bar{S}^C({\beta_1}^C, {\beta_2}^C) &= \sum_{i=1}^{n} g^C(R^C(e_i^C, {\beta_1}^C){\beta_2}^C, e_i^C) + g^C((\phi\beta_1)^C, (\phi\beta_2)^C) \\ &- \Phi^C({\beta_1}^C, {\beta_2}^C)g^V((\phi e_i)^C, e_i^C) - \Phi^V({\beta_1}^C, {\beta_2}^C)g^C((\phi e_i)^C, e_i^C) \\ &+ \eta^C({\beta_2}^C)\eta^V({\beta_1}^C) + \eta^V({\beta_2}^C)\eta^C({\beta_1}^C) \\ &- n\{\eta^C({\beta_1}^C)\eta^V({\beta_2}^C) + \eta^V({\beta_1}^C)\eta^C({\beta_2}^C)\}. \quad (56) \end{aligned}$$

Theorem 3.5. *Let* $\forall {\beta_1}^C, {\beta_2}^C \in \Im^1_0(TM)$. *Then*

$$\begin{aligned} \bar{S}^C({\beta_1}^C, {\beta_2}^C) &= S^C({\beta_1}^C, {\beta_2}^C) + g^C({\beta_1}^C, {\beta_2}^C) - \beta\Phi^C({\beta_1}^C, {\beta_2}^C) \\ &- n\{\eta^C({\beta_1}^C)\eta^V({\beta_2}^C) + \eta^V({\beta_1}^C)\eta^C({\beta_2}^C)\}, \quad (57) \\ \bar{r}^C &= r^C - \beta^2, \quad (58) \end{aligned}$$

where $\beta = trace(\Phi^C)$.

Proof. Since S^C is defined by

$$S^C({\beta_1}^C, {\beta_2}^C) = \sum_{i=1}^{n} g^C(R^C(e_i^C, {\beta_1}^C){\beta_2}^C, e_i^C),$$

then apply (26) and (56) we deduce (57). Also in view of 26) and (57) we deduce (58). From (57), we conclude:

Corollary 3.6. *The $\bar{S}^C$ of $\bar{\nabla}^C$ on TM is symmetric.*

By using (34),(26), (24), (35), (36), (57), we have the following equations

$$\begin{aligned}\bar{S}^C(\beta_1{}^C,\xi^C) &= (2-2n)\eta^C(\beta_1{}^C), &(59)\\ \bar{S}^C((\phi\beta_1)^C,(\phi\beta_2)^C) &= \bar{S}^C(\beta_1{}^C,\beta_2{}^C)+2(n-1)\{\eta^C(\beta_1{}^C)\eta^V(\beta_2{}^C)\\ &+ \eta^V(\beta_1{}^C)\eta^C(\beta_2{}^C) &(60)\\ \bar{S}^C((\phi\beta_1)^C,(\phi\beta_2)^C) &= S^C(\beta_1{}^C,\beta_2{}^C)+g^C(\beta_1{}^C,\beta_2{}^C)-\beta\Phi^C(\beta_1{}^C,\beta_2{}^C)\\ &+ (n-2)\{\eta^C(\beta_1{}^C)\eta^V(\beta_2{}^C)+\eta^V(\beta_1{}^C)\eta^C(\beta_2{}^C). &(61)\end{aligned}$$

Theorem 3.7. *If $\bar{S}^C$ of $\bar{\nabla}^C$ on TM, then TM is an η^C Einstein manifold wrt $\bar{\nabla}^C$.*

Proof. Let $\bar{R}^C(\beta_1{}^C,\beta_2{}^C)\bar{S}^C=0$ on TM, then we acquire

$$\bar{S}^C(\bar{R}^C(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C,U^C)+\bar{S}^C(\beta_3{}^C,\bar{R}^C(\beta_1{}^C,\beta_2{}^C)U^C)=0. \quad (62)$$

Setting $\beta_3=\beta_1=\xi$ in (62), we infer

$$\bar{S}^C(\bar{R}^C(\xi^C,\beta_2{}^C)\xi^C,U^C)+\bar{S}^C(\xi^C,\bar{R}^C(\xi^C,\beta_2{}^C)U^C)=0. \quad (63)$$

Making use of (53), (54) and (59) in (63), we deduce

$$\bar{S}^C(\beta_2{}^C,U^C)=(1-n)\{g^C(\beta_2{}^C,U^C)+\eta^C(\beta_2{}^C)\eta^V(U^C)+\eta^V(\beta_2{}^C)\eta^C(U^C)\}. \quad (64)$$

Hence the proof of the theorem is completed.

4. Proposed theorems for the concircular curvature tensor on *TM*

The concircular curvature tensor of M wrt $\bar{\nabla}$ is given by [6]

$$\bar{C}^*(\beta_1,\beta_2)\beta_3 = \bar{R}(\beta_1,\beta_2)\beta_3-\frac{\bar{r}}{n(n-1)}\{g(\beta_2,\beta_3)\beta_1-g(\beta_1,\beta_3)\beta_2\} \quad (65)$$

Employing the complete lift on (65), we acquire

$$\begin{aligned}\bar{C}^{*C}(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= \bar{R}^C(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C-\frac{\bar{r}}{n(n-1)}\{g^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V+g^V(\beta_2{}^C,\beta_3{}^C)\beta_1{}^C\\ &- g^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V-g^V(\beta_1{}^C,\beta_3{}^C)\beta_2{}^C\} &(66)\end{aligned}$$

By using (48), (58) and (66), we infer

$$\begin{aligned}\bar{C}^{*C}(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= C^{*C}(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C\\ &+ \Phi^C(\beta_1{}^C,\beta_3{}^C)(\phi\beta_2)^V+\Phi^V(\beta_1{}^C,\beta_3{}^C)(\phi\beta_2)^C\\ &- \Phi^C(\beta_2{}^C,\beta_3{}^C)(\phi\beta_1)^V-\Phi^V(\beta_2{}^C,\beta_3{}^C)(\phi\beta_1)^C\\ &+ \eta^C(\beta_3{}^C)\eta^C(\beta_1{}^C)\beta_2{}^V+\eta^C(\beta_3{}^C)\eta^V(\beta_1{}^C)\beta_2{}^C+\eta^V(\beta_3{}^C)\eta^C(\beta_1{}^C)\beta_2{}^C\\ &- \eta^C(\beta_3{}^C)\eta^C(\beta_2{}^C)\beta_1{}^V+\eta^C(\beta_3{}^C)\eta^V(\beta_2{}^C)\beta_1{}^C+\eta^V(\beta_3{}^C)\eta^C(\beta_2{}^C)\beta_1{}^C\\ &- \frac{\beta^2}{n(n-1)}\{g^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V+g^V(\beta_2{}^C,\beta_3{}^C)\beta_1{}^C\\ &- g^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V-g^V(\beta_1{}^C,\beta_3{}^C)\beta_2{}^C\}, &(67)\end{aligned}$$

where C^{*C} is the complete lift of the concircular curvature tensor C^* wrt a Levi-Civita connection ∇^C. If we consider $\bar{C}^{*C}=C^{*C}$, putting β_2 by ξ in (67), from (23) and (27) we deduce

$$g^C(\beta_1{}^C,\beta_3{}^C)=\eta^C(\beta_1{}^C)\eta^V(\beta_3{}^C)+\eta^V(\beta_1{}^C)\eta^C(\beta_3{}^C). \quad (68)$$

In view of (25), (26) and (68) we infer

$$\Phi^C({\beta_1}^C,{\beta_3}^C)=0. \tag{69}$$

In view of (57), (68) and (70) we conclude the following theorem

Theorem 4.1. *If C^{*C} on TM is invariant under $\bar{\nabla}^C$, then*

$$\bar{S}^C({\beta_1}^C,{\beta_2}^C)=S^C({\beta_1}^C,{\beta_2}^C)+(1-n)\{\eta^C({\beta_1}^C)\eta^V({\beta_2}^C)+\eta^V({\beta_1}^C)\eta^V({\beta_2}^C)\}, \tag{70}$$

$\forall {\beta_1}^C,{\beta_2}^C\in\Im^1_0(TM)$.

Definition 4.2. *The TM is ξ^C-concirculary flat wrt $\bar{\nabla}^C$ if $\bar{C}^{*C}({\beta_1}^C,{\beta_2}^C)\xi^C=0$ on TM.*

Setting $\beta_3=\xi$ in (66) and using (23), (25), (27), (24) and (35) we infer

$$\begin{aligned}\bar{C}^{*C}({\beta_1}^C,{\beta_2}^C)\xi^C &= [2+\frac{r^C-\beta^2}{n(n-1)}]\{\eta^C({\beta_1}^C)\eta^V({\beta_2}^C)+\eta^V({\beta_1}^C)\eta^V({\beta_2}^C)\\ &- \eta^C({\beta_2}^C)\eta^V({\beta_1}^C)-\eta^V({\beta_2}^C)\eta^V({\beta_1}^C)\}.\end{aligned} \tag{71}$$

Thus (71) we conclude the following theorem

Theorem 4.3. *The TM is ξ^C-concirculary flat wrt $\bar{\nabla}^C$ iff $r^C=\beta^2-2n(n-1)$.*

Definition 4.4. *The TM is ϕ^C- concirculary flat wrt $\bar{\nabla}^C$ if $g(\bar{C}^*((\phi\beta_1)^C,(\phi\beta_2)^C)(\phi\beta_3)^C,(\phi u_4)^C)=0$ on TM.*

Taking the inner product of (66) with u_4, we get

$$\begin{aligned}\bar{C}'^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,{u_4}^C) &= \bar{K}^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,{u_4}^C)\\ &- \frac{\bar{r}^C}{n(n-1)}\{g^C({\beta_2}^C,{\beta_3}^C)g^C({\beta_1}^V,{u_4}^C)\\ &+ g^V({\beta_2}^C,{\beta_3}^C)g^C({\beta_1}^C,{u_4}^C)\\ &- g^C({\beta_1}^C,{\beta_3}^C)g^C({\beta_2}^V,{u_4}^C)\\ &- g^V({\beta_1}^C,{\beta_3}^C)g^C({\beta_2}^C,{u_4}^C)\},\end{aligned} \tag{72}$$

where $\bar{C}'^C\in\Im^4_0(TM)$ and
$\bar{C}'^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,{u_4}^C)=g^C(\bar{C}^{*C}({\beta_1}^C,{\beta_2}^C){\beta_3}^C,{u_4}^C)$.

In view of (72) we infer

$$\begin{aligned}\bar{C}'^C((\phi\beta_1)^C,(\phi\beta_2)^C,(\phi\beta_3)^C,(\phi u_4)^C) &= \bar{K}^C((\phi\beta_1)^C,(\phi\beta_2)^C,(\phi\beta_3)^C,(\phi u_4)^C)\\ &- \frac{\bar{r}^C}{n(n-1)}\{g^C((\phi\beta_2)^C,(\phi\beta_3)^C)g^C((\phi\beta_1)^V,(\phi u_4)^C)\\ &+ g^V((\phi\beta_2)^C,(\phi\beta_3)^C)g^C((\phi\beta_1)^C,(\phi u_4)^C)\\ &- g^C((\phi\beta_1)^C,(\phi\beta_3)^C)g^C((\phi\beta_2)^V,(\phi u_4)^C)\\ &- g^V((\phi\beta_1)^C,(\phi\beta_3)^C)g^C((\phi\beta_2)^C,(\phi u_4)^C)\},\end{aligned} \tag{73}$$

Making use of (73), ϕ^C- concirculary flatness implies

$$\begin{aligned}\bar{K}^C((\phi\beta_1)^C,(\phi\beta_2)^C,(\phi\beta_3)^C,(\phi u_4)^C) &= \frac{\bar{r}^C}{n(n-1)}\{g^C((\phi\beta_2)^C,(\phi\beta_3)^C)g^C((\phi\beta_1)^V,(\phi u_4)^C)\\ &+ g^V((\phi\beta_2)^C,(\phi\beta_3)^C)g^C((\phi\beta_1)^C,(\phi u_4)^C)\\ &- g^C((\phi\beta_1)^C,(\phi\beta_3)^C)g^C((\phi\beta_2)^V,(\phi u_4)^C)\\ &- g^V((\phi\beta_1)^C,(\phi\beta_3)^C)g^C((\phi\beta_2)^C,(\phi u_4)^C)\}.\end{aligned} \tag{74}$$

Let $\{(\phi e_i)^C, i = 1, ..., n-1, (\phi\xi)^C\}$ be a local orthonormal basis on TM where $\{e_i, i = 1, ..., n-1, \xi\} \in M$. Setting $\beta_1 = u_4 = e_i$, then we infer

$$\begin{aligned}\sum_{i=1}^{n-1} \bar{K}^C((\phi e_i)^C, (\phi\beta_2)^C, (\phi\beta_3)^C, (\phi e_i)^C) &= \frac{\bar{r}^C}{n(n-1)} \sum_{i=1}^{n-1} \{g^C((\phi\beta_2)^C, (\phi\beta_3)^C) g^C((\phi e_i)^V, (\phi e_i)^C) \\ &+ g^V((\phi\beta_2)^C, (\phi\beta_3)^C) g^C((\phi e_i)^C, (\phi e_i)^C) \\ &- g^C((\phi e_i)^C, (\phi\beta_3)^C) g^C((\phi\beta_2)^V, (\phi e_i)^C) \\ &- g^V((\phi e_i)^C, (\phi\beta_3)^C) g^C((\phi\beta_2)^C, (\phi e_i)^C)\}. \end{aligned} \tag{75}$$

The equation (75) becomes

$$\bar{S}^C((\phi\beta_2)^C, (\phi\beta_3)^C) = \frac{\bar{r}(n-2)}{n(n-1)} g^C((\phi\beta_2)^C, (\phi\beta_3)^C). \tag{76}$$

From (23), (25), (27), (24), (58) and (61) we obtain

$$\begin{aligned}\bar{S}^C({\beta_2}^C, {\beta_3}^C) &= -g^C({\beta_2}^C, {\beta_3}^C) + \beta\Phi^C({\beta_2}^C, {\beta_3}^C) \\ &- (n-2)\{\eta^C({\beta_2}^C)\eta^V({\beta_3}^C) + \eta^V({\beta_2}^C)\eta^C({\beta_3}^C)\} \\ &+ \frac{r^C \bar{-} \beta^2(n-2)}{n(n-1)} g^C((\phi\beta_2)^C, (\phi\beta_3)^C). \end{aligned} \tag{77}$$

Then contracting the equation (77) over β_2 and β_3 and from (23), (25), (27), (24) we get

$$r^C = \beta^2 - n(n-1). \tag{78}$$

Then we conclude the following theorem:

Theorem 4.5. *If TM is ϕ^C-concirculary flat wrt $\bar{\nabla}^C$, then $r^C = \beta^2 - n(n-1)$.*

5. Some calculations for the projective curvature tensor on *TM*

Let TM be the tangent bundle of M. Projective curvature tensor $P \in \Im_1^3(M)$ wrt $\bar{\nabla}$ is given as [6]

$$\bar{P}(\beta_1, \beta_2)\beta_3 = \bar{R}(\beta_1, \beta_2)\beta_3 - \frac{1}{n-1}\{\bar{S}(\beta_2, \beta_3)\beta_1 - \bar{S}(\beta_1, \beta_3)\beta_2\}. \tag{79}$$

Employing the complete lift on (79), we acquire

$$\begin{aligned}\bar{P}^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C &= \bar{R}^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C \\ &- \frac{1}{n-1}\{\bar{S}^C({\beta_2}^C, {\beta_3}^C){\beta_1}^C - \bar{S}^C({\beta_1}^C, {\beta_3}^C){\beta_2}^C\}. \end{aligned} \tag{80}$$

In view of (48) and (57), using (80) we get

$$\begin{aligned}\bar{P}^C(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= P^C(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C \\ &- \frac{1}{n-1}\{g^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V + g^V(\beta_2{}^C,\beta_3{}^C)\beta_1{}^C \\ &- g^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V - g^V(\beta_1{}^C,\beta_3{}^C)\beta_2{}^C \\ &+ trace(\Phi)\Phi^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V + trace(\Phi)\Phi^V(\beta_2{}^C,\beta_3{}^C)\beta_1{}^C \\ &- trace(\Phi)\Phi^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V - trace(\Phi)\Phi^V(\beta_1{}^C,\beta_3{}^C)\beta_2{}^C \\ &- n\eta^C(\beta_2{}^C)\eta^C(\beta_3{}^C)\beta_1{}^V - n\eta^C(\beta_2{}^C)\eta^V(\beta_3{}^C)\beta_1{}^C \\ &- n\eta^V(\beta_2{}^C)\eta^C(\beta_3{}^C)\beta_1{}^C + n\eta^C(\beta_1{}^C)\eta^C(\beta_3{}^C)\beta_2{}^V \\ &+ n\eta^C(\beta_1{}^C)\eta^V(\beta_3{}^C)\beta_2{}^C + n\eta^V(\beta_1{}^C)\eta^C(\beta_3{}^C)\beta_2{}^C\} \\ &+ \Phi^C(\beta_2{}^C,\beta_3{}^C)(\Phi\beta_1)^V + \Phi^V(\beta_2{}^C,\beta_3{}^C)(\Phi\beta_1)^C \\ &- \Phi^C(\beta_1{}^C,\beta_3{}^C)(\Phi\beta_2)^V - \Phi^V(\beta_1{}^C,\beta_3{}^C)(\Phi\beta_2)^C \\ &- n\eta^C(\beta_2{}^C)\eta^C(\beta_3{}^C)\beta_1{}^V - n\eta^C(\beta_2{}^C)\eta^V(\beta_3{}^C)\beta_1{}^C \\ &- n\eta^V(\beta_2{}^C)\eta^C(\beta_3{}^C)\beta_1{}^C + n\eta^C(\beta_1{}^C)\eta^C(\beta_3{}^C)\beta_2{}^V \\ &+ n\eta^C(\beta_1{}^C)\eta^V(\beta_3{}^C)\beta_2{}^C + n\eta^V(\beta_1{}^C)\eta^C(\beta_3{}^C)\beta_2{}^C, \end{aligned} \tag{81}$$

where

$$\begin{aligned}P^C(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= R^C(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C \\ &- \frac{1}{n-1}\{S^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V + S^V(\beta_2{}^C,\beta_3{}^C)\beta_1{}^C \\ &- S^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V - S^V(\beta_1{}^C,\beta_3{}^C)\beta_2{}^C\}. \end{aligned} \tag{82}$$

Definition 5.1. *The TM is said to be ξ^C-projectively flat wrt $\bar{\nabla}^C$ if $\bar{P}^C(\beta_1{}^C,\beta_2{}^C)\xi^C = 0$ on TM.*

In view of (54), (59) and (80) we have

$$\bar{P}^C(\beta_1{}^C,\beta_2{}^C)\xi^C = 0. \tag{83}$$

Thus we conclude the following theorem:

Theorem 5.2. *The TM is ξ^C-projectively flat wrt $\bar{\nabla}^C$.*

Definition 5.3. *Let TM be the tangent bundle of a P-Sasakian manifold M with a quarter symmetric non-metric connection $\bar{\nabla}^C$. Then TM is said to be ϕ^C- projectively flat wrt a quarter symmetric non-metric connection $\bar{\nabla}^C$ if $g^C(P^C((\phi\beta_1)^C,(\phi\beta_2)^C)(\phi\beta_3)^C,(\phi u_4)^C) = 0$ on TM.*

From (80)

$$\begin{aligned}\bar{P}^C((\phi\beta_1)^C,(\phi\beta_2)^C)(\phi\beta_3)^C &= \bar{R}^C((\phi\beta_1)^C,(\phi\beta_2)^C)(\phi\beta_3)^C \\ &- \frac{1}{n-1}\{\bar{S}^C((\phi\beta_2)^C,(\phi\beta_3)^C)(\phi\beta_1)^V \\ &+ \bar{S}^V((\phi\beta_2)^C,(\phi\beta_3)^C)(\phi\beta_1)^C \\ &- \bar{S}^C((\phi\beta_1)^C,(\phi\beta_3)^C)(\phi\beta_2)^V \\ &- \bar{S}^V((\phi\beta_1)^C,(\phi\beta_3)^C)(\phi\beta_2)^C\}. \end{aligned} \tag{84}$$

In view of (61) and (84), ϕ^C-projectively flat, then we infer

$$\begin{aligned}\bar{K}^C((\phi\beta_1)^C,(\phi\beta_2)^C,(\phi\beta_3)^C,(\phi u_4)^C) &= \frac{1}{n-1}\{S^C({\beta_2}^C,{\beta_3}^C)\\ &+ g^C({\beta_2}^C,{\beta_3}^C)-\beta\Phi^C({\beta_2}^C,{\beta_3}^C)\\ &+ (n-2)\eta^C({\beta_2}^C)\eta^C({\beta_3}^C)]g^V((\phi\beta_1)^C,(\phi U)^C)\\ &+ S^V({\beta_2}^C,{\beta_3}^C)+g^V({\beta_2}^C,{\beta_3}^C)\\ &- \beta\Phi^V({\beta_2}^C,{\beta_3}^C)+(n-2)\eta^C({\beta_2}^C)\eta^V({\beta_3}^C)\\ &+ (n-2)\eta^V({\beta_2}^C)\eta^C({\beta_3}^C)]g^C((\phi\beta_1)^C,(\phi u_4)^C)\\ &+ [S^C({\beta_1}^C,{\beta_3}^C)+g^C({\beta_1}^C,{\beta_3}^C)\\ &- \beta\Phi^C({\beta_1}^C,{\beta_3}^C)\\ &+ (n-2)\eta^C({\beta_1}^C)\eta^C({\beta_3}^C)]g^V((\phi\beta_2)^C,(\phi U)^C)\\ &+ S^V({\beta_1}^C,{\beta_3}^C)+g^V({\beta_1}^C,{\beta_3}^C)\\ &- \beta\Phi^V({\beta_1}^C,{\beta_3}^C)+(n-2)\eta^C({\beta_1}^C)\eta^V({\beta_3}^C)\\ &+ (n-2)\eta^V({\beta_1}^C)\eta^C({\beta_3}^C)]g^C((\phi\beta_2)^C,(\phi u_4)^C)\}.\end{aligned} \tag{85}$$

Setting $\beta_1 = u_4 = e_i$ and summing up wrt $i = 1, 2, ..., n-1$ we deduce

$$\begin{aligned}\sum_{i=1}^{n-1}\bar{K}^C((\phi e_i)^C,(\phi\beta_2)^C,(\phi\beta_3)^C,(\phi e_i)^C) &= S^C({\beta_2}^C,{\beta_3}^C)\\ &+ g^C({\beta_2}^C,{\beta_3}^C)-\beta\Phi^C({\beta_2}^C,{\beta_3}^C)\\ &+ (n-2)\eta^C({\beta_2}^C)\eta^C({\beta_3}^C)]\\ &+ (n-2)\eta^C({\beta_2}^C)\eta^V({\beta_3}^C)\\ &+ (n-2)\eta^V({\beta_2}^C)\eta^C({\beta_3}^C)]\\ &+ \frac{1}{n-1}\{\sum_{i=1}^{n-1}[S^C(e_i^C,{\beta_3}^C)g^V((\phi\beta_2)^C),(\phi e_i)^C)\\ &+ S^V(e_i^C,{\beta_3}^C)g^C((\phi\beta_2)^C),(\phi e_i)^C)]\\ &+ g^C((\phi\beta_2)^C,(\phi\beta_3)^C)-\beta\Phi^C({\beta_2}^C,{\beta_3}^C)\}.\end{aligned} \tag{86}$$

Since

$$\begin{aligned}\{\sum_{i=1}^{n-1}[S^C(e_i^C,{\beta_3}^C)g^V((\phi\beta_2)^C),(\phi e_i)^C) &+ S^V(e_i^C,{\beta_3}^C)g^C((\phi\beta_2)^C),(\phi e_i)^C)]\\ &= S^C((\phi\beta_2)^C,(\phi\beta_3)^C).\end{aligned} \tag{87}$$

On applying (57), (61), (87) in (86), then (86) becomes

$$\sum_{i=1}^{n-1}\bar{K}^C((\phi e_i)^C,(\phi e_i)^C,(\phi e_i)^C,(\phi e_i)^C) = \frac{n-2}{n-1}\bar{S}^C((\phi\beta_2)^C,(\phi\beta_3)^C). \tag{88}$$

and

$$\begin{aligned}\sum_{i=1}^{n-1}\bar{K}^C((\phi e_i)^C,(\phi e_i)^C,(\phi e_i)^C,(\phi e_i)^C) &= \bar{S}^C((\phi\beta_2)^C,(\phi\beta_3)^C)\\ &- g^C(\bar{R}^C(\xi^C,(\phi\beta_2)^C)(\phi\beta_3)^C,\xi^C).\end{aligned} \tag{89}$$

In view of (32) and (89) we infer

$$\sum_{i=1}^{n-1} \bar{K}^C((\phi e_i)^C,(\phi e_i)^C,(\phi e_i)^C,(\phi e_i)^C) = \bar{S}^C((\phi\beta_2)^C,(\phi\beta_3)^C) - g^C((\phi\beta_2)^C,(\phi\beta_3)^C). \tag{90}$$

Furthermore, in view of (60), (88) and (90) we infer

$$\bar{S}^C({\beta_2}^C,{\beta_3}^C) = (n-1)\{g^C({\beta_2}^C,{\beta_3}^C) - 3\eta^C({\beta_2}^C)\eta^V({\beta_3}^C) - 3\eta^V({\beta_2}^C)\eta^C({\beta_3}^C)\} \tag{91}$$

Hence

Theorem 5.4. *If TM is ϕ^C-projectively flat wrt $\bar{\nabla}^C$, then TM is an η^C-Einstein manifold wrt $\bar{\nabla}^C$.*

6. Proposed theorem for conformal curvature tensor on *TM*

Let *TM* be the tangent bundle of a *P*-Sasakian manifold *M* with a quarter symmetric non-metric connection $\bar{\nabla}^C$. The conformal curvature tensor of *M* wrt a quarter symmetric non-metric connection ∇ is given

$$\begin{aligned}\bar{C}(\beta_1,\beta_2,\beta_3,U) &= \bar{K}(\beta_1,\beta_2,\beta_3,U) - \frac{1}{n-1}\{g(\beta_2,\beta_3)\bar{S}(\beta_1,U) - g(\beta_1,\beta_3)\bar{S}(\beta_2,U) \\ &+ \bar{S}(\beta_2,\beta_3)g(\beta_1,U) - \bar{S}(\beta_1,\beta_3)g(\beta_2,U)\} \\ &+ \frac{\bar{r}}{(n-1)(n-2)}\{g(\beta_2,\beta_3)g(\beta_1,U) - g(\beta_1,\beta_3)g(\beta_2,U)\}.\end{aligned} \tag{92}$$

Employing the complete lift on (92), we infer

$$\begin{aligned}\bar{C}^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) &= \bar{K}^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) - \frac{1}{n-1}\{g^C({\beta_2}^C,{\beta_3}^C)\bar{S}^V({\beta_1}^C,U^C) \\ &+ g^V({\beta_2}^C,{\beta_3}^C)\bar{S}^C({\beta_1}^C,U^C) - g^C({\beta_1}^C,{\beta_3}^C)\bar{S}^V({\beta_2}^C,U^C) \\ &- g^V({\beta_1}^C,{\beta_3}^C)\bar{S}^C({\beta_2}^C,U^C) + \bar{S}^C({\beta_2}^C,{\beta_3}^C)g^V({\beta_1}^C,U^C) \\ &+ \bar{S}^V({\beta_2}^C,{\beta_3}^C)g^C({\beta_1}^C,U^C) \\ &- \bar{S}^C({\beta_1}^C,{\beta_3}^C)g^V({\beta_2}^C,U^C) - \bar{S}^V({\beta_1}^C,{\beta_3}^C)g^C({\beta_2}^C,U^C)\} \\ &+ \frac{\bar{r}^C}{(n-1)(n-2)}\{g^C({\beta_2}^C,{\beta_3}^C)g^V({\beta_1}^C,U^C) \\ &+ g^V({\beta_2}^C,{\beta_3}^C)g^C({\beta_1}^C,U^C) - g^C({\beta_1}^C,{\beta_3}^C)g^V({\beta_2}^C,U^C) \\ &- g^V({\beta_1}^C,{\beta_3}^C)g^C({\beta_2}^C,U^C)\}.\end{aligned} \tag{93}$$

Assume that $C^C({\beta_1}^C,{\beta_2}^C,{\beta_3}^C,U^C) = 0$. Putting $\beta_2 = \beta_3 = \xi$ in (93) and using (54), (58) and (59) we obtain

$$\begin{aligned}\bar{S}^C({\beta_1}^C,U^C) &= \{\frac{r-\beta^2}{n-1} + 6 - 2n\}g^C({\beta_1}^C,U^C) \\ &+ \{\frac{\beta^2 - r}{n-1} - 4\}\{\eta^C({\beta_1}^C)\eta^V(U^C) + \eta^V({\beta_1}^C)\eta^C(U^C)\}.\end{aligned} \tag{94}$$

Hence we concludethe following theorem:

Theorem 6.1. *If TM is ϕ^C-conformally flat with $\bar{\nabla}^C$, then TM is Einstein manifold wrt $\bar{\nabla}^C$.*

Acknowledgement:

I would like to thank the Reviewers and the Editor for reviewing the paper carefully and their valuable comments to improve the overall quality of the paper.

References

[1] T. Adati and T. Miyazawa, On P-Sasakian manifolds satisfying certain conditions, Tensor N. S., 33(2) (1979), 173-178.
[2] T. Adati and K. Matsumoto, On conformally recurrent and conformally symmetric P-Sasakian manifolds, TRU Math., 13 (1977), 25-32 .
[3] M. Altunbas, L. Bilen and A. Gezer, Remarks about the Kaluza-Klein metric on tangent bundle, Int. J. Geo. Met. Mod. Phys., 16 (3) (2019), 1950040.
[4] S. Ali and R. Nivas, On submanifolds immersed in a manifold with quarter symmetric connection, Rivista di Matematica della Universita di Parma, 6(3) (2000), 11–23.
[5] A. Barman and G. Ghosh, Concircular curvature tensor of a semi-symmetric non-metric connection on P-Sasakian manifolds, Analele Universitatii de Vest Timisoara Seria Matematica-informatica, LIV, 2 (2016), 47-58.
[6] O. Bahadir, *P*-Sasakian manifold with quarter-symmetric non-metric connection, Universal Journal of Applied Mathematics, 6(4) (2018), 123-133.
[7] S. C. Biswas and U. C. De, Quarter-symmetric metric connection in an SP-Sasakian manifold, Communications, Faculty of Sciences. University of Ankara Series A, 46(1-2) (1997) 49–56.
[8] L. S. Das and M. N. I. Khan, Symmetric and Ricci LP-Sasakian manifold, Mathematical Sciences Research Journal, 17 (10) (2013), 263-268.
[9] L. S. Das and M. N. I. Khan, Almost *r*-contact structures on the Tangent bundle, Differential Geometry-Dynamical Systems, 7 (2005), 34-41.
[10] U. C. De, Y. L. Han and P. B. Zhao, A special type of semi-symmetric non-metric connection of a Riemannian manifold, Facta Universitis (NIS) Ser. Math. Inform, 31(2) (2016), 529–541.
[11] H. M. Dida, and F. Hathout, Ricci soliton on the tangent bundle with semi-symmetric metric connection, Bulletin of the Transilvania University of Brasov Series III: Mathematics and Computer Science, 1(63)(2) (2021), 37-52.
[12] H. M. Dida, and A. Ikemakhen, A class of metrics on tangent bundles of pseudo-Riemannian manifolds, Archivum Mathematicum (BRNO) Tomus, 47 (2011), 293–308.
[13] H. M. Dida, F. Hathout and M. Djaa, On the geometry of the second order tangent bundle with the diagonal lift metric, Int. Journal of Math. Analysis, 3(9) (2009), 443 - 456.
[14] A. Friedmann and J. A. Schouten, AUber die geometrie der halbsymmetrischen Aubertragung, Math. Zeitschr. 21(1924), 211-223.
[15] S. Golab, On semi-symmetric and quarter-symmetric linear connections, Tensor N.S., 29 (1975), 249-254.
[16] I. Sato, On a structure similar to the almost contact structure, Tensor new series, 30(3) (1976), 219-224.
[17] H. A. Hayden, Subspaces of a space with torsion, Proc. London Math. Soc. 34 (1932), 27-50.
[18] M. N. I. Khan, Lifts of hypersurfaces with quarter-symmetric semi-metric connection to tangent bundles, Afrika Matematika, 27 (2014), 475-482.
[19] M. N. I. Khan and U. C. De, Velimirovic, L.S., Lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle. Mathematics, 11, 53, 2023.
[20] A. Kazan, and H. B. Karadag, Locally decomposable golden tangent bundles with CheegerGromoll metric, Miskolc Math. Not., 17 (1) (2016), 399-411.
[21] M. N. I. Khan, Lifts of semi-symmetric non-metric connection on a Kähler manifold, Afrika Matematika, 27(3) (2016), 345-352.
[22] M. N. I. Khan, Tangent bundle endowed with quarter-symmetric non-metric connection on an almost Hermitian manifold, Facta Universitis (NIS) Ser. Math. Inform, 35(1) (2020), 167-178.
[23] M. N. I. Khan, Quarter-symmetric semi-metric connection on a Sasakian manifold, Tensor, N.S., 68(2) (2007), 154-157.
[24] M. N. I. Khan, Novel theorems for metallic structures on the frame bundle of the second order, Filomat, 36(13) (2022), 4471-4482.
[25] M. N. I. Khan and U. C. De, Lifts of metallic structure on a cross-section, 36(18) (2022), 6369-6373.
[26] M. N. I. Khan, Proposed theorems for lifts of the extended almost complex structures on the complex manifold, Asian-European Journal of Mathematics, 15(11) 2250200 (13 pages), 2022.
[27] M. N. I. Khan, Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold, Chaos, Solitons & Fractals, 146, May 2021, 110872.
[28] A. K. Mondal and U. C. De, Quarter-symmetric non-metric connection on *P*-sasakian manifolds, ISRN Geometry, (2012), 14 pages.
[29] R. S. Mishra and S. Pandey, On quarter symmetric metric F-connections, Tensor, N.S., 34 (1980), 1-7.
[30] M. D. Maksimović, Quarter-symmetric non-metric connection, math.DG arXiv, 2022. https://arxiv.org/abs/2210.01509
[31] N. Poyraz and H. I. Yolda, Chen Inequalities for Submanifolds of Real Space Forms with a Ricci Quarter-Symmetric Metric Connection, International Electronic Journal of Geometry 12(1) (2019), 102-110.
[32] E. Peyghan, F. Firuzi and U. C. De, Golden Riemannian structures on the tangent bundle with g-natural metrics, Filomat, 33 (8) (2019), 2543-2554.
[33] J. A. Schouten, Ricci calculus, Springer, 1954.
[34] S. Sular, C. Ozgur and U. C. De, Quarter-symmetric metric connection in a Kenmotsu manifold, SUT Journal of Mathematics, 44(2) (2008), 297–306.

[35] S. L. J. Velimirović, S. M. Minčić, M. S. Stanković, Infinitesimal deformations of curvature tensor of non-symmetric affine connection space, Mat. Vesnic, 54 (2002), 219-226.
[36] K. Yano and S. Ishihara, Tangent and cotangent bundles, Marcel Dekker, Inc., New York, 1973.
[37] 28 K. Yano and T. Imai, Quarter-symmetric connections and their curvature tensors, Tensor, N.S., 38 (1982), 13-18.