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Some results on invariant submanifolds of a paracontact (κ, μ, ν)-space

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Abstract. In this paper, we have characterized an invariant submanifold of a paracontact (κ, μ, ν)-space. Besides this, we have researched some geometric conditions for an invariant submanifold of a paracontact (κ, μ, ν)-space to be totally geodesic.

1. Introduction

The study of paracontact geometry was initiated by Kaneyuki and Williams[8]. After then, Zamkovoy started working paracontact metric manifolds and their subclasses [15]. Since several geometers interested paracontact metric manifolds and researched various important properties of these manifolds and some interesting results have been found.

The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. One of the class of paracontact manifolds for which the characteristic vector field ξ-belongs to the (κ, μ)-nullity condition for some real constants κ and μ. Such manifolds are known as (κ, μ)-paracontact metric manifolds [13].

I. Küpeli Erken and C. Murathan showed that a paracontact metric $(\kappa, \mu, \nu)-$manifold with $\kappa = -1$ is not necessary para-Sasakian. They found examples about paracontact metric $(\kappa, \mu, \nu)-$manifolds according to the cases $\kappa > -1$, $\kappa < -1$. They researched a relation between non-Sasakian $(\kappa, \mu, \nu = const.)-$contact metric manifold with the Boeckx invariant $I_M = \frac{1-\frac{M}{2}}{\sqrt{1-k}}$ is constant along the integral curves of $\xi(I_M) = 0$ [10].

In [1], M. Atçeken studied how the functions κ, μ and ν behave on the submanifold. He investigated necassary and sufficient conditions for an invariant submanifold of an almost Kenmotsu $(\kappa, \mu, \nu)-$ space to be totally geodesic under some conditions.

In addition to the studies I mentioned above, many authors have examined invariant submanifolds and many important properties of different manifolds in their studies, [2–7, 9, 11, 12, 14].

Recently, we have studied an invariant submanifold of a (κ, μ, ν) paracontact metric manifold and obtained some new results. In this paper, we research the conditions under which invariant pseudoparallel submanifolds of a (κ, μ, ν)-paracontact space are totally geodesic.

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2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}$ is said to be a paracontact metric manifold if it admits a (1,1)-type tensor field ϕ, a unit spacelike vector field ξ, 1-form η and a semi-Riemannian metric tensor g which satisfy

$$\begin{aligned} \phi^2 x_1 &= x_1 - \eta(x_1)\xi, \ \ \eta(x_1) = g(x_1, \xi) && (1) \\ g(\phi x_1, \phi x_2) &= -g(x_1, x_2) + \eta(x_1)\eta(x_2), \ \ \eta \circ \phi = 0 && (2) \end{aligned}$$

and

$$d\eta(x_1, x_2) = g(x_1, \phi x_2),$$

for all $x_1, x_2 \in \Gamma(T\widetilde{M})$, where $\Gamma(T\widetilde{M})$ denote the set of the differentiable vector fields on $\widetilde{M}$.

In a paracontact metric manifold $(\widetilde{M}, \phi, \eta, \xi, g)$, we define a $(1,1)$-type tensor field by $h = \frac{1}{2}\ell_\xi \phi$, where ℓ denotes the Lie-derivative. One can easily to see that h is a symmetric and satisfies

$$\begin{aligned} h\xi &= 0, \ \ h\phi = -\phi h \ \ and \ \ Trh = 0. && (3) \\ 2hx_1 &= (\ell_\xi \phi)x_1 = \ell_\xi \phi x_1 - \phi \ell_\xi x_1 = [\xi, \phi x_1] - \phi[\xi, x_1]. && (4) \end{aligned}$$

By $\widetilde{\nabla}$, we denote the Levi-Civita connection of g, then we have

$$\widetilde{\nabla}_{x_1}\xi = -\phi x_1 + \phi h x_1, \ \ \widetilde{\nabla}_\xi \phi = 0. \tag{5}$$

for all $x_1 \in \Gamma(T\widetilde{M})$.
Moreover, $h = 0$ if and only if ξ is a Killing vector field and this case $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is said to be K-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. In any para-Sasakian manifold

$$\widetilde{R}(x_1, x_2)\xi = -(\eta(x_2)x_1 - \eta(x_1)x_2) \tag{6}$$

holds, but unlike contact metric geometry the condition (6) not necessarily implies that the manifold is para-Sasakian.

A paracontact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is said to be a (κ, μ)-space form if its the Riemannian curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(x_1, x_2)\xi = \kappa\{\eta(x_2)x_1 - \eta(x_1)x_2\} + \mu\{\eta(x_2)hx_1 - \eta(x_1)hx_2\}, \tag{7}$$

for all $x_1, x_2 \in \Gamma(T\widetilde{M})$, where κ, μ are real constant.

A $(2n+1)-$dimensional paracontact metric (κ, μ, ν)-manifold is a paracontact metric manifold for which the curvature tensor field satisfies

$$\begin{aligned} \widetilde{R}(x_1, x_2)\xi &= \kappa\{\eta(x_2)x_1 - \eta(x_1)x_2\} + \mu\{\eta(x_2)hx_1 - \eta(x_1)hx_2\} \\ &\quad + \nu\{\eta(x_2)\phi h x_1 - \eta(x_1)\phi h x_2\}, && (8) \end{aligned}$$

for all $x_1, x_2 \in \Gamma(T\widetilde{M})$, where κ, μ, ν are smooth functions on $\widetilde{M}$.

Lemma 2.1. *Let $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ an paracontact metric (κ, μ, ν)-manifold. Then the following identities hold:*

$$\begin{aligned} h^2 &= (1+\kappa)\phi^2, \ \ for \ \kappa \neq -1, && (9) \\ \xi(\kappa) &= -2\nu(1+\kappa), && (10) \\ Q\xi &= 2nk\xi, && (11) \\ (\widetilde{\nabla}_{x_1}\phi)x_2 &= -g(x_1 - hx_1, x_2)\xi + \eta(x_2)(x_1 - hx_1), && (12) \\ S(x_1, \xi) &= 2nk\eta(x_1), && (13) \\ \widetilde{R}(\xi, x_1)x_2 &= \kappa\{g(x_1, x_2)\xi - \eta(x_2)x_1\} + \mu\{g(hx_1, x_2)\xi - \eta(x_2)hx_1\} \\ &\quad + \nu\{g(\phi h x_1, x_2)\xi - \eta(x_2)\phi h x_1\}, && (14) \end{aligned}$$

for any vector fields x_1, x_2 *on* $\widetilde{M}$, *where S and Q denote the Ricci tensor and Ricci operatory defined* $S(x_1, x_2) = g(Qx_1, x_2)$.

Now, let M be an immersed submanifold of a (κ, μ, ν)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$, by ∇ and $\nabla^{\perp}$, we denote the induced connections on $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively. Then the Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}_{x_1} x_2 = \nabla_{x_1} x_2 + \sigma(x_1, x_2) \tag{15}$$

and

$$\widetilde{\nabla}_{x_1} x_5 = -A_{x_5} x_1 + \nabla^{\perp}_{x_1} x_5, \tag{16}$$

for all $x_1, x_2 \in \Gamma(TM)$ and $x_5 \in \Gamma(T^{\perp}M)$, where σ and A are called the second fundamental form and shape operator of M, respectively.
They are related by

$$g(A_{x_5} x_1, x_2) = g(\sigma(x_1, x_2), x_5). \tag{17}$$

The first covariant derivative of the second fundamental form σ is defined by

$$(\widetilde{\nabla}_{x_1}\sigma)(x_2, x_3) = \nabla^{\perp}_{x_1}\sigma(x_2, x_3) - \sigma(\nabla_{x_1} x_2, x_3) - \sigma(x_2, \nabla_{x_1} x_3), \tag{18}$$

for all $x_1, x_2 \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, then the submanifold is said to be its second fundamental form is parallel. By R, we denote the Riemannian curvature tensor of the submanifold M, we have the following Gauss equation

$$\begin{aligned} \widetilde{R}(x_1, x_2)x_3 &= R(x_1, x_2)x_3 + A_{\sigma(x_1,x_3)} x_2 - A_{\sigma(x_2,x_3)} x_1 + (\widetilde{\nabla}_{x_1}\sigma)(x_2, x_3) \\ &\quad -(\widetilde{\nabla}_{x_2}\sigma)(x_1, x_3), \end{aligned} \tag{19}$$

for all $x_1, x_2, x_3 \in \Gamma(TM)$.
$\widetilde{R} \cdot \sigma$ is given by

$$\begin{aligned} (\widetilde{R}(x_1, x_2) \cdot \sigma)(x_4, x_5) &= R^{\perp}(x_1, x_2)\sigma(x_4, x_5) - \sigma(R(x_1, x_2)x_4, x_5) \\ &\quad -\sigma(x_4, R(x_1, x_2)x_5), \end{aligned} \tag{20}$$

where the Riemannian curvature tensor of normal bundle $\Gamma(T^{\perp}M)$ is given

$$R^{\perp}(x_1, x_2) = [\nabla^{\perp}_{x_1}, \nabla^{\perp}_{x_2}] - \nabla^{\perp}_{[x_1,x_2]}.$$

On a semi-Riemannian manifold (M, g), for a (o, k)-type tensor field T and $(0, 2)$-type tensor field A, $(0, k+2)$-type tensor field $Q(A, T)$ is defined as

$$\begin{aligned} Q(A, T)(x_{11}, x_{12}, ..., x_{1k}; x_1, x_2) &= -T((x_1 \wedge_A x_2)x_{11}, x_{12}, ..., x_{1k}) \\ &- T(x_{11}, (x_1 \wedge_A x_2)x_{12}, x_{13}, ..., x_{1k}) \\ &\cdot \\ &\cdot \\ &\cdot \\ &- T(x_{11}, x_{12}, ..., (x_1 \wedge_A x_2)x_{1k}), \end{aligned} \tag{21}$$

for all $x_{11}, x_{12}, ..., x_{1k}, x_1, x_2 \in \Gamma(TM)$, where

$$(x_1 \wedge_A x_2)x_{11} = A(x_2, x_{11})x_1 - A(x_1, x_{11})x_2. \tag{22}$$

Definition 2.2. *Let M be a submanifold of a Riemannian manifold $(\widetilde{M}, g)$. If there exist functions L_1, L_2, L_3 and L_4 on $\widetilde{M}$ such that*

$$\begin{aligned} \widetilde{R}\cdot\sigma &= L_1 Q(g,\sigma), && (23)\\ \widetilde{R}\cdot\widetilde{\nabla}\sigma &= L_2 Q(g,\widetilde{\nabla}\sigma), && (24)\\ \widetilde{R}\cdot\sigma &= L_3 Q(S,\sigma), && (25)\\ \widetilde{R}\cdot\widetilde{\nabla}\sigma &= L_4 Q(S,\widetilde{\nabla}\sigma), && (26)\end{aligned}$$

then M is, respectively, pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel submanifold. In particular, if $L_1 = 0$ (resp., $L_2 = 0$), then M is said to be semiparallel (resp. 2-semiparallel) [5].

3. Invariant submanifolds of a paracontact (κ, μ, ν)-space

For an immersed submanifold M of a (κ,μ,ν)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$, M is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕx_1 is tangent to M for all $x_1 \in \Gamma(TM)$ at every point on M, that is, $\phi(T_{x_1}M) \subseteq T_{x_1}M$ at each point $x_1 \in M$. We will assume that M is an invariant submanifold in the rest of this paper unless say otherwise.

Lemma 3.1. *Let M be an invariant submanifold of a (κ,μ,ν)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then the following relations hold.*

$$\begin{aligned} \nabla_{x_1}\xi &= -\phi x_1 + \phi h x_1 && (27)\\ \sigma(\phi x_1, x_2) &= \sigma(x_1,\phi x_2) = \phi\sigma(x_1,x_2) && (28)\\ \sigma(x_1,\xi) &= 0, && (29)\end{aligned}$$

for all $x_1, x_2 \in \Gamma(TM)$.

Proof. Since the proof is a result of direct calculations, we will omit to it. □

Theorem 3.2. *Let M be an invariant submanifold of a paracontact (κ,μ,ν)-space $M^{2n+1}(\phi,\eta,\xi,g)$. Then the second fundamental form σ of M is parallel if and only if M is totally geodesic provided $\kappa \neq 0$.*

Proof. Let us assume that σ is parallel. From (19), implies that

$$\begin{aligned} (\widetilde{\nabla}_{x_1}\sigma)(x_2,x_3) &= \nabla^{\perp}_{x_1}\sigma(x_2,x_3) - \sigma(\nabla_{x_1}x_2,x_3)\\ -\sigma(x_2,\nabla_{x_1}x_3) &= 0, && (30)\end{aligned}$$

for all vector fields x_1, x_2 and x_3 on M^{2n+1}. Setting $x_3 = \xi$ in (30) and taking into account (27) and (28), we get

$$\sigma(x_2,\nabla_{x_1}\xi) = \sigma(x_2, -\phi x_1 + \phi h x_1) = 0,$$

that is,

$$\sigma(x_2,\phi X) - \sigma(x_2,\phi h x_1) = 0. \tag{31}$$

Substituting x_1 by hx_1 in (31) and making use of (10) and (12), we obtain

$$\begin{aligned} \phi\sigma(x_2,hx_1) - \phi\sigma(x_2,h^2x_1) &= 0,\\ \phi\sigma(x_2,hx_1) + (1+k)\phi\sigma(x_1,x_2) &= 0. && (32)\end{aligned}$$

From (31) and (32), we conclude that $\kappa\sigma(x_1,x_2) = 0$, which proves our assertion. □

Theorem 3.3. *Let M be an invariant submanifold of a paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. If M is a pseudoparallel submanifold of a paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then M is either totally geodesic or the function L_1 satisfies*

$$L_1 = \kappa \pm \sqrt{(\mu^2+\nu^2)(\kappa+1)}, \quad (\kappa+1)\mu\nu = 0. \tag{33}$$

Proof. Let M be an invariant pseudoparrallel submanifold of an paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. This implies that

$$L_1 Q(g,\sigma)(x_4,x_5;x_1,x_2) = (\widetilde{R}(x_1,x_2)\cdot\sigma)(x_4,x_5),$$

for all $x_1,x_2,x_4,x_5 \in \Gamma(TM)$. This yields to

$$\begin{aligned}&-L_1\{\sigma((x_1\wedge_g x_2)x_4,x_5)+\sigma(x_4,(x_1\wedge_g x_2)x_5)\} = R^{\perp}(x_1,x_2)\sigma(x_4,x_5)\\&-\sigma(R(x_1,x_2)x_4,x_5)-\sigma(x_4,R(x_1,x_2)x_5).\end{aligned} \tag{34}$$

In (34), putting $x_1 = x_4 = \xi$ and taking into account (8), (27) and (12), we obtain

$$L_1\sigma(x_2,x_5) = -\sigma(R(\xi,x_2)\xi,x_5) = \kappa\sigma(x_2,x_5)+\mu\sigma(hx_2,x_5)+\nu\sigma(\phi hx_2,x_5),$$

that is,

$$(L_1-\kappa)\sigma(x_2,x_5) = \mu\sigma(hx_2,x_5)+\nu\phi\sigma(hx_2,x_5). \tag{35}$$

If hx_2 is written instead of x_2 at (35) and using (9), (18), we get

$$\begin{aligned}(L_1-\kappa)\sigma(hx_2,x_5) &= \mu\sigma(h^2x_2,x_5)+\nu\phi\sigma(h^2x_2,x_5)\\&= (1+\kappa)[\mu\sigma(x_2,x_5)+\nu\phi\sigma(x_2,x_5)].\end{aligned} \tag{36}$$

From (35) and (36), we conclude that

$$[(L_1-\kappa)^2-(1+\kappa)(\mu^2+\nu^2)]\sigma(x_2,x_5)-2\mu\nu(1+\kappa)\phi\sigma(x_2,x_5) = 0.$$

This completes the proof. □

Corollary 3.4. *Let M be an invariant submanifold of a paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. M is a pseudoparallel submanifold if and only if M is totally geodesic provided*

$$\kappa^2+(\kappa+1)(\mu^2+\nu^2)\neq 0 \qquad \text{or} \qquad (\kappa+1)\mu\nu\neq 0.$$

Theorem 3.5. *Let M be an invariant submanifold of a paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. If M is a Ricci-generalized pseudoparallel submanifold of a paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then M is either totally geodesic or the function L_3 satisfies*

$$L_3 = \frac{1}{2n}\left[1 \pm \frac{\sqrt{(\mu^2+\nu^2)(\kappa+1)}}{\kappa}\right], \quad (\kappa+1)\mu\nu = 0. \tag{37}$$

Proof. If M is an invariant Ricci-generalized pseudoparallel of a paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$, that means

$$L_3 Q(S,\sigma)(x_4,x_5;x_1,x_2) = (\widetilde{R}(x_1,x_2)\cdot\sigma)(x_4,x_5),$$

for all $x_1,x_2,x_4,x_5 \in \Gamma(TM)$, which implies that

$$\begin{aligned}&-L_3\{\sigma((x_1\wedge_S x_2)x_4,x_5)+\sigma(x_4,(x_1\wedge_S x_2)x_5)\} = R^{\perp}(x_1,x_2)\sigma(x_4,x_5)\\&-\sigma(R(x_1,x_2)x_4,x_5)-\sigma(x_4,R(x_1,x_2)x_5).\end{aligned} \tag{38}$$

In (38), setting $x_1 = x_5 = \xi$ and making use of (27), (28), we arrive

$$\begin{aligned} 2n\kappa L_3\sigma(x_4, x_2) &= -\sigma(R(\xi, x_2)\xi, x_4) = \kappa\sigma(x_2, x_4) + \mu\sigma(hx_2, x_4) \\ &\quad +\nu\sigma(\phi hx_2, x_4). \end{aligned} \tag{39}$$

In view of (39), it follows that

$$\kappa(2nL_3 - 1)\sigma(x_4, x_2) - \mu\sigma(x_4, hx_2) - \nu\phi\sigma(x_4, hx_2) = 0. \tag{40}$$

Substituting hx_2 for x_2 in (39) and using (8), (9), we get

$$\kappa(2nL_3 - 1)\sigma(x_4, hx_2) - (1 + \kappa)\mu\sigma(x_4, x_2) - (1 + \kappa)\nu\phi\sigma(x_4, x_2) = 0. \tag{41}$$

From (40) and (41), we reach at

$$\left[\kappa^2(2nL_3 - 1)^2 - (1 + \kappa)(\mu^2 + \nu^2)\right]\sigma(x_4, x_2) - 2\mu\nu(1 + \kappa)\phi\sigma(x_4, x_2) = 0.$$

This completes the proof. □

Theorem 3.6. *Let M be an invariant submanifold of a paracontact (κ, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$. If M is a 2-pseudoparallel submanifold of $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$, then M is either totally geodesic or the function L_2 satisfies*

$$L_2 = \kappa \pm \sqrt{(\mu^2 + \nu^2)(\kappa + 1)}, \quad (\kappa + 1)\mu\nu = 0. \tag{42}$$

Proof. There exists a function L_2 such that

$$L_2 Q(g, \widetilde{\nabla}\sigma)(x_4, x_5, x_3; x_1, x_2) = (\widetilde{R}(x_1, x_2) \cdot \widetilde{\nabla}\sigma)(x_4, x_5, x_3),$$

for all $x_1, x_2, x_4, x_5, x_3 \in \Gamma(TM)$. This yields to

$$\begin{aligned} &-L_2\{(\widetilde{\nabla}_{(x_1\wedge_g x_2)x_4}\sigma)(x_5, x_3) + (\widetilde{\nabla}_{x_4}\sigma)((x_1 \wedge_g x_2)x_5, x_3) \\ &+(\widetilde{\nabla}_{x_4}\sigma)(x_5, (x_1 \wedge_g x_2)x_3)\} \\ = \ & R^{\perp}(x_1, x_2)(\widetilde{\nabla}_{x_4}\sigma)(x_5, x_3) - (\widetilde{\nabla}_{R(x_1,x_2)x_4}(x_5, x_3) \\ &-(\widetilde{\nabla}_{x_4}\sigma)(R(x_1, x_2)x_5, x_3) - (\widetilde{\nabla}_{x_4}\sigma)(x_5, R(x_1, x_2)x_3). \end{aligned} \tag{43}$$

In (43), taking $x_1 = x_5 = \xi$, we have

$$\begin{aligned} &-L_2\{(\widetilde{\nabla}_{(\xi\wedge_g x_2)x_4}\sigma)(\xi, x_3) + (\widetilde{\nabla}_{x_4}\sigma)((\xi \wedge_g x_2)\xi, x_3) \\ &+(\widetilde{\nabla}_{x_4}\sigma)(\xi, (\xi \wedge_g x_2)x_3)\} \\ = \ & R^{\perp}(\xi, x_2)(\widetilde{\nabla}_{x_4}\sigma)(\xi, x_3) - (\widetilde{\nabla}_{R(\xi,x_2)x_4}(\xi, x_3)) \\ &-(\widetilde{\nabla}_{x_4}\sigma)(R(\xi, x_2)\xi, x_3) - (\widetilde{\nabla}_{x_4}\sigma)(\xi, R(\xi, x_2)x_3). \end{aligned} \tag{44}$$

Now we will calculate them separately. In view of (18), (22), (27) and (28), we can derive

$$\begin{aligned} (\widetilde{\nabla}_{(\xi\wedge_g x_2)x_4}\sigma)(\xi, x_3) &= -\sigma(\nabla_{(\xi\wedge_g x_2)x_4}\xi, x_3) \\ &= \sigma(\phi(\xi \wedge_g x_2)x_4 - \phi h(\xi \wedge_g x_2)x_4, x_3) \\ &= \sigma(\phi(g(x_2, x_4)\xi - \eta(x_4)x_2), x_3) \\ &\quad -\sigma(\phi h(g(x_2, x_4)\xi - \eta(x_4)x_2), x_3) \\ &= \eta(x_4)\{\sigma(\phi hx_2, x_3) - \sigma(\phi x_2, x_3)\}. \end{aligned} \tag{45}$$

In the same way,

$$\begin{aligned}(\widetilde{\nabla}_{x_4}\sigma)((\xi\wedge_g x_2)\xi,x_3) &= (\widetilde{\nabla}_{x_4}\sigma)(\eta(x_2)\xi-x_2,x_3)\\ &= \nabla^{\perp}_{x_4}\sigma(\eta(x_2)\xi,x_3)-\sigma(\nabla_{x_4}x_3,\eta(x_2)\xi)\\ &\quad -\sigma(x_3,\nabla_{x_4}\eta(x_2)\xi)-(\widetilde{\nabla}_{x_4}\sigma)(x_2,x_3)\\ &= \eta(x_2)\{\sigma(x_3,\phi x_4)-\sigma(x_3,\phi hx_4)\}\\ &\quad -(\widetilde{\nabla}_{x_4}\sigma)(x_2,x_3), \end{aligned} \tag{46}$$

$$\begin{aligned}(\widetilde{\nabla}_{x_4}\sigma)(\xi,(\xi\wedge_g x_2)x_3) &= -\sigma(\nabla_{x_4}\xi,(\xi\wedge_g x_2)x_3)\\ &= -\sigma(-\phi x_4+\phi hx_4,g(x_2,x_3)\xi-\eta(x_3)x_2)\\ &= \eta(x_3)\{\sigma(\phi hx_4,x_2)-\sigma(\phi x_4,x_2)\}. \end{aligned} \tag{47}$$

For the right side of (44), by view of (18), (20) and (14), we have

$$\begin{aligned}R^{\perp}(\xi,x_2)(\widetilde{\nabla}_{x_4}\sigma)(\xi,x_3) &= R^{\perp}(\xi,x_2)\{\nabla^{\perp}_{x_4}\sigma(\xi,x_3)\\ &\quad -\sigma(\nabla_{x_4}\xi,x_3)-\sigma(\xi,\nabla_{x_4}x_3)\}\\ &= -R^{\perp}(\xi,x_2)\sigma(\nabla_{x_4}\xi,x_3)\\ &= R^{\perp}(\xi,x_2)\{\sigma(\phi x_4-\phi hx_4,x_3)\}. \end{aligned} \tag{48}$$

Also, making use of (3) and (8), we obtain

$$\begin{aligned}(\widetilde{\nabla}_{R(\xi,x_2)x_4}(\xi,x_3)) &= -\sigma(x_3,\nabla_{R(\xi,x_2)x_4}\xi)\\ &= -\sigma(x_3,-\phi R(\xi,x_2)x_4+\phi hR(\xi,x_2)x_4)\\ &= -\eta(x_4)\{\kappa\sigma(\phi x_2,x_3)+\mu\sigma(\phi hx_2,x_3)\\ &\quad +\nu(1+\kappa)\sigma(hx_2,x_3)-\kappa\sigma(\phi hx_2,x_3)\\ &\quad -\mu(1+\kappa)\sigma(\phi x_2,x_3)\\ &\quad -\nu(1+\kappa)\sigma(x_2,x_3)\} \end{aligned} \tag{49}$$

and we have

$$(\widetilde{\nabla}_{x_4}\sigma)(R(\xi,x_2)\xi,x_3) = (\widetilde{\nabla}_{x_4}\sigma)\{\kappa(\eta(x_2)\xi-x_2)-\mu hx_2-\nu\phi hx_2,x_3\}. \tag{50}$$

Finally,

$$\begin{aligned}(\widetilde{\nabla}_{x_4}\sigma)(\xi,R(\xi,x_2)x_3) &= -\sigma(\nabla_{x_4}\xi,R(\xi,x_2)x_3)\\ &= -\sigma(\phi x_4,\kappa\eta(x_3)x_2+\mu\eta(x_3)hx_2\\ &\quad +\nu\eta(x_3)\phi hx_2)+\sigma(\phi hx_4,\kappa\eta(x_3)x_2\\ &\quad +\mu\eta(x_3)hx_2+\nu\eta(x_3)\phi hx_2)\\ &= -\eta(x_3)\{\kappa\sigma(\phi x_4,x_2)-\mu\sigma(\phi x_4,hx_2)\\ &\quad -\nu\sigma(\phi x_4,\phi hx_2)-\kappa\sigma(\phi hx_4,x_2)\\ &\quad -\mu\sigma(\phi hx_4,hx_2)-\nu\sigma(\phi hx_4,\phi hx_2)\} \end{aligned} \tag{51}$$

Consequently, the values of (45)-(51) are put in (44), we arrive

$$\begin{aligned}
& -L_2\{\eta(x_4)\sigma(\phi x_2, x_3) + \eta(x_4)\sigma(\phi h x_2, x_3) \\
& +\eta(x_2)\sigma(x_3, \phi x_4) - \eta(x_2)\sigma(x_3, \phi h x_4) \\
& -(\widetilde{\nabla}_{x_4}\sigma)(x_3, x_2) - \eta(x_3)\sigma(\phi x_4, x_2) + \eta(x_3)\sigma(\phi h x_4, x_2)\} \\
= \;& R^{\perp}(\xi, x_2)\sigma(\phi x_4 - \phi h x_4, x_3) + \eta(x_4)\{\kappa\sigma(\phi x_2, x_3) \\
& +\mu\sigma(\phi h x_2, x_3) + \nu(1+\kappa)\sigma(h x_2, x_3) - \kappa\sigma(\phi h x_2, x_3) \\
& -\mu(1+\kappa)\sigma(\phi x_2, x_3) - \nu(1+\kappa)\sigma(x_2, x_3)\} \\
& -(\widetilde{\nabla}_{x_4}\sigma)\{\kappa[\eta(x_2)\xi - x_2] - \mu h x_2 - \nu\phi h x_2, x_3\} \\
& +\eta(x_3)\{\kappa\sigma(\phi x_4, x_2) + \mu\sigma(\phi x_4, h x_2) \\
& +\nu\sigma(\phi x_4, \phi h x_2) - \kappa\sigma(\phi h x_4, x_2) \\
& -\mu\sigma(\phi h x_4, h x_2) - \nu\sigma(\phi h x_4, \phi h x_2)\}. \qquad (52)
\end{aligned}$$

In (52), putting $x_3 = \xi$ and taking into account (8), we have

$$\begin{aligned}
& L_2\{(\widetilde{\nabla}_{x_4}\sigma)(\xi, x_2) + \sigma(\phi x_4, x_2) - \sigma(\phi h x_4, x_2)\} \\
= \;& -(\widetilde{\nabla}_{x_4}\sigma)(\kappa(\eta(x_2)\xi - x_2) - \mu h x_2 - \nu\phi h x_2, \xi) \\
& +\sigma(\phi x_4, \kappa x_2 + \mu h x_2 + \nu\phi h x_2) \\
& -\sigma(\phi h x_4, \kappa x_2 + \mu h x_2 + \nu\phi h x_2), \qquad (53)
\end{aligned}$$

where, by direct calculations, one can easily see that

$$\begin{aligned}
& (\widetilde{\nabla}_{x_4}\sigma)(\kappa[\eta(x_2)\xi - x_2] - \mu h x_2 - \nu\phi h x_2, \xi) \\
= \;& -\sigma(\nabla_{x_4}\xi, \kappa[\eta(x_2)\xi - x_2] - \mu h x_2 - \nu\phi h x_2) \\
= \;& \sigma(\phi x_4 - \phi h x_4, \kappa[\eta(x_2)\xi - x_2] - \mu h x_2 - \nu\phi h x_2) \\
= \;& -\kappa\sigma(\phi x_4, x_2) - \mu\sigma(\phi x_4, h x_2) - \nu\phi\sigma(\phi x_4, h x_2) \\
& +\kappa\sigma(\phi h x_4, x_2) + \mu\sigma(\phi h x_4, h x_2) \\
& -\nu\phi\sigma(\phi h x_4, h x_2) \qquad (54)
\end{aligned}$$

and we get

$$\begin{aligned}
(\widetilde{\nabla}_{x_4}\sigma)(\xi, x_2) &= -\sigma(\nabla_{x_4}\xi, x_2) = \sigma(\phi x_4 - \phi h x_4, x_2) \\
&= \sigma(\phi x_4, x_2) - \sigma(\phi h x_4, x_2). \qquad (55)
\end{aligned}$$

If (45)-(55) are put in (53), we obtain

$$\begin{aligned}
& [\phi(L_2 - \kappa) + (1+\kappa)(\mu\phi + \nu)]\sigma(x_4, x_2) \\
& +[\phi(L_2 + \mu - \kappa) + \nu]\sigma(x_4, h x_2) = 0. \qquad (56)
\end{aligned}$$

In (56), substituting hx_2 instead of x_2 and by virtue of (9), we reach at

$$\begin{aligned}
& [\phi(L_2 - \kappa) + (1+\kappa)(\mu\phi + \nu)]\sigma(x_4, h x_2) \\
& +(1+\kappa)[\phi(L_2 + \mu - \kappa) + \nu]\sigma(x_4, x_2) = 0. \qquad (57)
\end{aligned}$$

From (56) and (57), provided $\kappa \neq 0$, we can infer

$$\kappa[(1+\kappa)(\mu^2 + \nu^2) - (L_2 - \kappa)^2]\sigma(x_4, x_2) - 2\mu\nu(1+\kappa)\phi\sigma(x_4, x_2) = 0.$$

This implies that M is either totally geodesic or (42) is satisfied. So, the proof is completed. □

Corollary 3.7. *Let M be an invariant submanifold of a paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. M is a 2-pseudoparallel submanifold if and only if M is totally geodesic provided*

$$\kappa^2+(\kappa+1)(\mu^2+\nu^2)\neq 0 \quad or \quad (\kappa+1)\mu\nu\neq 0.$$

Theorem 3.8. *Let M be an invariant submanifold of a paracontact (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. If M is a 2-Ricci-generalized pseudoparallel submanifold of $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then there M is either totally geodesic or the function L_4 satisfies*

$$L_4=\frac{1}{2n}\pm\frac{\sqrt{(\mu^2+\nu^2)(\kappa+1)}}{2nk},\quad (\kappa+1)\mu\nu=0. \tag{58}$$

Proof. Let M be an invariant 2-Ricci-generalized pseudoparallel submanifold of a paracontact $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)-$space. Then there exists a function L_4 such that

$$L_4Q(S,\widetilde{\nabla}\sigma)(x_4,x_5,x_3;x_1,x_2)=(\widetilde{R}(x_1,x_2)\cdot\widetilde{\nabla}\sigma)(x_4,x_5,x_3),$$

for all $x_1,x_2,x_4,x_5,x_3\in\Gamma(TM)$, that is,

$$\begin{aligned}&-L_4\{(\widetilde{\nabla}_{(x_1\wedge_S x_2)x_4}\sigma)(x_5,x_3)+(\widetilde{\nabla}_{x_4}\sigma)((x_1\wedge_S x_2)x_5,x_3)\\&+(\widetilde{\nabla}_{x_4}\sigma)(x_5,(x_1\wedge_S x_2)x_3)\}\\=\;&R^{\perp}(x_1,x_2)(\widetilde{\nabla}_{x_4}\sigma)(x_5,x_3)-(\widetilde{\nabla}_{R(x_1,x_2)x_4}(x_5,x_3)\\&-(\widetilde{\nabla}_{x_4}\sigma)(R(x_1,x_2)x_5,x_3)-(\widetilde{\nabla}_{x_4}\sigma)(x_5,R(x_1,x_2)x_3).\end{aligned} \tag{59}$$

In (59), using $x_1=x_5=\xi$, we have

$$\begin{aligned}&-L_4\{(\widetilde{\nabla}_{(\xi\wedge_S x_2)x_4}\sigma)(\xi,x_3)+(\widetilde{\nabla}_{x_4}\sigma)((\xi\wedge_S x_2)\xi,x_3)\\&+(\widetilde{\nabla}_{x_4}\sigma)(\xi,(\xi\wedge_S x_2)x_3)\}\\=\;&R^{\perp}(\xi,x_2)(\widetilde{\nabla}_{x_4}\sigma)(\xi,x_3)-(\widetilde{\nabla}_{R(\xi,x_2)x_4}(\xi,x_3))\\&-(\widetilde{\nabla}_{x_4}\sigma)(R(\xi,x_2)\xi,x_3)-(\widetilde{\nabla}_{x_4}\sigma)(\xi,R(\xi,x_2)x_3).\end{aligned} \tag{60}$$

Now we will calculate them separately. In view of (9), (13), (18) and (28), we can derive

$$\begin{aligned}(\widetilde{\nabla}_{(\xi\wedge_S x_2)x_4}\sigma)(\xi,x_3)&=-\sigma(\nabla_{(\xi\wedge_S x_2)x_4}\xi,x_3)\\&=\sigma(\phi(\xi\wedge_S x_2)x_4-\phi h(\xi\wedge_S x_2)x_4,x_3)\\&=\sigma(\phi(S(x_2,x_4)\xi-2n\kappa\eta(x_4)x_2),x_3)\\&\quad-\sigma(\phi h(S(x_2,x_4)\xi-2n\kappa\eta(x_4)x_2),x_3)\\&=-2n\kappa\eta(x_4)\{\sigma(\phi hx_2,x_3)\\&\quad-\sigma(\phi x_2,x_3)\}.\end{aligned} \tag{61}$$

In the same way,

$$\begin{aligned}(\widetilde{\nabla}_{x_4}\sigma)((\xi\wedge_S x_2)\xi,x_3)&=(\widetilde{\nabla}_{x_4}\sigma)(2nk\eta(x_2)\xi-2nkx_2,x_3)\\&=2n\kappa\{(\widetilde{\nabla}_{x_4}\sigma)(\eta(x_2)\xi,x_3)-(\widetilde{\nabla}_{x_4}\sigma)(x_2,x_3)\}\\&=2n\kappa\{\eta(x_2)\sigma(-\phi x_4,x_3)+\eta(x_2)\sigma(\phi hx_4,x_3)\\&\quad-(\widetilde{\nabla}_{x_4}\sigma)(x_2,x_3)\},\end{aligned} \tag{62}$$

$$\begin{aligned}(\widetilde{\nabla}_{x_4}\sigma)(\xi,(\xi\wedge_S x_2)x_3)&=-\sigma(\nabla_{x_4}\xi,(\xi\wedge_S x_2)x_3)\\&=-\sigma(-\phi x_4+\phi hx_4,S(x_2,x_3)\xi-2nk\eta(x_3)x_2)\\&=2n\kappa\eta(x_3)\{\sigma(\phi hx_4,x_2)-\sigma(\phi x_4,x_2)\}.\end{aligned} \tag{63}$$

For the right side of (63), by view of (18), (20) and (14), we have

$$\begin{aligned} R^{\perp}(\xi, x_2)(\widetilde{\nabla}_{x_4}\sigma)(\xi, x_3) &= R^{\perp}(\xi, x_2)\{\nabla^{\perp}_{x_4}\sigma(\xi, x_3) - \sigma(\nabla_{x_4}\xi, x_3) \\ &\quad -\sigma(\xi, \nabla_{x_4}x_3)\} \\ &= -R^{\perp}(\xi, x_2)\sigma(\nabla_{x_4}\xi, x_3) \\ &= R^{\perp}(\xi, x_2)\{\sigma(\phi x_4 - \phi h x_4, x_3)\}. \end{aligned} \tag{64}$$

Also, making use of (3) and (8), we obtain

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,x_2)x_4}(\xi, x_3)) &= -\sigma(x_3, \nabla_{R(\xi,x_2)x_4}\xi) \\ &= -\sigma(x_3, -\phi R(\xi, x_2)x_4 + \phi h R(\xi, x_2)x_4) \\ &= -\eta(x_4)\{\kappa\sigma(\phi x_2, x_3) + \mu\sigma(\phi h x_2, x_3) \\ &\quad +\nu\sigma(h x_2, x_3) - \kappa\sigma(\phi h x_2, x_3) \\ &\quad -\mu(1+\kappa)\sigma(\phi x_2, x_3) \\ &\quad -\nu(1+\kappa)\sigma(x_2, x_3)\} \end{aligned} \tag{65}$$

and we get

$$(\widetilde{\nabla}_{x_4}\sigma)(R(\xi, x_2)\xi, x_3) = (\widetilde{\nabla}_{x_4}\sigma)\{\kappa(\eta(x_2)\xi - x_2) - \mu h x_2 - \nu\phi h x_2, x_3\}. \tag{66}$$

Finally,

$$\begin{aligned} (\widetilde{\nabla}_{x_4}\sigma)(\xi, R(\xi, x_2)x_3) &= -\sigma(\nabla_{x_4}\xi, R(\xi, x_2)x_3) = -\sigma(\phi x_4, \kappa\eta(x_3)x_2 \\ &\quad +\mu\eta(x_3)h x_2 + \nu\eta(x_3)\phi h x_2) \\ &\quad +\sigma(\phi h x_4, \kappa\eta(x_3)x_2 + \mu\eta(x_3)h x_2 + \nu\eta(x_3)\phi h x_2) \\ &= -\eta(x_3)\{\kappa\sigma(\phi x_4, x_2) + \mu\sigma(\phi x_4, h x_2) \\ &\quad +\nu\sigma(\phi x_4, \phi h x_2) - \kappa\sigma(\phi h x_4, x_2) \\ &\quad -\mu\sigma(\phi h x_4, h x_2) - \nu\sigma(\phi h x_4, \phi h x_2)\}. \end{aligned} \tag{67}$$

Consequently, statements (61)-(67) are put in (60), we arrive

$$\begin{aligned} &2n\kappa L_4\{\eta(x_4)\sigma(\phi x_2, x_3) - \eta(x_4)\sigma(\phi h x_2, x_3) \\ &+\eta(x_2)\sigma(x_3, \phi x_4) - \eta(x_2)\sigma(x_3, \phi h x_4) \\ &+(\widetilde{\nabla}_{x_4}\sigma)(x_3, x_2) + \eta(x_3)\sigma(\phi x_4, x_2) - \eta(x_3)\sigma(\phi h x_4, x_2)\} \\ = \; &R^{\perp}(\xi, x_2)\sigma(\phi x_4 - \phi h x_4, x_3) + \eta(x_4)\{\kappa\sigma(\phi x_2, x_3) \\ &+\mu\sigma(\phi h x_2, x_3) + \nu\sigma(h x_2, x_3) - \kappa\sigma(\phi h x_2, x_3) \\ &-\mu(1+\kappa)\sigma(\phi x_2, x_3) - \nu(1+\kappa)\sigma(x_2, x_3)\} \\ &-(\widetilde{\nabla}_{x_4}\sigma)\{\kappa[\eta(x_2)\xi - x_2] - \mu h x_2 - \nu\phi h x_2, x_3\} \\ &+\eta(x_3)\{\kappa\sigma(\phi x_4, x_2) + \mu\sigma(\phi x_4, h x_2) \\ &+\nu\sigma(\phi x_4, \phi h x_2) - \kappa\sigma(\phi h x_4, x_2) \\ &-\mu\sigma(\phi h x_4, h x_2) - \nu\sigma(\phi h x_4, \phi h x_2)\}. \end{aligned} \tag{68}$$

In (68), putting $x_3 = \xi$ and taking into account (13), we obtain

$$\begin{aligned} &2n\kappa L_4\{(\widetilde{\nabla}_{x_4}\sigma)(\xi, x_2) + \sigma(\phi x_4, x_2) - \sigma(\phi h x_4, x_2)\} \\ = \; &-(\widetilde{\nabla}_{x_4}\sigma)(\kappa(\eta(x_2)\xi - x_2) - \mu h x_2 - \nu\phi h x_2, \xi) \\ &+\sigma(\phi x_4, \kappa x_2 + \mu h x_2 + \nu\phi h x_2) \\ &-\sigma(\phi h x_4, \kappa x_2 + \mu h x_2 + \nu\phi h x_2), \end{aligned} \tag{69}$$

where, by direct calculations, one can easily see that

$$\begin{aligned} & (\widetilde{\nabla}_{x_4}\sigma)(\kappa[\eta(x_2)\xi - x_2] - \mu h x_2 - \nu\phi h x_2, \xi) \\ = & -\sigma(\nabla_{x_4}\xi, \kappa[\eta(x_2)\xi - x_2] - \mu h x_2 - \nu\phi h x_2) \\ = & \ \sigma(\phi x_4 - \phi h x_4, \kappa[\eta(x_2)\xi - x_2] - \mu h x_2 - \nu\phi h x_2) \\ = & -\kappa\sigma(\phi x_4, x_2) - \mu\sigma(\phi x_4, h x_2) \\ & -\nu\phi\sigma(\phi x_4, h x_2) + \kappa\sigma(\phi h x_4, x_2) \\ & +\mu\sigma(\phi h x_4, h x_2) + \nu\phi\sigma(\phi h x_4, h x_2) \end{aligned} \tag{70}$$

and

$$\begin{aligned} (\widetilde{\nabla}_{x_4}\sigma)(\xi, x_2) &= -\sigma(\nabla_{x_4}\xi, x_2) = \sigma(\phi x_4 - \phi h x_4, x_2) \\ &= \sigma(\phi x_4, x_2) - \sigma(\phi h x_4, x_2). \end{aligned} \tag{71}$$

If (70) and (71) are put in (69), we obtain

$$\begin{aligned} & [\phi(2n\kappa L_4 - \kappa + \mu(1+\kappa)) + (1+\kappa)\nu]\sigma(x_4, x_2) \\ & -[\phi(2n\kappa L_4 + \mu - \kappa) + \nu]\sigma(x_4, h x_2) = 0. \end{aligned} \tag{72}$$

In (72), substituting hx_2 instead of x_2 and by virtue of (9), we reach at

$$\begin{aligned} & [\phi(2n\kappa L_4 - \kappa + \mu(1+\kappa)) + (1+\kappa)\nu]\sigma(x_4, h x_2) \\ & -(1+\kappa)[\phi(2n\kappa L_4 + \mu - \kappa) + \nu]\sigma(x_4, x_2) = 0. \end{aligned} \tag{73}$$

From (72) and (73), provided $\kappa \neq 0$, we can infer

$$\kappa[(1+\kappa)(\mu^2 + \nu^2) - (2n\kappa L_4 - \kappa)^2]\sigma(x_4, x_2) - 2\mu\nu(1+\kappa)\phi\sigma(x_4, x_2) = 0.$$

This implies the proof is completed. □

Example 3.9. *We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3,\ z \neq 0\}$, where (x, y, z) are standart coordinates of $\mathbb{R}^3$. The vector fields*

$$e_1 = 2x^5\frac{\partial}{\partial x} + \frac{8}{3}z^3\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \ \ e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_2) &= g(e_1, e_3) = g(e_2, e_3) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = 1, \ \ g(e_3, e_3) = -1 \end{aligned}$$

Let η be the 1-form defined by $\eta(x_1) = g(x_1, e_2)$ for any $x_1 \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by

$$\phi(e_2) = 0, \qquad \phi(e_3) = -e_1, \qquad \phi(e_1) = -e_3.$$

Let ∇ be the Levi-Civita connection with respect to the metric tensor g. Then we get

$$[e_3, e_1] = -8z^2 e_2, \ \ [e_1, e_2] = 0, \quad [e_2, e_3] = 0.$$

Then we have

$$\eta(e_2) = g(e_2, e_2) = 1, \ \ \phi^2 x_1 = x_1 - \eta(x_1)e_1, \quad g(\phi x_1, \phi x_2) = -g(x_1, x_2) + \eta(x_1)\eta(x_2),$$

for any x_1 , $x_2 \in \chi(M)$. Hence, (ϕ, ξ, η, g) defines a paracontact metric structure on M for $e_2 = \xi$.

The Levi-Civita connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned}2g(\nabla_{x_1}x_2, x_3) &= x_1g(x_2, x_3) + x_2g(x_3, x_1) - x_3g(x_1, x_2)\\ &\quad -g(x_1, [x_2, x_3]) - g(x_2, [x_1, x_3]) + g(x_3, [x_1, x_2]).\end{aligned}$$

Using the above formula, we obtain.

$$\begin{aligned}\nabla_{e_1}e_1 &= 0, & \nabla_{e_2}e_1 &= -4z^2e_3, & \nabla_{e_3}e_1 &= 4z^2e_2,\\ \nabla_{e_1}e_2 &= -4z^2e_3, & \nabla_{e_2}e_2 &= 0, & \nabla_{e_3}e_2 &= -4z^2e_1,\\ \nabla_{e_1}e_3 &= -4z^2e_2, & \nabla_{e_2}e_3 &= -4z^2e_1, & \nabla_{e_3}e_3 &= 0.\end{aligned}$$

Comparing the above relations with $\nabla x_1 e_2 = -\phi x_1 + \phi h x_1$, *we get*

$$he_1 = -(4z^2 + 1)e_2, \quad he_3 = -(4z^2 + 1)e_3, \quad he_2 = 0.$$

Using the formula $R(x_1, x_2)x_3 = \nabla x_1 \nabla x_2 x_3 - \nabla x_2 \nabla x_1 x_3 - \nabla_{[x_1,x_2]}x_3$, *we calculate the following:*

$$\begin{aligned}R(e_2, e_1)e_2 &= \nabla_{e_2}\nabla_{e_1}e_2 - \nabla_{e_1}\nabla_{e_2}e_2 - \nabla_{[e_2,e_1]}e_2\\ &= \nabla_{e_2}(-4z^2e_3) = 16z^4e_1\end{aligned}$$

$$\begin{aligned}R(e_2, e_3)e_2 &= \nabla_{e_2}\nabla_{e_3}e_2 - \nabla_{e_3}\nabla_{e_2}e_2 - \nabla_{[e_2,e_3]}e_2\\ &= \nabla_{e_2}(-4z^2e_1) = 16z^4e_3\end{aligned}$$

$$\begin{aligned}R(e_1, e_3)e_2 &= \nabla_{e_1}\nabla_{e_3}e_2 - \nabla_{e_3}\nabla_{e_1}e_2 - \nabla_{[e_1,e_3]}e_2\\ &= 0.\end{aligned}$$

By direct calculations, we get

$$\begin{aligned}R(e_2, e_1)e_2 &= \left[(4z^2 + 1)^2 - 1\right]\{\eta(e_1)e_2 - \eta(e_2)e_1\} + 8z^2\{\eta(e_1)he_2 - \eta(e_2)he_1\}\\ &\quad +0\{\eta(e_1)\phi he_2 - \eta(e_2)\phi he_1\}\\ &= 16z^4e_1\end{aligned}$$

$$\begin{aligned}R(e_2, e_3)e_2 &= \left[(4z^2 + 1)^2 - 1\right]\{\eta(e_3)e_2 - \eta(e_2)e_3\} + 8z^2\{\eta(e_3)he_2 - \eta(e_2)he_3\}\\ &\quad +0\{\eta(e_3)\phi he_2 - \eta(e_2)\phi he_3\}\\ &= 16z^4e_3\end{aligned}$$

$$\begin{aligned}R(e_1, e_3)e_2 &= \left[(4z^2 + 1)^2 - 1\right]\{\eta(e_3)e_1 - \eta(e_1)e_3\} + 8z^2\{\eta(e_3)he_1 - \eta(e_1)he_3\}\\ &\quad +0\{\eta(e_3)\phi he_1 - \eta(e_1)\phi he_3\}\\ &= 0.\end{aligned}$$

By the above expressions of the curvature tensor and using (9), we conclude that M is a $(k, \mu, \nu)-$*paracontact metric manifold with* $\kappa = \left[(4z^2 + 1)^2 - 1\right]$, $\mu = 8z^2$ *and* $\nu = 0$.

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