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Analysis of stable currents and homology of biwarped product submanifolds in the Euclidean space

Meraj Ali Khan[a], Cenep Ozel[b], Mohd Danish Siddiqi[c], Sudhakar Kumar Chaubeyi[d], Sadia Samar Ali[e]

[a] *Department of Mathemaatics and Statistics, Imam Muhammad Ibn Saud Islamic University, Riyadh, 11566, Saudi Arabia*
[b] *Department of Mathematics, College of Science King Abdulaziz University Jeddah, Saudi Arabia*
[c] *Department of Mathematics, College of Science Jazan University Jazan,45142 Saudi Arabia*
[d] *Section of Mathematics, Department of Information Technology University of Technology and Applied Sciences P.O. box 77, Postal Code 324, Shinas, Oman.*
[e] *Department of Industrial Engineering Faculty of Engineering King Abdul-Aziz University, Jeddah Saudi Arabia .*

Abstract. This paper examines the topological features of the compact biwarped product submanifolds of a space form with vanishing constant sectional curvature. More precisely, we show that stable integral $p-$ current does not exist in a compact oriented biwarped product submanifold in an Euclidean space that meets some geometric conditions based on Laplacian of warping functions, slant functions. Simultaneously, it is shown that their homology group are zero under these geometric conditions. Additionally, some special cases are also described.

1. Introduction

Federer-Fleming was the first to establish the concept of integral currents [14]. By connecting the geometric structure of differentiable manifolds and homological groups with integral coefficients, the concept of an integral current plays a significant role in presenting the topological informations. The criterion for the non-existence of stable currents for the submanifolds in the sphere S^n by using second fundamental form was established by Lawson-Simons [15] in the year 1970.

Alternatively, in [18], non-existence of stable integral currents as well as the vanishing of Homology were found in a contact CR-warped product submanifold in an odd dimensional sphere, based on Lawson-Simon result [15], the authors concluded that the homology groups were trivial and that there were no stable currents in a contact CR-warped product submanifold immersed in a sphere of odd dimension. Further, F. Sahin [20, 21] showed that the CR-warped product submanifold in Euclidean spaces and the nearly Kaehler six sphere S^6 have equivalent conclusions. Influenced by prior studies, Ali et al [6] adjusted the warping function and point wise slant functions for a warped product submanifold on the unit sphere with the trivial homology groups on the point wise slant fiber. Fu and Xu [13] explored some topological properties for the

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Email addresses: mskhan@mamu.edu.sa (Meraj Ali Khan), cenap.ozel@gmail.com (Cenep Ozel), msiddiqi@jazanu.edu.sa (Mohd Danish Siddiqi), sudhakar.chaubey@shct.edu.om (Sudhakar Kumar Chaubeyi), ssaali@kau.edu.sa (Sadia Samar Ali)

submanifolds immersed in a hyperbolic space and proved the topological space theorem. Motivated by the study of Fu and Xu Ali et al [8] obtained some characterization for the non-existence of stable currents for the CR-warped product submanifolds of the complex hyperbolic space and simultaneously, vanishing of homology groups for these submanifolds were studied. Various topological sphere theorems have been extended in [8]. In [7], the authors extend the same work to Lagrangian warped product submanifolds of the six dimensional sphere. Several researchers derived various conclusions concerning topological and differentiable structures of the submanifolds by putting specific requirements on the second fundamental form ([16], [17], [9], [22]-[24]).

2. Preliminaries

Let $(\bar{\Gamma}, g)$ be an almost Hermitian manifold with an almost complex structure J. The manifold $(\bar{\Gamma})$ is called Kaehler manifold if the almost complex structure J is parallel with respect to *Levi − Civita* connection $\bar{\nabla}$ on Γ, i.e.,

$$(\bar{\nabla}_{e_1} J)e_2 = 0, \tag{1}$$

for all $e_1, e_2 \in \bar{\Gamma}\bar{U}$, then $(\bar{\Gamma}J, g)$ is called a *Kaehler manifold*.

Let Γ be an $n-$dimensional Riemannian manifold isometrically immersed in a $m-$dimensional Riemannian manifold $\bar{\Gamma}$. Then the Gauss and Weingarten formulas are

$$\bar{\nabla}_{e_1} e_2 = \nabla_{e_1} e_2 + \sigma(e_1, e_2)$$

and

$$\bar{\nabla}_{e_1} \xi = -A_\xi e_1 + \nabla^{\perp}_{e_1} \xi,$$

respectively, for all $e_1, e_2 \in T\Gamma$ and $\xi \in T^{\perp}\Gamma$. Where ∇ is the induced Levi-Civita connection on Γ, ξ is a vector field normal to Γ, σ is the second fundamental form of Γ, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}\Gamma$ and A_ξ is the shape operator of the second fundamental form. The second fundamental form h and the shape operator are associated by the following formula

$$g(\sigma(e_1, e_2), \xi) = g(A_\xi e_1, e_2). \tag{2}$$

The equation of Gauss is given by

$$R(e_1, e_2, e_3, e_4) = \bar{R}(e_1, e_2, e_3, e_4) + g(\sigma(e_1, e_4), \sigma(e_2, e_3)) - g(\sigma(e_1, e_3), \sigma(e_2, e_4)), \tag{3}$$

for all $e_1, e_2, e_3, e_4 \in T\Gamma$. Where, $\bar{R}$ and R are the curvature tensors of $\bar{\Gamma}$ and Γ respectively.

For any $e \in T\Gamma$ and $N \in T^{\perp}\Gamma$, Je and JN can be decomposed as follows

$$Je = Pe + Fe \tag{4}$$

and

$$JN = tN + fN, \tag{5}$$

where Pe (resp. tN) is the tangential and Fe (resp. fN) is the normal component of Je (resp. JN).

The submanifold Γ of an almost Hermitian manifold $\bar{\Gamma}$ is called a pointwise slant submanifold if at each point $x \in \Gamma$, the wirtinger angle $\theta(e_1)$ between Je_1 and $T_x\Gamma$ is independent of the choice of the nonzero vector $e_1 \in T_x\Gamma$. In this case, the angle θ is treated as a function on Γ, which is called the slant function of the submanifold, and the submanifold is called the pointwise slant submanifold. If the slant function $\theta(X)$ is constant on Γ, then Γ is a slant submanifold [12]. Now, we have following characterization for the pointwise slant submanfold.

The submanifold Γ is pointwise slant submanifold if and only if the endomorphism T satisfies

$$T^2 = -\lambda I \tag{6}$$

for $\lambda \in [0, 1]$ such that, $\lambda = \cos^2\theta$. From (4) and (6), one can conclude

$$g(Te_1, Te_2) = \cos^2\theta g(e_1, e_2) \tag{7}$$

$$g(Fe_1, Fe_2) = \sin^2\theta g(e_1, e_2) \tag{8}$$

for any $e_1, e_2 \in T\Gamma$.

If $\theta(e_1) = 0$ on Γ globally, then the pointwise slant submanifold becomes invariant submanifold. Similarly, if $\theta(e_1) = \frac{\pi}{2}$ on Γ globally, then the pointwise slant submanifold becomes anti-invariant or totally real submanifold.

B. Y. Chen and F. Dillen [1] extended the concept of a warped product submanifold to multiple warped product manifolds in the following way.

Let $\{\Gamma_i\}$, $i = 1, 2, \ldots, k$ be Riemannian manifolds with respective Riemannian metrics $\{g_i\}_{i=1,2,\ldots,k}$ and $\{\lambda_i\}_{i=2,3,\ldots,k}$ are positive valued functions on Γ_1. Then there is the product manifold $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$ which is equipped with the Riemannian metric g

$$g = \phi_1^*(g_1) + \sum_{i=2}^{k} (\lambda_i \circ \phi_1)^2 \phi_i^*(g_i)$$

is called multiply warped product manifold and denoted by $\Gamma = \Gamma_1 \times_{\lambda_2} \Gamma_2 \times \cdots \times_{\lambda_k} \Gamma_k$ where $\phi_i (i = 1, 2, \ldots, k)$ are the projection maps of Γ onto Γ_i respectively. The functions λ_i are called the warping functions [1]. If the warping functions are constants, the warped product is simply Riemannian product of manifolds. As a special case of multiply warped product manifolds, Biwarped product manifolds can be defined, for $i = 3$. For $i = 2$, multiply warped product manifold becomes the single warped product manifold. Let $\Gamma = \Gamma_0 \times_{\lambda_1} \Gamma_1 \times_{\lambda_2} \Gamma_2$ be a biwarped product submanifold with the Levi-Civita connection of Γ_i for $i = 0, 1, 2$. For biwarped product submanifolds, we now have the following result.

Lemma 2.1. [2] *Let $\Gamma = \Gamma_0 \times_{\lambda_1} \Gamma_1 \times_{\lambda_2} \Gamma_2$ be a biwarped product manifold. Then we have*

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = e_1(ln\lambda_i) e_2 \tag{9}$$

for $e_1 \in T\Gamma_0$ and $e_2 \in T\Gamma_i$, for $i = 1, 2$.

For a smooth function λ on a Riemannian manifold Γ with Riemannian metric g, the gradient of λ is denoted by $\nabla\lambda$ and is defined as

$$g(\nabla\lambda, e) = e\lambda, \tag{10}$$

for all $e \in T\Gamma$.

From the Proposition 2.4 of [10], for the biwarped product submanifold $\Gamma_T \times_{\lambda_1} \Gamma_\theta \times_{\lambda_2} \Gamma_\perp$, we can conclude the following relations

$$R(e_1, e_3)e_2 = \frac{H^{\lambda_1}(e_1, e_2)}{\lambda_1} e_3, \tag{11}$$

$$R(e_1, e_4)e_2 = \frac{H^{\lambda_2}(e_1, e_2)}{\lambda_2} e_4, \tag{12}$$

where $e_1, e_2 \in T\Gamma_T$, $e_3 \in T\Gamma_\theta$, $e_4 \in T\Gamma_\perp$, and H^{λ_i} is the Hessian of the function λ_i, $i = 1, 2$.

For the Laplacian $\Delta(ln\lambda_i)$ of the warping functions λ_i, $i = 1, 2$, we have

$$\Delta(ln\lambda_i) = -div(\frac{\nabla\lambda_i}{\lambda_i}) = -g(\nabla\frac{1}{\lambda_i}, \nabla\lambda_i) - \lambda_i div(\nabla\lambda_i) = \|\nabla ln\lambda_i\|^2 + \frac{\Delta\lambda_i}{\lambda_i}. \tag{13}$$

From above equation, we deduce

$$\frac{\Delta ln\lambda_i}{\lambda_i} = \Delta ln\lambda_i - \|\nabla ln\lambda_i\|^2 \tag{14}$$

3. Main Results

We need the following information from Lawson and Simons [15] in order to establish our main results

Lemma 3.1. [15, 22] *For the second fundamental form σ and any positive integers p, q with $p+q = n$, if the inequality*

$$\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}(2\|\sigma(u_\alpha, u_\beta)\|^2 - g(\sigma(u_\alpha, u_\alpha), \sigma(u_\beta, u_\beta))) < pqc$$

is satisfied for an $n-$dimensional compact submanifold Γ^n in a space form $\bar{\Gamma}(c)$ of constant curvature $c \geq 0$, then there is no stable $p-$current i Γ^n and $H_p(\Gamma^n, Z) = H_q(\Gamma^n, Z) = 0$, where $H_\alpha(\Gamma^n, Z)$ is the $\alpha - th$ homology group of Γ^n with integer coefficients and $\{e_\alpha\}_{1\leq\alpha\leq}$ is the orthonormal basis of M^n.

The study of biwarped submanifolds in the Kaehler manifolds has been done by H. M. Tastan [3] which was followed by M. A. Khan and K. Khan [4]. Basically, M. A. Khan and K. Khan looked on biwarped product submanifolds of the type $\Gamma = \Gamma_T \times_{\lambda_1} \Gamma_\perp \times_{\lambda_2} \Gamma_\theta$ in the frame of complex space forms, where Γ_T, $\Gamma_\perp$ and Γ_θ are the invariant, totally real and pointwise slant submanifolds respectively. Across this paper we consider $n-$dimensional biwarped product submanifold $\Gamma^n = \Gamma_T^p \times_{\lambda_2} \Gamma_\perp^t \times_{\lambda_3} \Gamma_\theta^s$ of a complex space form, where p, t, s are the dimensions of the invariant, totally real and pointwise slant submanifolds. If $\Gamma_\theta^s = \{0\}$ then the biwarped product submanifold becomes the CR-warped product submanifold. Similarly, if $\Gamma_\perp^{k_2} = \{0\}$ then the biwarped product submanifold reduces to pointwise semi-slant warped product submanifold.

Now, we have some initial results

Lemma 3.2. [3] *Let $\Gamma = \Gamma_T \times_{\lambda_1} \Gamma_\perp \times_{\lambda_2} \Gamma_\theta$ be a nontrivial biwarped product submanifold of a Kaehler manifold $(\bar{\Gamma}, J, g)$, then we have*

(i) $g(\sigma(e_1, e_2), Je_3) = 0$,
(ii) $g(\sigma(e_2, e_3), Je_3) = -Je_2(ln\lambda_1)\|e_3\|^2$,
(iii) $g(\sigma(e_2, e_4), Je_3) = 0$,

for any $e_1, e_2 \in D^T$, $e_3 \in D^\perp$, and $e_4 \in D_\theta$.

Lemma 3.3. [3] *Let $\Gamma = \Gamma_T \times_{\lambda_1} \Gamma_\perp \times_{\lambda_2} \Gamma_\theta$ be a nontrivial biwarped product submanifold of a Kaehler manifold $(\bar{\Gamma}, J, g)$, then we have*

(i) $g(\sigma(e_1, e_2), Fe_4) = 0$,

(ii) $g(\sigma(e_2, e_3), Fe_4) = 0$,

(iii) $g(\sigma(e_2, e_4), Fe_4) = -Je_2 ln\lambda_2 \|e_4\|^2$,

for any $e_1, e_2 \in D^T$, $e_3 \in D^\perp$, *and* $e_4 \in D_\theta$.

It is well known that the even dimensional Euclidean space with zero constant sectional curvature is a complex space form. As a result, given a biwarped product submanifold $\Gamma = \Gamma_T \times_{\lambda_1} \Gamma_\perp \times_{\lambda_2} \Gamma_\theta$ in a flat space or the Euclidean space, we get the following nonexistence stable integral p-currents theorem.

Theorem 3.4. *Let* $\Gamma^n = \Gamma_T^p \times_{\lambda_1} \Gamma_\perp^t \times_{\lambda_2} \Gamma_\theta^s$ *be a compact orientable biwarped product submanifold in the Euclidean space* $R^{p+2t+2s}$ *with* $n = p + q$. *If the following condition holds*

$$t\Delta ln\lambda_1 + q\Delta ln\lambda_2 > t(2-t)\|\nabla ln\lambda_1\|^2 + s(1 - s + \csc^2\theta + \cot^2\theta)\|\nabla ln\lambda_2\|^2 - \\ - qg(\nabla ln\lambda_1, \nabla ln\lambda_2). \tag{15}$$

Then there does not exist integral $p-$*current in* Γ^n *and* $H_p(\Gamma^n, Z) = H_q(\Gamma^n, Z) = 0$, *where* $H_i(\Gamma^n, Z)$ *is the ith homology group of* Γ^n *with integer coefficient, where p, t and s are the dimensions of the submanifolds* Γ_T^p, $\Gamma_\perp^t$ *and* Γ_θ^s *respectively, with* $q = t + s$.

Proof. Let $dim(\Gamma_T^p) = p = 2\alpha$, $dim(\Gamma_\perp^t) = t$, and $dim(\Gamma_\theta^s) = s = 2\beta$, where N_T, $N_\perp$ and N_θ are the integral manifolds of the distributions D_T, $D_\perp$ and D_θ, respectively. Consider $\{u_1, u_2, \ldots, u_\alpha, u_{\alpha+1} = Ju_1, \ldots, u_{2\alpha} = Ju_\alpha\}$, $\{u_{2\alpha+1} = \hat{u}_1, \ldots, u_{2\alpha+t} = \hat{u}_t\}$, and $\{u_{2\alpha+t+1} = u_1^*, \ldots, u_{2\alpha+t+\beta} = u_\beta^*, u_{2\alpha+t+\beta+1} = u_{\beta+1}^* = \sec\theta Pu_1^*, \ldots, u_{2\alpha+t+2\beta} = u_{(q=2\beta)}^* = sec\theta Pu_\beta^*\}$ be the orthonormal frames of $T\Gamma_\perp^t$, $T\Gamma_\perp^t$ and $T\Gamma_\theta^s$, respectively. Therefore, the orthonormal basis of the normal subbundle $JD^\perp$ and FD^θ are $\{u_{n+1} = \tilde{e}_1 = J\hat{e}_1, \ldots, u_{n+t} = \tilde{e}_t = J\hat{e}_t\}$ and $\{e_{t+1} = \bar{e}_1 = \csc\theta Fu_1^*, \ldots, e_{t+\beta} = \bar{e}_\beta = \csc\theta Fe_1^*, e_{t+\beta+1} = \bar{e}_{\beta+1} = \csc\theta\sec\theta FTu_1^*, \ldots, e_{t+2\beta} = \bar{e}_{2\beta} = \csc\theta\sec\theta FTu_\beta^*\}$ respectively. Thus, we have

$$\begin{aligned}
\sum_{i=1}^{p}\sum_{j=1}^{n}\{2\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\} &= \sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(\sigma_{ij}^r)^2 + \\
&+ \sum_{i=1}^{p}\sum_{j=p+1}^{n}\{\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\} \\
&= \sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=p+1}^{t}(\sigma_{ij}^r)^2 + \sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=t+1}^{n}(\sigma_{ij}^r)^2 + \\
&+ \sum_{i=1}^{p}\sum_{j=p+1}^{t}\{\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\} \\
&+ \sum_{i=1}^{p}\sum_{j=t+1}^{s}\{\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\}.
\end{aligned}$$

Then by the Gauss equation (3) for the Euclidean space $R^{p+2t+2q}$, we get

$$\begin{aligned}
\sum_{i=1}^{p}\sum_{j=1}^{n}\{2\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\} &= \sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(\sigma_{ij}^r)^2 + \\
&+ \sum_{i=1}^{p}\sum_{j=p+1}^{n}\{\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\} \\
&= \sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=p+1}^{t}(\sigma_{ij}^r)^2 + \sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=t+1}^{n}(\sigma_{ij}^r)^2 + \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{t} g(R(u_i,u_j)u_i,u_j) + \sum_{i=1}^{p}\sum_{j=t+1}^{s} g(R(u_i,u_j)u_i,u_j).
\end{aligned} \tag{16}$$

We can conclude the following relations from (11) and (12)

$$\sum_{i=1}^{p}\sum_{j=1}^{t} g(R(u_i,u_j)u_i,u_j) = \frac{t}{\lambda_1}\sum_{i=1}^{p} g(\nabla_{e_i}\nabla\lambda_1,u_i), \tag{17}$$

and

$$\sum_{i=1}^{p}\sum_{j=1}^{s} g(R(u_i,u_j)u_i,u_j) = \frac{s}{\lambda_2}\sum_{i=1}^{p} g(\nabla_{u_i}\nabla\lambda_2,u_i). \tag{18}$$

Combining (16), (17), and (18), we get

$$\begin{aligned}
\sum_{i=1}^{p}\sum_{j=1}^{n}\{2\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\} &= \frac{t}{\lambda_1}\sum_{i=1}^{p} g(\nabla_{u_i}\nabla\lambda_1,u_i) \\
&+ \frac{s}{\lambda_2}\sum_{i=1}^{p} g(\nabla_{u_i}\nabla\lambda_2,u_i) + \sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=p+1}^{t}(\sigma_{ij}^r)^2 \\
&+ \sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=t+1}^{n}(\sigma_{ij}^r)^2.
\end{aligned} \tag{19}$$

First we compute the terms $\Delta\lambda_1$ and $\Delta\lambda_2$, these are the Laplacian of the warping functions λ_1 and λ_2

$$\begin{aligned}
\Delta\lambda_1 = -\sum_{k=1}^{n} g(\nabla_{u_k}(\nabla\lambda_1),u_k) &= -\sum_{i=1}^{p} g(\nabla_{u_i}(\nabla\lambda_1),u_i) - \sum_{j=1}^{t} g(\nabla_{\hat{u}_j}(\nabla\lambda_1),\hat{u}_j) \\
&- \sum_{r=1}^{s} g(\nabla_{u_r^*}(\nabla\lambda_1),u_r^*) \\
&= -\sum_{i=1}^{p} g(\nabla_{u_i}(\nabla\lambda_1),u_i) - \frac{1}{\lambda_1}\sum_{j=1}^{t} g(\hat{u}_j,\hat{u}_j)\|\nabla\lambda_1\|^2 \\
&- \sum_{r=1}^{\beta} g(\nabla_{u_r^*}(\nabla\lambda_1),u_r^*) - \sec^2\theta\sum_{r=1}^{\beta} g(\nabla_{Pu_r^*}\nabla\lambda_1,Pu_r^*).
\end{aligned} \tag{20}$$

On using equation (4), the above relation reduces to

$$\Delta\lambda_1 = -\sum_{i=1}^{p} g(\nabla_{u_i}(\nabla\lambda_1), u_i) - \frac{t}{\lambda_1}\|\nabla\lambda_1\|^2 - \frac{s}{\lambda_2} g(\nabla\lambda_1, \nabla\lambda_2). \tag{21}$$

Similarly, we can calculate

$$\Delta\lambda_2 = -\sum_{i=1}^{p} g(\nabla_{u_i}(\nabla\lambda_2), u_i) - \frac{t}{\lambda_1} g(\nabla\lambda_1, \nabla\lambda_2) - \frac{s}{\lambda_2}\|\nabla\lambda_2\|^2. \tag{22}$$

Multiplying (21) by $\frac{1}{\lambda_1}$, we get

$$\frac{\Delta\lambda_1}{\lambda_1} = -\frac{1}{\lambda_1}\sum_{i=1}^{p} g(\nabla_{u_i}(\nabla\lambda_1), u_i) - t\|\nabla ln\lambda_1\|^2 - sg(\nabla ln\lambda_1, \nabla ln\lambda_2), \tag{23}$$

using the equation (14), the above equation turn up

$$\Delta ln\lambda_1 - \|\nabla ln\lambda_1\|^2 = -\frac{1}{\lambda_1}\sum_{i=1}^{p} g(\nabla_{u_i}(\nabla\lambda_1), u_i) - t\|\nabla ln\lambda_1\|^2 - sg(\nabla ln\lambda_1, \nabla ln\lambda_2), \tag{24}$$

rearranging the terms, we get

$$\frac{1}{\lambda_1}\sum_{i=1}^{p} g(\nabla_{u_i}(\nabla\lambda_1), u_i) = -\Delta ln\lambda_1 + (1-t)\|\nabla ln\lambda_1\|^2 - sg(\nabla ln\lambda_1, \nabla ln\lambda_2). \tag{25}$$

In a similar fashion, we derive

$$\frac{1}{\lambda_2}\sum_{i=1}^{p} g(\nabla_{u_i}(\nabla\lambda_1), u_i) = -\Delta ln\lambda_2 + (1-s)\|\nabla ln\lambda_2\|^2 - tg(\nabla ln\lambda_1, \nabla ln\lambda_2). \tag{26}$$

On the other hand

$$\begin{aligned}\sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(\sigma_{ij}^r)^2 &= \sum_{r=1}^{2\beta}\sum_{i=1}^{p}\sum_{j=1}^{s} g(\sigma(u_i, u_j^*), \bar{u}_r)^2 + \sum_{r=1}^{t}\sum_{i=1}^{p}\sum_{j=1}^{t} g(\sigma(u_i, \hat{u}_j), \tilde{u}_r)^2 \\ &= \sum_{i=1}^{p}\sum_{j,r=1}^{\beta}\{g(\sigma(u_i, u_j^*), \csc\theta F u_r^*)^2 + g(\sigma(u_i, u_j^*), \csc\theta\sec\theta FPu_r^*)^2\} \\ &+ \sum_{i=1}^{p}\sum_{j,r=1}^{t} g(\sigma(u_i, \hat{u}_j), J\hat{u}_r)^2.\end{aligned} \tag{27}$$

In view of Lemma 3.2 and 3.3, the above equation gives

$$\sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(\sigma_{ij}^{r})^2 = 2(\csc^2\theta+\cot^2\theta)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(u_i ln\lambda_2)^2 g(u_j^*,u_j^*)^2 + 2(\csc^2\theta+\cot^2\theta)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(Ju_i ln\lambda_2)^2 g(u_j^*,u_j^*)^2 + \sum_{i=1}^{p}\sum_{j=1}^{t}(Ju_i ln\lambda_1)^2 g(\hat{u}_j,\hat{u}_j)^2. \tag{28}$$

After some routine computations, we arrive

$$\sum_{r=n+1}^{p+2t+2s}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(\sigma_{ij}^{r})^2 = t\|\nabla ln\lambda_1\|^2 + s(\csc^2\theta+\cot^2\theta)\|\nabla ln\lambda_2\|^2. \tag{29}$$

Putting the values from equations (25), (26), and (29) in (19), we get

$$\sum_{i=1}^{p}\sum_{j=1}^{n}\{2\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\} = -t\Delta ln\lambda_1 - s\Delta ln\lambda_2 + t(1-t)\|\nabla ln\lambda_1\|^2 + s(1-s)\|\nabla ln\lambda_2\|^2 - qg(\nabla ln\lambda_1,\nabla ln\lambda_2) + t\|\nabla ln\lambda_1\|^2 + s(\csc^2\theta+\cot^2\theta)\|\nabla ln\lambda_2\|^2. \tag{30}$$

If the equation (15) in Theorem 3.4 satisfies, then from above equation, we get

$$\sum_{i=1}^{p}\sum_{j=1}^{n}\{2\|\sigma(u_i,u_j)\|^2 - g(\sigma(u_i,u_i),\sigma(u_j,u_j))\} < 0. \tag{31}$$

The final conclusion of our theorem is obtained by applying Lemma 3.1 to the Euclidean space. □

Remark 3.5. *If $\Gamma_\theta^s = \{0\}$, then the biwarped product submanifold $\Gamma^n = \Gamma_T^p \times_{\lambda_1} \Gamma_\perp^t \times_{\lambda_2} \Gamma_\theta^s$ becomes the CR-warped product submanifold the type $\Gamma^n = \Gamma_T^p \times_{\lambda_1} \Gamma_\perp^t$ for more details one can see [11]. Moreover, if $\Gamma_\perp^t = \{0\}$, then the biwarped product submanifold $\Gamma^n = \Gamma_T^p \times_{\lambda_1} \Gamma_\perp^t \times_{\lambda_2} \Gamma_\theta^s$ becomes the pointwise semi-slant warped product submanifold $\Gamma^n = \Gamma_T^p \times_{\lambda_2} \Gamma_\theta^s$, as defined in [19].*

In the light of above arguments, we have the following results

Corollary 3.6. *Let $\Gamma^n = \Gamma_T^p \times_{\lambda_1} \Gamma_\perp^t$ be a compact orientable CR-warped product submanifold in the Euclidean space R^{p+2t} with $n = p + t$. If the following condition holds*

$$t\Delta ln\lambda_1 > t(2-t)\|\nabla ln\lambda_1\|^2. \tag{32}$$

Then there does not exist integral $p-$current in Γ^n and $H_p(\Gamma^n, Z) = H_q(\Gamma^n, Z) = 0$, where $H_i(\Gamma^n, Z)$ is the ith homology group of Γ^n with integer coefficient, where p and t are the dimensions of the submanifolds Γ_T^p and $\Gamma_\perp^t$ respectively.

Corollary 3.7. (Theorem 3.3 [5]) *Let $\Gamma^n = \Gamma_T^p \times_{\lambda_1} \Gamma_\theta^s$ be a compact orientable pointwise semi-slant warped product submanifold in the Euclidean space R^{p+2s} with $n = p + s$. If the following condition holds*

$$s\Delta ln\lambda_2 > s(1-s+\csc^2\theta+\cot^2\theta)\|\nabla ln\lambda_2\|^2. \tag{33}$$

Then there does not exist integral $p-$current in Γ^n and $H_p(\Gamma^n, Z) = H_q(\Gamma^n, Z) = 0$, where $H_i(\Gamma^n, Z)$ is the ith homology group of Γ^n with integer coefficient, where p and s are the dimensions of the submanifolds Γ_T^p and Γ_θ^s respectively.

4. Conclusion

In the context of Kaehler manifolds, there exist two well known classes of warped product submanifolds: CR-warped product submanifolds and pointwise semi-slant warped product submanifolds. These classes are quite different from each other and it has always been investigated to explore that how far the homology of CR-warped product submanifolds differ or resemble with that of pointwise semi-slant warped product submanifolds. The frame of biwarped product submanifolds in a way unifies the two classes of these warped products. By studying the homology of biwarped product submanifolds in the setting of Kaehler manifold, one clearly finds out the deviations in the geometric behavior of the homology of CR-warped product submanifolds and the pointwise semi-slant warped product submanifolds in the setting of Kaehler manifold. With this motivation, in the present paper we study the homology of biwarped product submanifolds in the setting of Kaehler manifolds. The present study investigates the geometric homology of three well known classes of warped product submanifolds, that is, biwarped products, CR-warped products, and the pointwise semi-slant warped products.

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