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# Global existence for nonlinear diffusion with the conformable operator using Banach fixed point theorem

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**Abstract.** In this work, we are interested in a fractional diffusion equation with a conformable derivative that contains the time dependent coefficients which occurs in many application models. By using some given assumptions, we consider the global solution to the problem. Moreover, the convergence of the mild solution when fractional order tends to $1^-$ is presented. This research can be considered as one of the first results on the topic related to conformable problem with time-dependent coefficients. In the simple case of coefficient, we show the global regularity for the mild solution in $L^p$ space. The main techniques of this work are to use Banach fixed point theorem, $L^p - L^q$ heat semigroup and some complex evaluations and techniques.

## 1. Introduction

During the past decades Fractional calculus has been studied extensively during the past decades and is now approximately 325 years old. Fractional derivatives have numerous definitions, each with a unique set of characteristics. We are aware of a number of definitions for fractional derivatives and integrals at the moment, including Riemann-Liouville, Caputo, Hadamard, Riesz, Grunwald-Letnikov, Marchaud, etc. The public is becoming more interested in certain works, including [8–10, 16–20] and the references therein. In this work, we are interested in models with conformable derivatives because of its application and urgency. The conformable derivative is understood as an extension of the classical limit definitions of derivatives of a function, which were proposed by Khalil and his colleagues [21]. The interesting thing about this derivative is that it responds to a lot of well-known properties of integer derivatives. Over the years, there has been a lot of research work on equations with conformable derivatives.

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Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. We consider the fractional diffusion equation with time dependent coefficient as follows

$$\begin{cases} D_C^\alpha y + a(t)(-\Delta)^\beta y = G(y), & (x,t) \in \Omega \times (0,T), \\ y = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = \theta(x), & x \in \Omega, \end{cases} \tag{1}$$

where the functions $a, \theta, G$ are defined later. The conformable derivative $D_C^\alpha$ which is defined in Definition 2.1. A quick observation is that if $\alpha = 1$ and $a(t) = 1, \beta = 1$ then the Problem (1) becomes the classical heat equation. Let us try to state some previous results concerning conformable diffusion equation. In [13], the authors investigated a Kirchhoff-type conformable derivative diffusion equation. They showed the global existence and uniqueness of mild solutions. Moreover, some regularity results for the mild solution and its derivatives are established. The main tool for analysis in their paper is of using Banach fixed point theory and Sobolev embeddings. In [14], the authors considered the backward problem for the nonlinear diffusion equation in the case of discrete data and multidimensional with a conformable derivative. They showed that this problem is ill-posed and then the authors establish stable approximate solutions by using two different regularization methods: the quasi-boundary value and Fourier truncated methods. In [15], the authors studied conformable stochastic differential equations. By applying the Picard iteration method, they derived the existence and uniqueness of solutions of nonlinear conformable stochastic differential equations.

According to common logic, the thermal conductivity coefficients $a$ are typically constant. However, the coefficient $a$ will frequently depend on time when the process is affected by outside influences and due to the presence of memory. For this investigation, we used model (1) for another reason. To the best of our knowledge, Problem (1) with non-constant coefficients has not yet received any attention or has received very little attention. There is not any result concerning on Problem (1). Our result is one of the foundation results for this direction.

The global existence and uniqueness of the mild solution for Problem (1) are our main goals in this work. In recent articles [1, 2], we discovered a very intriguing way to get around these obstacles.

- If $\alpha = 1$, we easily get the solution by the explicit formula when solving first differential equation $y'(t) - a(t)y(t) = 0$. However, when using the conformable derivative, tt will face some difficulties to obtain an explicit solution for the first order fractional differential equation $D_C^\alpha y(t) - a(t)y(t) = 0$. We need to employ a transformation to get around this problem so that the left side of the new equation looks like a constant coefficient.

- Mathematically, evaluating and proving global solutions is fundamentally challenging. We apply the Lemma derived from the work of Atienza [11] to get past these obstacles.

In addition, there have been numerous investigations on diffusion equations in recent years, these studies can be found in references such as [4–7, 22–27].

There are three main contributions of this study. The first contribution is to prove the existence of a global solution and evaluate the regularity of the mild solution. The second result of this paper is to investigate the mild solution when fractional order $\beta \to 1^-$. Our results can be considered a new approach considering the mild solution in $L^p$. In the last result, under the case $a = \beta = 1$, our paper also studies the regularity of the gradient of the mild solution. The main technique in our paper is to use $L^p - L^q$ estimate for the heat semigroup, for example, lemmas 2.1 and 2.2.

The structure of this article is as follows. Section 2 offers some introductory and mild solutions. The diffusion equation with a constant coefficient is the main topic of Section 2. In Section 3, we deal with the global existence of the Problem (33). The convergence result for the mild solution is given when $\beta \to 1^-$.

## 2. Main results for diffusion problem with constant coefficient

At the beginning of this section, we present some important theories to obtain results for the diffusion problem with constant coefficient.

**Definition 2.1.** *(Conformable derivative) Let us take B be a Banach space, and consider the function* $f : [0,\infty) \to B$. *Notation* ${}_{\mathrm{Co}}D_t^\alpha$ *be a conformable derivative of order* $0 < \alpha \le 1$ *which is defined by*

$$ {}_{\mathrm{Co}}D_t^\alpha f := \lim_{h\to 0} \frac{f(t+ht^{1-\alpha}) - f(t)}{h} $$

*for each* $t > 0$ *and some more knowledge about this definition in [21].*

**Lemma 2.2.** *(see [11]). For any positive constants m and r such that* $1 \le m \le r$, *there exists C is a positive constant satisfies*

$$ \left\| e^{-z\mathcal{A}} f \right\|_{L^r(\Omega)} \le C(m,r) z^{-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})} \|f\|_{L^m(\Omega)}, \ \ z > 0, f \in L^m(\Omega). \tag{2} $$

**Lemma 2.3.** *(see [11]). Let* $a_1 > -1$, $a_2 > -1$ *such that* $a_1 + a_2 \ge -1$, $\rho > 0$ *and* $t \in [0,T]$. *For* $h > 0$, *the following limit holds*

$$ \lim_{\rho\to\infty} \left( \sup_{t\in[0,T]} t^h \int_0^1 \nu^{a_1} (1-\nu)^{a_2} e^{-\rho t(1-\nu)} \mathrm{d}\nu \right) = 0. $$

**Definition 2.4.** *Let* $\mathbf{Q}_{a,m,\alpha}((0,T];B)$ *denote the weighted space of all the functions* $\psi \in C((0,T];X)$ *such that*

$$ \|\psi\|_{\mathbf{Q}_{a,m,\alpha}((0,T];B)} := \sup_{t\in(0,T]} t^a e^{-mt^\alpha} \|\psi(t,\cdot)\|_B < \infty, $$

*where* $a, m > 0$ *and* $0 < \alpha \le 1$ *(see [11]).*

**Theorem 2.5.** *In Problem* (1), *let* $a = \beta = 1$. *Let G be the function such that*

$$ \left\| Gu - Gv \right\|_{L^p(\Omega)} \le \mathbb{K} \left\| u - v \right\|_{L^q(\Omega)} \tag{3} $$

*for* $1 \le p \le q$. *Let us assume that*

$$ \frac{1}{p} - \frac{1}{q} \le \frac{1}{N} $$

*where*

$$ \frac{\alpha N}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) \le b < \min\Big(\alpha,\ \alpha - \frac{\alpha N}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big)\Big). $$

*Then Problem* (1) *has a unique solution* $y \in L^\theta(0,T;L^q(\Omega))$. *In addition, we get that*

$$ \left\| y \right\|_{L^\theta(0,T;L^q(\Omega))} \lesssim \|z_0\|_{L^q(\Omega)}, \tag{4} $$

*where* $1 < \theta < \dfrac{1}{b}$.

*Proof.* Let us define the mapping

$$ \mathbb{J}(t)\varphi = e^{-\frac{t^\alpha}{\alpha}\mathcal{A}} z_0 + \int_0^t r^{\alpha-1} e^{\frac{r^\alpha - t^\alpha}{\alpha}\mathcal{A}} G(\varphi(r)) dr. \tag{5} $$

If $\varphi = 0$, since $G(0) = 0$, we get the following equality

$$ \mathbb{J}(t)\varphi = e^{-\frac{t^\alpha}{\alpha}\mathcal{A}} z_0. $$

Since Lemma 2.2 with $z = \frac{t^\alpha}{\alpha}$, we derive that

$$\left\| e^{-\frac{t^\alpha}{\alpha}\mathcal{A}} z_0 \right\|_{L^q(\Omega)} \leq C(p,q)(\frac{t^\alpha}{\alpha})^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \|z_0\|_{L^q(\Omega)} = C(p,q)\alpha^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} t^{-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{q})} \|z_0\|_{L^q(\Omega)}. \tag{6}$$

Multiplying bothsides to $t^b e^{-mt^\alpha}$, we get

$$t^b e^{-mt^\alpha} \left\| e^{-\frac{t^\alpha}{\alpha}\mathcal{A}} z_0 \right\|_{L^q(\Omega)} \leq C(p,q)\alpha^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} t^{b-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{q})} \|z_0\|_{L^q(\Omega)}. \tag{7}$$

Since the fact that $b - \frac{\alpha N}{2}(\frac{1}{p} - \frac{1}{q}) \geq 0$, we know that

$$t^b e^{-mt^\alpha} \left\| e^{-\frac{t^\alpha}{\alpha}\mathcal{A}} z_0 \right\|_{L^q(\Omega)} \leq C(p,q)\alpha^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} T^{b-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{q})} \|z_0\|_{L^q(\Omega)}. \tag{8}$$

Therefore, we can claim that $Jw \in \mathbf{Q}_{a,m,\alpha}((0,T];L^q(\Omega))$ if $w \in \mathbf{Q}_{a,m,\alpha}((0,T];L^q(\Omega))$.
Let any $\varphi_1, \varphi_2 \in L^q(\Omega)$. Then we get that

$$\begin{aligned} \left\| \mathbb{J}(t)\varphi_1 - \mathbb{J}(t)\varphi_2 \right\|_{L^q(\Omega)} &= \left\| \int_0^t r^{\alpha-1} e^{\frac{r^\alpha - t^\alpha}{\alpha}\mathcal{A}} \left( G(\varphi_1(r)) - G(\varphi_2(r)) \right) \right\|_{L^q(\Omega)} dr \\ &\leq \int_0^t r^{\alpha-1} \left\| e^{\frac{r^\alpha - t^\alpha}{\alpha}\mathcal{A}} \left( G(\varphi_1(r)) - G(\varphi_2(r)) \right) \right\|_{L^q(\Omega)} dr. \end{aligned} \tag{9}$$

Using Lemma 2.2 with $z = \frac{t^\alpha - r^\alpha}{\alpha}$, we obtain the following bound

$$\begin{aligned} &\left\| e^{\frac{r^\alpha - t^\alpha}{\alpha}\mathcal{A}} \left( G(\varphi_1(r)) - G(\varphi_2(r)) \right) \right\|_{L^q(\Omega)} \\ &\quad \leq C(p,q) \left( \frac{t^\alpha - r^\alpha}{\alpha} \right)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \left\| G(\varphi_1(r)) - G(\varphi_2(r)) \right\|_{L^p(\Omega)} dr \\ &\quad \leq \mathbb{K}C(p,q)\alpha^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \left( t^\alpha - r^\alpha \right)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \left\| \varphi_1(r) - \varphi_2(r) \right\|_{L^q(\Omega)}. \end{aligned} \tag{10}$$

Combining (9) and (10), we derive that

$$\begin{aligned} &\left\| \mathbb{J}(t)\varphi_1 - \mathbb{J}(t)\varphi_2 \right\|_{L^q(\Omega)} \\ &\leq \mathbb{K}C(p,q)\alpha^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \int_0^t r^{\alpha-1} \left( t^\alpha - r^\alpha \right)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \left\| \varphi_1(r) - \varphi_2(r) \right\|_{L^q(\Omega)} dr. \end{aligned} \tag{11}$$

Multiplying bothsides to $t^b e^{-mt^\alpha}$, we get

$$\begin{aligned} &t^b e^{-mt^\alpha} \left\| \mathbb{J}(t)\varphi_1 - \mathbb{J}(t)\varphi_2 \right\|_{L^q(\Omega)} \\ &\leq \mathbb{K}C(p,q) t^b e^{-mt^\alpha} \int_0^t r^{\alpha-1} \left( t^\alpha - r^\alpha \right)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \left\| \varphi_1(r) - \varphi_2(r) \right\|_{L^q(\Omega)} dr \\ &= \mathbb{K}C(p,q) t^b \int_0^t r^{\alpha-1-b} \left( t^\alpha - r^\alpha \right)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} e^{-m(t^\alpha - r^\alpha)} r^b e^{-mr^\alpha} \left\| \varphi_1(r) - \varphi_2(r) \right\|_{L^q(\Omega)} dr. \end{aligned} \tag{12}$$

Noting the fact that

$$r^b e^{-mr^\alpha} \left\| \varphi_1(r) - \varphi_2(r) \right\|_{L^q(\Omega)} = \|\varphi_1 - \varphi_2\|_{\mathbf{Q}_{a,m,\alpha}((0,T];L^q(\Omega))}$$

for any $0 \le r \le T$. This implies that

$$\sup_{0\le t\le T} t^b e^{-mt^\alpha} \Big\| \mathbb{J}(t)\varphi_1 - \mathbb{J}(t)\varphi_2 \Big\|_{L^q(\Omega)} \le \left( \sup_{0\le t\le T} t^b \int_0^t r^{\alpha-1-b} \left(t^\alpha - r^\alpha\right)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-m(t^\alpha - r^\alpha)} dr \right) \Big\| \varphi_1 - \varphi_2 \Big\|_{\mathbf{Q}_{a,m,\alpha}((0,T];L^q(\Omega))}. \tag{13}$$

Let us contitute to consider the integral term on the right above. Let us change variable

$$r = ts^{\frac{1}{\alpha}}.$$

Then we get immediately that

$$t^b \int_0^t r^{\alpha-1-b} \left(t^\alpha - r^\alpha\right)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-m(t^\alpha - r^\alpha)} dr = \frac{1}{\alpha} t^{\alpha - \frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \int_0^1 s^{-\frac{b}{\alpha}} (1-s)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-mt^\alpha(1-s)} ds. \tag{14}$$

Let us need to check the condition of Lemma 2.3. Since $\frac{1}{p} - \frac{1}{q} < \frac{2}{N}$, $0 < b < \alpha$ and we can verify that

$$\alpha - \frac{\alpha N}{2}\left(\frac{1}{p} - \frac{1}{q}\right) > 0, \quad -\frac{b}{\alpha} > -1, \quad -\frac{N}{2}\left(\frac{1}{p} - \frac{1}{q}\right) > -1, \quad -\frac{b}{\alpha} - \frac{N}{2}\left(\frac{1}{p} - \frac{1}{q}\right) > -1.$$

Since the above conditions, we apply Lemma 2.3 to deduce that

$$\lim_{m\to+\infty} \left[ \sup_{0\le t\le T} t^b \int_0^t r^{\alpha-1-b} \left(t^\alpha - r^\alpha\right)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-m(t^\alpha - r^\alpha)} dr \right] = 0. \tag{15}$$

Hence, there exists a positive number $m_0$ such that

$$\sup_{0\le t\le T} t^b \int_0^t r^{\alpha-1-b} \left(t^\alpha - r^\alpha\right)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-m_0(t^\alpha - r^\alpha)} dr \le \frac{1}{2}. \tag{16}$$

Combining (13) and (16), we deduce that

$$\sup_{0\le t\le T} t^b e^{-m_0 t^\alpha} \Big\| \mathbb{J}(t)\varphi_1 - \mathbb{J}(t)\varphi_2 \Big\|_{L^q(\Omega)} \le \frac{1}{2} \Big\| \varphi_1 - \varphi_2 \Big\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))}. \tag{17}$$

Thus, one has the following inequality

$$\Big\| \mathbb{J}\varphi_1 - \mathbb{J}\varphi_2 \Big\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))} \le \frac{1}{2} \Big\| \varphi_1 - \varphi_2 \Big\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))}. \tag{18}$$

By applying Banach fixed point theory, we deduce that the mapping $\mathbb{J}$ have a fixed point $u$ in $\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))$. Hence, we can show that $u$ is a mild solution to Problem (1). Next, we claim the regularity of the mild solution $u$. It is obvious to see that $y(t) = \mathbb{J}(t)y$, so

$$\Big\| y \Big\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))} \le \Big\| \mathbb{J}y - \mathbb{J}(u=0) \Big\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))} + \Big\| \mathbb{J}(y=0) \Big\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))} \le \frac{1}{2} \Big\| y \Big\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))} + \Big\| e^{-\frac{t^\alpha}{\alpha}\mathcal{A}} z_0 \Big\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))}. \tag{19}$$

This combines with (8) yields that

$$\left\|y\right\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))} \leq 2\left\|e^{-\frac{t^\alpha}{\alpha}\mathcal{A}}z_0\right\|_{\mathbf{Q}_{a,m_0,\alpha}((0,T];L^q(\Omega))}$$
$$\leq C(p,q)\alpha^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}T^{b-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{q})}\|z_0\|_{L^q(\Omega)}. \tag{20}$$

Hence, we obtain that

$$\left\|y(t)\right\|_{L^q(\Omega)} \lesssim t^{-b}\|z_0\|_{L^q(\Omega)}. \tag{21}$$

Since $1 < \theta < \frac{1}{b}$, we follows above observation that $y \in L^\theta(0,T;L^q(\Omega))$ and

$$\left\|y\right\|_{L^\theta(0,T;L^q(\Omega))} \lesssim \|z_0\|_{L^q(\Omega)}.$$

The proof is completed. □

**Theorem 2.6.** *Let $G$ be as in Theorem 2.5. Let $z_0 \in L^q(\Omega)$. Then we get the following estimate*

$$\left\|\nabla y(t)\right\|_{L^q(\Omega)} \lesssim t^{-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}}\|z_0\|_{L^q(\Omega)} + \frac{T^{\alpha-\frac{\alpha\mu N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha\mu}{2}}}{1-\frac{\mu N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}}\|z_0\|_{L^q(\Omega)}, \tag{22}$$

*where*

$$1 < \mu < \frac{2}{N(\frac{1}{p}-\frac{1}{q})+1}.$$

*Proof.* Let us recall that

$$y(t) = e^{-\frac{t^\alpha}{\alpha}\mathcal{A}}z_0 + \int_0^t r^{\alpha-1}e^{\frac{r^\alpha-t^\alpha}{\alpha}\mathcal{A}}G(y(r))dr. \tag{23}$$

This implies that

$$\nabla y(t) = \nabla e^{-\frac{t^\alpha}{\alpha}\mathcal{A}}z_0 + \nabla\left(\int_0^t r^{\alpha-1}e^{\frac{r^\alpha-t^\alpha}{\alpha}\mathcal{A}}G(y(r))dr\right). \tag{24}$$

In view of Lemma 2.2 with $z = \frac{t^\alpha}{\alpha}$, we infer that

$$\left\|\nabla e^{-\frac{t^\alpha}{\alpha}\mathcal{A}}z_0\right\|_{L^q(\Omega)} \leq C(p,q)(\frac{t^\alpha}{\alpha})^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|z_0\|_{L^q(\Omega)}$$
$$= C(p,q)\alpha^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}t^{-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}}\|z_0\|_{L^q(\Omega)}. \tag{25}$$

By using Lemma 2.2, we have that

$$\left\|\nabla\left(\int_0^t r^{\alpha-1}e^{\frac{r^\alpha-t^\alpha}{\alpha}\mathcal{A}}G(y(r))dr\right)\right\|_{L^q(\Omega)}$$
$$\leq C(p,q)\int_0^t r^{\alpha-1}\left(\frac{t^\alpha-r^\alpha}{\alpha}\right)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\left\|G(y(r))\right\|_{L^p(\Omega)}dr. \tag{26}$$

Using global Lipschitz of $G$ and (21), we get that

$$\left\|G(y(r))\right\|_{L^p(\Omega)} \leq K\left\|y(r)\right\|_{L^q(\Omega)} \leq Kr^{-b}\|z_0\|_{L^q(\Omega)}. \tag{27}$$

Combining the two above observations, we derive that

$$\int_0^t r^{\alpha-1}\left(\frac{t^\alpha - r^\alpha}{\alpha}\right)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \left\|G(u(r))\right\|_{L^p(\Omega)} dr$$

$$\leq K\alpha^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}} \|z_0\|_{L^q(\Omega)} \left(\int_0^t r^{\alpha-1-b}(t^\alpha - r^\alpha)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} dr\right). \tag{28}$$

Let us consider the integral term on the right above. Indeed, applying Hölder inequality, we find that

$$\int_0^t r^{\alpha-1-b}(t^\alpha - r^\alpha)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} dr = \int_0^t r^{\frac{(\mu-1)(\alpha-1)}{\mu}-b} r^{\frac{\alpha-1}{\mu}} (t^\alpha - r^\alpha)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} dr$$

$$\leq \Big(\int_0^t r^{\alpha-1-\frac{b\mu}{\mu-1}} dr\Big)^{\frac{\mu-1}{\mu}} \Big(\int_0^t r^{\alpha-1}(t^\alpha - r^\alpha)^{-\frac{\mu N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}} dr\Big)^{\frac{1}{\mu}}, \tag{29}$$

for any $\mu > 1$. It is obvious to see that $\int_0^t r^{\alpha-1-\frac{b\mu}{\mu-1}} dr = \frac{t^{\alpha-\frac{b\mu}{\mu-1}}}{\alpha - \frac{b\mu}{\mu-1}}$ where we note that $b < \frac{\mu-1}{\mu}\alpha$. Set $\xi = t^\alpha - r^\alpha$. Then we have $d\xi = -\alpha r^{\alpha-1} dr$. Then

$$\int_0^t r^{\alpha-1}(t^\alpha - r^\alpha)^{-\frac{\mu N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}} dr = \frac{1}{\alpha}\int_0^{t^\alpha} \xi^{-\frac{\mu N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}} d\xi = \frac{t^{\alpha-\frac{\alpha\mu N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha\mu}{2}}}{1-\frac{\mu N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}}$$

where we note that

$$\frac{\mu N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{\mu}{2} < 1.$$

□

## 3. The time dependent problem

In this section, we consider the fractional diffusion equation with time dependent coefficient

$$\begin{cases} D_C^\alpha y + a(t)(-\Delta)^\beta y = G(y), & (x,t) \in \Omega \times (0,T), \\ y = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = \theta(x), & x \in \Omega, \end{cases} \tag{30}$$

where the functions $a, \theta, G$ are defined later.

**Theorem 3.1.** *Let G be a function such that*

$$\left\|G(u) - G(v)\right\|_{\mathbb{H}^a(\Omega)} \leq K_f \left\|u - v\right\|_{\mathbb{H}^b(\Omega)}, \tag{31}$$

*where $K_f$ is an independent constant with u and v. Let us assume that $a \in L^\infty(0,T)$ and $\theta \in \mathbb{H}^b(\Omega)$. Then problem* (1) *has a unique solution $y \in L^p(0,T;\mathbb{H}^b(\Omega))$. In addition, we have*

$$\left\|y\right\|_{L^p(0,T;\mathbb{H}^b(\Omega))} \lesssim \left\|\theta\right\|_{\mathbb{H}^b(\Omega)}, \tag{32}$$

*for $1 < p < \frac{1}{d}$, $0 < \beta < \alpha$.*

*Proof.* Let us define the space $\mathbf{X}_{\alpha,d,q}((0,T];\mathbb{H}^\nu(\Omega))$ denotes the weighted space of all functions $v \in L^\infty((0,T];\mathbb{H}^\nu(\Omega))$ such that

$$\|w\|_{\mathbf{X}_{\alpha,d,q}((0,T];\mathbb{H}^\nu(\Omega))} := \sup_{t\in(0,T]} t^d e^{-qt^\alpha} \|w(t,\cdot)\|_{\mathbb{H}^\nu(\Omega)} < \infty,$$

where $p > 0$. Let us first to give the explicit formula of the mild solution to Problem (1). It is obvious and not difficult to transform problem (1) into the following problem

$$\begin{cases} D_C^\alpha y + (-\Delta)^s y = F(y) + \Big(1 - a(x,t)\Big)(-\Delta)^\beta y, \ \ (x,t) \in \Omega \times (0,T), \\ y = 0, \ \ (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = \theta(x). \end{cases} \tag{33}$$

For convenience, we denote by a new source function

$$H(y(x,t)) = F(y) + \Big(1 - a(x,t)\Big)(-\Delta)^\beta y.$$

By using the separation of variables, the solution of (1) is given by Fourier series

$$y(x,t) = \sum_{n\in\mathbb{N}} \left(\int_\Omega y(x,t)\psi_n(x)dx\right)\psi_n(x), \quad y_n(t) = \int_\Omega y(x,t)\psi_n(x)dx.$$

Solving the above system, we deduce that the formula of $y_n$

$$\begin{aligned} y_n(t) &= \exp\Big(-\lambda_n \frac{t^\alpha}{\alpha}\Big)\left(\int_\Omega \theta(x)\psi_n(x)dx\right) \\ &\quad + \int_0^t r^{\alpha-1} \exp\Big(-\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\left(\int_\Omega F(y(x,t))\psi_n(x)dx\right) \\ &\quad + \int_0^t r^{\alpha-1} \exp\Big(-\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\left(\int_\Omega \Big(1 - a(t)\Big)(-\Delta)^\beta y\psi_n(x)dx\right) \end{aligned} \tag{34}$$

where $y_n(t)$ is the Fourier coefficient of the function $y$. Thus, we get the following equality

$$y(x,t) = \mathcal{M}_0(x,t) + \mathcal{M}_1(y(x,t)) + \mathcal{M}_2(y(x,t)), \tag{35}$$

where

$$\mathcal{M}_0(x,t) = \sum_{n=1}^\infty \exp\Big(-\lambda_n \frac{t^\alpha}{\alpha}\Big)\left(\int_\Omega \theta(x)\psi_n(x)dx\right)\psi_n(x),$$

and

$$\mathcal{M}_1(y(x,t)) = \sum_{n=1}^\infty \left[\int_0^t r^{\alpha-1} \exp\Big(-\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\left(\int_\Omega G(y(x,t))\psi_n(x)dx\right)\right]\psi_n(x) \tag{36}$$

$$\mathcal{M}_2(y(x,t)) = \sum_{n=1}^\infty \left[\int_0^t r^{\alpha-1} \exp\Big(-\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\left(\int_\Omega \Big(1 - a(t)\Big)(-\Delta)^\beta y\psi_n(x)dx\right)\right]\psi_n(x). \tag{37}$$

Our main goal is to show that the nonlinear equation $\mathcal{M}y = y$ has a unique solution $y$ where $\mathcal{M}$ is defined by

$$\mathcal{M}(y(x,t)) = \mathcal{M}_0(x,t) + \mathcal{M}_1(y(x,t)) + \mathcal{M}_2(y(x,t)).$$

Let $y = 0$ then

$$\Big\|\mathcal{M}(y(x,t)=0)\Big\|^2_{\mathbb{H}^b(\Omega)} = \Big\|\mathcal{M}_0(.,t)\Big\|^2_{\mathbb{H}^b(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^{2b} \exp\Big(-2\lambda_n \frac{t^\alpha}{\alpha}\Big)\Big(\int_\Omega \theta(x)\psi_n(x)dx\Big)^2. \tag{38}$$

Since $\theta \in \mathbb{H}^b(\Omega)$ and $\exp\Big(-2\lambda_n \frac{t^\alpha}{\alpha}\Big) \le 1$, we deduce that

$$\Big\|\mathcal{M}(y(x,t)=0)\Big\|_{\mathbb{H}^b(\Omega)} \le \sum_{n=1}^{\infty} \lambda_n^{2b} \Big(\int_\Omega \theta(x)\psi_n(x)dx\Big)^2 = \Big\|\theta\Big\|^2_{\mathbb{H}^b(\Omega)}. \tag{39}$$

This inequality implies that $\mathcal{M}y \in C([0,T];\mathbb{H}^b(\Omega))$ if $y \in C([0,T];\mathbb{H}^b(\Omega))$. Our next goal is to show that $\mathcal{M}$ is a contraction.

*Step 1. Estimate of* $\Big\|\mathcal{M}_1(y_1(x,t)) - \mathcal{M}_1(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)}$

Let $y_1, y_2 \in \mathbb{H}^b(\Omega)$. Then using Parseval's equality and Hölder inequality, we get

$$\begin{aligned}
&\Big\|\mathcal{M}_1(y_1(x,t)) - \mathcal{M}_1(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)} \\
&= \sum_{n=1}^{\infty} \lambda_n^{2b} \Big[\int_0^t r^{\alpha-1} \exp\Big(-\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\big(G_n(y_1(r)) - G_n(y_2(r))\big)\,dr\Big]^2 \\
&\le \Big(\int_0^t r^{\alpha-1}dr\Big) \sum_{n=1}^{\infty} \lambda_n^{2b} \Big(\int_0^t r^{\alpha-1} \exp\Big(-2\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\big(G_n(y_1(r)) - G_n(y_2(r))\big)^2\,dr\Big)
\end{aligned} \tag{40}$$

where we denote

$$G_n(y(r)) = \int_\Omega G(y(x,r))\psi_n(x)dx.$$

Using the inequality $e^{-z} \le C_\beta z^{-\beta}$ for any $\beta > 0$, we find that

$$\exp\Big(-2\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big) \le C_{\alpha,\beta} \lambda_n^{-2\beta} (t^\alpha - r^\alpha)^{-2\beta}. \tag{41}$$

This implies that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_n^{2b} \Big(\int_0^t r^{\alpha-1} \exp\Big(-2\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\big(G_n(y_1(r)) - G_n(y_2(r))\big)^2\,dr\Big) \\
&\le C_{\alpha,\beta} \sum_{n=1}^{\infty} \lambda_n^{2b-2\beta} \Big(\int_0^t r^{\alpha-1}(t^\alpha - r^\alpha)^{-2\beta}\big(G_n(y_1(r)) - G_n(y_2(r))\big)^2\,dr\Big).
\end{aligned} \tag{42}$$

Combining (40) and (42) and noting that $b - \beta \le a$, we derive that

$$\begin{aligned}
\Big\|\mathcal{M}_1(y_1(x,t)) - \mathcal{M}_1(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)} &\le C(\alpha,\beta,T)\Big(\int_0^t r^{\alpha-1}(t^\alpha - r^\alpha)^{-2\beta}\Big\|G(y_1) - G(y_2)\Big\|^2_{\mathbb{H}^{b-\beta}(\Omega)}dr\Big) \\
&\le C(a,\alpha,\beta,T)\Big(\int_0^t r^{\alpha-1}(t^\alpha - r^\alpha)^{-2\beta}\Big\|G(y_1) - G(y_2)\Big\|^2_{\mathbb{H}^a(\Omega)}dr\Big) \\
&\le C_1\Big(\int_0^t r^{\alpha-1}(t^\alpha - r^\alpha)^{-2\beta}\Big\|y_1 - y_2\Big\|^2_{\mathbb{H}^b(\Omega)}dr\Big)
\end{aligned} \tag{43}$$

where $C_1$ depends on $a, \alpha, \beta, T, K_f$. Then we get the following bound

$$\begin{aligned} & t^{2d} e^{-2qt^\alpha} \Big\| \mathcal{M}_1(y_1(x,t)) - \mathcal{M}_1(y_2(x,t)) \Big\|^2_{\mathbb{H}^b(\Omega)} \\ & \le C_1 t^{2d} e^{-2qt^\alpha} \bigg( \int_0^t r^{\alpha-1} (t^\alpha - r^\alpha)^{-2\beta} \Big\| y_1 - y_2 \Big\|^2_{\mathbb{H}^b(\Omega)} dr \bigg) \\ & = C_1 t^{2d} \bigg( \int_0^t r^{\alpha-1-2d} (t^\alpha - r^\alpha)^{-2\beta} e^{-2q(t^\alpha - r^\alpha)} r^{2d} e^{-2qr^\alpha} \Big\| y_1(r) - y_2(r) \Big\|^2_{\mathbb{H}^b(\Omega)} dr \bigg). \end{aligned} \tag{44}$$

Therefore, we derive that

$$\begin{aligned} & \int_0^t r^{\alpha-1-2d} (t^\alpha - r^\alpha)^{-2\beta} e^{-2q(t^\alpha - r^\alpha)} r^{2d} e^{-2qr^\alpha} \Big\| y_1(r) - y_2(r) \Big\|^2_{\mathbb{H}^b(\Omega)} dr \\ & \le \Big\| y_1 - y_2 \Big\|^2_{\mathbf{X}_{\alpha,d,q}((0,T];\mathbb{H}^b(\Omega))} \bigg( \int_0^t r^{\alpha-1-2d} (t^\alpha - r^\alpha)^{-2\beta} e^{-2q(t^\alpha - r^\alpha)} dr \bigg). \end{aligned} \tag{45}$$

Since two latter observations, we find that

$$\begin{aligned} & \sup_{0 \le t \le T} t^{2d} e^{-2qt^\alpha} \Big\| \mathcal{M}_1(y_1(x,t)) - \mathcal{M}_1(y_2(x,t)) \Big\|^2_{\mathbb{H}^b(\Omega)} \\ & \le \Big\| y_1 - y_2 \Big\|^2_{\mathbf{X}_{\alpha,d,q}((0,T];\mathbb{H}^b(\Omega))} \sup_{0 \le t \le T} \bigg( t^{2d} \int_0^t r^{\alpha-1-2d} (t^\alpha - r^\alpha)^{-2\beta} e^{-2q(t^\alpha - r^\alpha)} dr \bigg). \end{aligned} \tag{46}$$

Let us to control the term

$$\mathcal{I}_q(t) = t^{2d} \int_0^t r^{\alpha-1-2d} (t^\alpha - r^\alpha)^{-2\beta} e^{-2q(t^\alpha - r^\alpha)} dr.$$

In order to control this term, we need to change variable $r = t\mu^{\frac{1}{\alpha}}$. Then we have $dr = \frac{t}{\alpha}\mu^{\frac{1}{\alpha}-1} d\mu$. By a simple computation, we have immediately that

$$\mathcal{I}_q(t) = \frac{t^{\alpha-2\alpha\beta}}{\alpha} \int_0^1 \mu^{-\frac{-2d}{\alpha}} (1-\mu)^{-2\beta} e^{-2qt^\alpha(1-\mu)} d\mu.$$

Let us look at Lemma 2.3. Since $\beta < 1/2$, $d < \frac{\alpha}{2}$, $d < \frac{\alpha(1-2\beta)}{2}$, we know that $\alpha - 2\alpha\beta > 0$ and

$$-\frac{-2d}{\alpha} > -1, \;\; 2\beta > -1, \;\; -\frac{-2d}{\alpha} - 2\beta > -1.$$

Using Lemma 2.3, we deduce that

$$\lim_{q \to +\infty} \Big( \sup_{0 \le t \le T} \mathcal{I}_q(t) \Big) = 0. \tag{47}$$

*Step 2. Estimate of* $\Big\| \mathcal{M}_2(y_1(x,t)) - \mathcal{M}_2(y_2(x,t)) \Big\|^2_{\mathbb{H}^b(\Omega)}$

For short, we denote by

$$\overline{G}(y)(x,t) = (1 - a(t)) \Big( (-\Delta)^\beta y(x,t).$$

Then using Parseval's equality and Hölder inequality, we get the following equality

$$
\begin{aligned}
&\Big\|\mathcal{M}_2(y_1(x,t)) - \mathcal{M}_2(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)} \\
&= \sum_{n=1}^{\infty} \lambda_n^{2b} \left[ \int_0^t r^{\alpha-1} \exp\Big(-\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big) \Big( \int_\Omega \overline{G}(y_1)(x,r) - \overline{G}(y_2)(x,r) \Big) e_n(x) dx \Big) dr \right]^2 \\
&\leq \Big( \int_0^t r^{\alpha-1} dr \Big) \sum_{n=1}^{\infty} \lambda_n^{2b} \Big( \int_0^t r^{\alpha-1} \exp\Big(-2\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big) \Big( \int_\Omega \overline{G}(y_1)(x,r) - \overline{G}(y_2)(x,r) \Big) e_n(x) dx \Big)^2 dr \Big).
\end{aligned} \tag{48}
$$

Using (41), we have that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_n^{2b} \Big( \int_0^t r^{\alpha-1} \exp\Big(-2\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big) \Big( \int_\Omega \overline{G}(y_1)(x,r) - \overline{G}(y_2)(x,r) \Big) e_n(x) dx \Big)^2 dr \Big) \\
&\leq C_{\alpha,\beta} \sum_{n=1}^{\infty} \lambda_n^{2b-2\beta} \left( \int_0^t r^{\alpha-1} (t^\alpha - r^\alpha)^{-2\beta} \Big(1 - a(r)\Big)^2 \left( \int_\Omega \Big( (-\Delta)^\beta y_1 - (-\Delta)^\beta y_2 \Big) e_n(x) dx \right)^2 dr \right).
\end{aligned} \tag{49}
$$

Noting that $\Big\|\Delta^\beta v\Big\|_{\mathbb{H}^s(\Omega)} = \Big\|v\Big\|_{\mathbb{H}^{s+\beta}(\Omega)}$, we can claim that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_n^{2b-2\beta} \left( \int_\Omega \Big( (-\Delta)^\beta y_1 - (-\Delta)^\beta y_2 \Big) e_n(x) dx \right)^2 \\
&\qquad\qquad = \Big\|(-\Delta)^\beta (y_1 - y_2)\Big\|^2_{\mathbb{H}^{b-\beta}(\Omega)} = \Big\|y_1 - y_2\Big\|^2_{\mathbb{H}^b(\Omega)}.
\end{aligned} \tag{50}
$$

From three above observations, we find that the following inequality

$$
\begin{aligned}
&\Big\|\mathcal{M}_2(y_1(x,t)) - \mathcal{M}_2(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)} \\
&\leq C(\alpha,\beta,T) \int_0^t r^{\alpha-1} (t^\alpha - r^\alpha)^{-2\beta} \Big(1 - a(r)\Big)^2 \Big\|y_1(r) - y_2(r)\Big\|^2_{\mathbb{H}^b(\Omega)} dr.
\end{aligned} \tag{51}
$$

Since the function $a \in L^\infty(0,T)$, we can deduce that

$$
\begin{aligned}
&\Big\|\mathcal{M}_2(y_1(x,t)) - \mathcal{M}_2(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)} \\
&\leq C(\alpha,\beta,T) \int_0^t r^{\alpha-1} (t^\alpha - r^\alpha)^{-2\beta} \Big(1 - a(r)\Big)^2 \Big\|y_1(r) - y_2(r)\Big\|^2_{\mathbb{H}^b(\Omega)} dr \\
&\leq C(\alpha,\beta,T) \Big(1 + \|a\|_{L^\infty(0,T)}\Big) \int_0^t r^{\alpha-1} (t^\alpha - r^\alpha)^{-2\beta} \Big\|y_1(r) - y_2(r)\Big\|^2_{\mathbb{H}^b(\Omega)} dr
\end{aligned} \tag{52}
$$

where we remind that

$$
\|a\|_{L^\infty(0,T)} = \sup_{0 \leq t \leq T} |a(t)|.
$$

Multiplying both sides of (52) with $t^{2d}e^{-2qt^\alpha}$, we get that

$$\begin{aligned}
&t^{2d}e^{-2qt^\alpha}\Big\|\mathcal{M}_2(y_1(x,t))-\mathcal{M}_2(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)}\\
&\le C(\alpha,\beta,T)\Big(1+\|a\|_{L^\infty(0,T)}\Big)t^{2d}e^{-2qt^\alpha}\left(\int_0^t r^{\alpha-1}(t^\alpha-r^\alpha)^{-2\beta}\Big\|y_1-y_2\Big\|^2_{\mathbb{H}^b(\Omega)}dr\right)\\
&= C(\alpha,\beta,T)\Big(1+\|a\|_{L^\infty(0,T)}\Big)t^{2d}\\
&\times\left(\int_0^t r^{\alpha-1-2d}(t^\alpha-r^\alpha)^{-2\beta}e^{-2q(t^\alpha-r^\alpha)}r^{2d}e^{-2qr^\alpha}\Big\|y_1(r)-y_2(r)\Big\|^2_{\mathbb{H}^b(\Omega)}dr\right).
\end{aligned} \tag{53}$$

By a similar claim as in (45), we get that

$$\begin{aligned}
&\sup_{0\le t\le T} t^{2d}e^{-2qt^\alpha}\Big\|\mathcal{M}_2(y_1(x,t))-\mathcal{M}_2(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)}\\
&\le \Big\|y_1-y_2\Big\|^2_{\mathbf{X}_{\alpha,d,q}((0,T];\mathbb{H}^b(\Omega))}\sup_{0\le t\le T}\left(t^{2d}\int_0^t r^{\alpha-1-2d}(t^\alpha-r^\alpha)^{-2\beta}e^{-2q(t^\alpha-r^\alpha)}dr\right).
\end{aligned} \tag{54}$$

From two steps, we can deduce that

$$\begin{aligned}
&\sup_{0\le t\le T} t^{2d}e^{-2qt^\alpha}\left(\Big\|\mathcal{M}_1(y_1(x,t))-\mathcal{M}_1(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)}+\Big\|\mathcal{M}_2(y_1(x,t))-\mathcal{M}_2(y_2(x,t))\Big\|^2_{\mathbb{H}^b(\Omega)}\right)\\
&\le\Big(\sup_{0\le t\le T}\mathcal{I}_q(t)\Big)\Big\|y_1-y_2\Big\|^2_{\mathbf{X}_{\alpha,d,q}((0,T];\mathbb{H}^b(\Omega))}.
\end{aligned} \tag{55}$$

Thus, we obtain that

$$\Big\|\mathcal{M}y_1-\mathcal{M}y_2\Big\|_{\mathbf{X}_{\alpha,d,q}((0,T];\mathbb{H}^b(\Omega))}\le\left(\sqrt{\Big(\sup_{0\le t\le T}\mathcal{I}_q(t)\Big)}\right)\Big\|y_1-y_2\Big\|_{\mathbf{X}_{\alpha,d,q}((0,T];\mathbb{H}^b(\Omega))}. \tag{56}$$

Since (47), we can find $q_0$ such that

$$\sup_{0\le t\le T}\mathcal{I}_{q_0}(t)\le\frac{1}{4}. \tag{57}$$

It follows from (56) that

$$\Big\|\mathcal{M}y_1-\mathcal{M}y_2\Big\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}\le\frac{1}{2}\Big\|y_1-y_2\Big\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}. \tag{58}$$

By using Banach fixed point theorem, we conclude that $\mathcal{M}$ has a fixed point $y$ which belongs to $\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))$. Let us claim the regularity result of $y$. Indeed, we get the following estimate

$$\begin{aligned}
\Big\|y\Big\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}&=\Big\|\mathcal{M}y\Big\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}\\
&\le\frac{1}{2}\Big\|y\Big\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}+\Big\|\mathcal{M}(y(x,t)=0)\Big\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}.
\end{aligned} \tag{59}$$

In view of (39), we derive that

$$\Big\|\mathcal{M}(y(x,t)=0)\Big\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}=\sup_{0\le t\le T}t^de^{-q_0t^\alpha}\Big\|\mathcal{M}(y(x,t)=0)\Big\|_{\mathbb{H}^b(\Omega)}\le T^d\Big\|\theta\Big\|_{\mathbb{H}^b(\Omega)}. \tag{60}$$

Combining (59) and (60), we derive that

$$\left\|y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} \leq 2T^d\left\|\theta\right\|_{\mathbb{H}^b(\Omega)}. \tag{61}$$

This implies that

$$\left\|y(.,t)\right\|_{\mathbb{H}^b(\Omega)} \leq 2e^{q_0 t^\alpha}T^d t^{-d}\left\|\theta\right\|_{\mathbb{H}^b(\Omega)}. \tag{62}$$

Since $0 < d < 1$, we can infer that $y \in L^p(0,T;\mathbb{H}^b(\Omega))$ for $1 < p < \frac{1}{d}$. Thus, we deduce that

$$\left\|y\right\|_{L^p(0,T;\mathbb{H}^b(\Omega))} = \Big(\int_0^T \left\|y(.,t)\right\|^p_{\mathbb{H}^b(\Omega)}dt\Big)^{1/p} \lesssim \left\|\theta\right\|_{\mathbb{H}^b(\Omega)}. \tag{63}$$

The proof is completed. □

**Theorem 3.2.** *Let $y_\beta$ be the mild solution to Problem (1) with $\beta > 1$. Let $y$ be the mild solution to Problem (1) with $\beta = 1$. Let us assume that the Cauchy data $\theta \in \mathbb{H}^{b+\beta+\sigma-\gamma}(\Omega)$ for $\sigma > 0$ and $0 < \gamma < \frac{1}{2}$. Then we get that*

$$\left\|y_\beta - y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} \lesssim (\beta-1)^\sigma\left\|y\right\|_{\mathbb{H}^{b+\beta+\sigma-\gamma}(\Omega)} \tag{64}$$

*Proof.* Let us remind that the following operator

$$\mathcal{M}_{2,\beta}(y(x,t)) = \sum_{n=1}^\infty\left[\int_0^t r^{\alpha-1}\exp\Big(-\lambda_n\frac{t^\alpha-r^\alpha}{\alpha}\Big)\bigg(\int_\Omega\Big(1-a(r)\Big)(-\Delta)^\beta y\psi_n(x)dx\bigg)\right]\psi_n(x)$$

and

$$\mathcal{M}_2^*(y(x,t)) = \sum_{n=1}^\infty\left[\int_0^t r^{\alpha-1}\exp\Big(-\lambda_n\frac{t^\alpha-r^\alpha}{\alpha}\Big)\bigg(\int_\Omega\Big(1-a(r)\Big)(-\Delta)y\psi_n(x)dx\bigg)\right]\psi_n(x).$$

Since $y_\beta$ is the mild solution to problem, we have that

$$y_\beta(x,t) = \mathcal{M}_0(x,t) + \mathcal{M}_1(y_\beta(x,t)) + \mathcal{M}_{2,\beta}(y_\beta(x,t)). \tag{65}$$

Since $y$ is the mild solution to the Problem (1), we get that

$$y(x,t) = \mathcal{M}_0(x,t) + \mathcal{M}_1(y(x,t)) + \mathcal{M}_2^*(y(x,t)). \tag{66}$$

From two above equality, one has

$$\begin{aligned} y_\beta(x,t) - y(x,t) &= \mathcal{M}_1\Big(y_\beta(x,t)-y(x,t)\Big) + \mathcal{M}_{2,\beta}\Big(y_\beta(x,t)-y(x,t)\Big) \\ &\quad + \Big(\mathcal{M}_{2,\beta}-\mathcal{M}_2^*\Big)y(x,t). \end{aligned} \tag{67}$$

Hence, using the triangle inequality, we have immediately that the following inequality

$$\begin{aligned} \left\|y_\beta - y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} &\leq \left\|\mathcal{M}_1\Big(y_\beta - y\Big)\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} \\ &\quad + \left\|\mathcal{M}_{2,\beta}\Big(y_\beta - y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} + \left\|\Big(\mathcal{M}_{2,\beta}-\mathcal{M}_2^*\Big)y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} \\ &\leq \frac{1}{2}\left\|y_\beta - y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} + \left\|\Big(\mathcal{M}_{2,\beta}-\mathcal{M}_2^*\Big)y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}. \end{aligned} \tag{68}$$

Thus, one has the following inequality

$$\left\|y_\beta - y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} \le 2\left\|\left(\mathcal{M}_{2,\beta} - \mathcal{M}_2^*\right) y\right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))}. \tag{69}$$

Our next task is to provide the upper bound of the right above. Using Parseval's equality and Hölder inequality, we get that

$$\begin{aligned}
&\left\|\left(\mathcal{M}_{2,\beta} - \mathcal{M}_2^*\right) y(x,t)\right\|^2_{\mathbb{H}^b(\Omega)} \\
&= \sum_{n=1}^{\infty} \lambda_n^{2b} \left[\int_0^t r^{\alpha-1} \exp\Big(-\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\Big(1-a(r)\Big)\left(\int_\Omega \Big((-\Delta)^\beta y - (-\Delta) y\Big)\psi_n(x)dx\right) dr\right]^2 \\
&\lesssim \sum_{n=1}^{\infty} \lambda_n^{2b} \left[\int_0^t r^{\alpha-1} \exp\Big(-2\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big)\Big(1-a(r)\Big)^2\left(\int_\Omega \Big((-\Delta)^\beta y - (-\Delta) y\Big)\psi_n(x)dx\right)^2 dr\right].
\end{aligned} \tag{70}$$

Using the inequality $e^{-z} \le C_\gamma z^{-\gamma}$ for any $\gamma > 0$, we find that

$$\exp\Big(-2\lambda_n \frac{t^\alpha - r^\alpha}{\alpha}\Big) \le C_{\alpha,\gamma} \lambda_n^{-2\gamma}(t^\alpha - r^\alpha)^{-2\gamma}. \tag{71}$$

Thus, we follows from (70) that

$$\begin{aligned}
&\left\|\left(\mathcal{M}_{2,\beta} - \mathcal{M}_2^*\right) y(x,t)\right\|^2_{\mathbb{H}^b(\Omega)} \\
&\lesssim \sum_{n=1}^{\infty} \lambda_n^{2b-2\gamma} \int_0^t r^{\alpha-1}(t^\alpha - r^\alpha)^{-2\gamma}\Big(1-a(r)\Big)^2\left(\int_\Omega \Big((-\Delta)^\beta y - (-\Delta) y\Big)\psi_n(x)dx\right)^2 dr \\
&\lesssim \sum_{n=1}^{\infty} \lambda_n^{2b-2\gamma} \int_0^t r^{\alpha-1}(t^\alpha - r^\alpha)^{-2\gamma}\left(\int_\Omega \Big((-\Delta)^\beta y - (-\Delta) y\Big)\psi_n(x)dx\right)^2 dr
\end{aligned} \tag{72}$$

where we note that

$$|1 - a(r)| \le 1 + \left\|a\right\|_{L^\infty(0,T)}.$$

It is obvious to see that

$$\sum_{n=1}^{\infty} \lambda_n^{2b-2\gamma}\left(\int_\Omega \Big((-\Delta)^\beta y - (-\Delta) y\Big)\psi_n(x)dx\right)^2 = \sum_{n=1}^{\infty} \lambda_n^{2b-2\gamma}\left(\lambda_n^\beta - \lambda_n\right)^2\left(\int_\Omega y\psi_n(x)dx\right)^2.$$

In view of the paper [12], we have the following observation. If $\lambda_n \le 1$ then under the assumption $\beta > 1$, we get

$$\left|\lambda_n^\beta - \lambda_n\right| \le C(\sigma)\lambda_n^{1-\sigma}(\beta-1)^\sigma, \ \ \sigma > 0. \tag{73}$$

If $\lambda_n > 1$ then

$$\left|\lambda_n^\beta - \lambda_n\right| \le C(\sigma)\lambda_n^{\beta+\sigma}(\beta-1)^\sigma, \ \ \sigma > 0. \tag{74}$$

From the above fact, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n^{2b-2\gamma}\left(\lambda_n^\beta - \lambda_n\right)^2\left(\int_\Omega y\psi_n(x)dx\right)^2 &\le (\beta-1)^{2\sigma}\sum_{n=1}^{\infty} \lambda_n^{2b+2\beta+2\sigma-2\gamma}\left(\int_\Omega y\psi_n(x)dx\right)^2 \\
&= (\beta-1)^{2\sigma}\left\|y\right\|^2_{\mathbb{H}^{b+\beta+\sigma-\gamma}(\Omega)}.
\end{aligned} \tag{75}$$

Combining (72) and (75), we obtain that

$$\left\| \left( \mathcal{M}_{2,\beta} - \mathcal{M}_2^* \right) y(x,t) \right\|_{\mathbb{H}^b(\Omega)} \lesssim (\beta-1)^\sigma \left\| y \right\|_{\mathbb{H}^{b+\beta+\sigma-\gamma}(\Omega)} \Big( \int_0^t r^{\alpha-1} (t^\alpha - r^\alpha)^{-2\gamma} dr \Big)^{1/2}. \tag{76}$$

Since the fact that $0 < \gamma < \frac{1}{2}$, we know that

$$\int_0^t r^{\alpha-1} (t^\alpha - r^\alpha)^{-2\gamma} dr = \frac{t^{\alpha(1-2\gamma)}}{\alpha(1-2\gamma)}. \tag{77}$$

Two latter estimates implies that

$$\begin{aligned} \left\| \left( \mathcal{M}_{2,\beta} - \mathcal{M}_2^* \right) y \right\|_{\mathbf{X}_{\alpha,d,q_0}((0,T];\mathbb{H}^b(\Omega))} &= \sup_{0 \le t \le T} t^d e^{-qt^\alpha} \left\| \left( \mathcal{M}_{2,\beta} - \mathcal{M}_2^* \right) y(.,t) \right\|_{\mathbb{H}^b(\Omega)} \\ &\lesssim (\beta-1)^\sigma \left\| y \right\|_{\mathbb{H}^{b+\beta+\sigma-\gamma}(\Omega)}. \end{aligned} \tag{78}$$

The proof is completed. □

**Remark 3.3.** *If* $\theta \in \mathbb{H}^{b+\beta+\sigma-\gamma}(\Omega)$ *then we can show that Problem* (1) *with* $\beta = 1$ *has a unique solution* $y \in C([0,T]; \mathbb{H}^{b+\beta+\sigma-\gamma}(\Omega))$.


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