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Some results for two classes of two-point local fractional proportional boundary value problems

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Abstract. In this paper, we consider two classes of boundary value problems in the frame of local proportional fractional derivatives. For both of these classes, we obtain the associated Green's functions and discuss their properties. Using these properties, we go about the uniqueness of the solutions. In addition, we establish Lyapunov-type and Hartman-Wintner-type inequalities and build sharp estimated for the unique solutions of the considered equations.

1. Introduction

The non-local fractional calculus have attracted the interest of many scientists working on different areas of science and engineering. In fact, this calculus permits differentiation and integration of any real or complex orders and thus generalizes the usual calculus that studies integrals and derivatives of integer orders. What makes the fractional calculus interesting is that there is a variety of fractional derivatives and thus a researcher can choose the most suitable derivative which may help in understanding and modeling a real world phenomena they are working on [1–7].

On the flip side, local fractional derivatives admits differentiation of non integer orders as well. One kind of these derivatives was developed in [8, 9] and was called the conformable derivative. The disadvantage of the conformable derivative is that it does not yield the function itself when the order is 0. To bypass this defect, Anderson et al. [10, 11] suggested a modification of the conformable derivative so that if the order of this derivative tends to 0, it gives the function itself and if the order tends to 1, it gives the first-order derivative of the function. Later the authors in [12] presented a new type of fractional operators generated from a certain class of the modified conformable derivatives and the authors investigated more properties of this modified conformable derivative in [13].

One of the most important integral inequalities that played a noticeable role in the development of differential equations is the Lyapunov inequality. The Russian mathematician A. M. Lyapunov [14] proved

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that if the boundary value problem (BVP)

$$\eta''(t) + q(t)\eta(t) = 0, \quad t \in (a,b), \quad \eta(a) = \eta(b) = 0,$$

where q is a real-valued continuous function, has a nontrivial solution, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}.$$

The constant 4 in the above inequality is sharp so that it cannot be replaced by a larger number. This inequality has effectuated many applications in various fields of sciences and engineering; see [15–20] and the references therein. On the top of this, many articles have discussed the extension of the Lyapunov inequality in the presence of non-local and local fractional derivatives [28–33] by using maximum value of a Green's function [34, 35]. Moreover, sharp estimates for the solutions of certain classes of boundary values problems were handled in [21–27].

Motivated by the aforementioned works on Lyapunov inequality and inspired by the work done in [21–27], we intend to obtain specific inequalities and establish sharp estimates for the existence of a unique solution for the following local fractional proportional differential equations with two-point boundary conditions

$$\begin{cases} (\mathfrak{D}y)(t) + f(t, y(t)) = 0, \ t \in [a,b], \\ y(a) = A, y(b) = B, \ A, B \in \mathbb{R}, \end{cases} \tag{1}$$

where $\mathfrak{D} \in \{{}_aD^{\rho_1}\ {}_aD^{\rho_2},\ {}_aD^{\rho}\}$ such that ${}_aD^{\sigma}$ denotes the local fractional proportional differential operator of order σ $(\sigma = \rho_1, \rho_2, \rho)$ with $0 < \rho_1, \rho_2 < 1, 1 < \rho_1 + \rho_2 < 2, \rho_1 \neq \rho_2, 1 < \rho < 2$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a known function.

Also, we investigate the Lyapunov-type and Hartman-Wintner-type inequalities for the following open problems presented by Abdeljawad in [13].

$$\begin{cases} (\mathfrak{D}y)(t) + q(t)y(t) = 0, \ t \in [a,b], \\ y(a) = 0 = y(b), \end{cases} \tag{2}$$

where $q : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

2. Preliminaries

In this section, we present some basic concepts of local fractional proportional integral and derivative.

Definition 2.1 ([3]). *Let $\alpha \geq 0$. The fractional integral of Reimann Liouville type of the function $\Psi \in L^1[a,b]$ is defined by $\left(\mathbf{I}_a^0\Psi\right)(t) = \Psi(t)$ and*

$$\left(\mathbf{I}_a^{\alpha}\Psi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \Psi(\tau)\, d\tau, \text{ for } \alpha > 0,$$

where $t \in [a,b]$ and $\Gamma(\alpha) = \int_0^{+\infty} \tau^{(\alpha-1)} e^{-\tau} d\tau$.

Definition 2.2 ([12]). *Let $\rho \in [0,1]$. The local fractional proportional integral of order ρ of the function Ψ is defined by $\left({}_aI^0\Psi\right)(t) = \Psi(t)$ and*

$$\left({}_aI^{\rho}\Psi\right)(t) = \frac{1}{\rho} \int_a^t e^{\frac{\rho-1}{\rho}(t-\tau)} \Psi(\tau)\, d\tau \text{ for } \rho \in (0,1], \tag{3}$$

where $t \in [a,b]$.

Definition 2.3 ([13]). *Let $\rho \in (n, n+1]$, $(n \in \mathbb{N})$ and $\nu = \rho - n$. The local fractional proportional integral of order ρ of the function Ψ is defined by*

$$\left({}_aI^{\rho}\Psi\right)(t) = \left(\mathbf{I}_a^n\, {}_aI^{\nu}\Psi\right)(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1}\left({}_aI^{\nu}\Psi\right)(\tau)\,d\tau, \tag{4}$$

where $t \in [a, b]$.

Remark 2.4. *Note that unfortunately ${}_aI^{\nu_1}\,{}_aI^{\nu_2} \neq_a I^{\nu_1+\nu_2}$.*

Definition 2.5 ([13]). *Let $\rho \in (n, n+1]$, $n \in \mathbb{N}$. The local fractional proportional derivative of order ρ of the function $\Psi \in C^{(n+1)}[a, b]$ is defined by*

$$({}_aD^{\rho}\Psi)(t) = ({}_aD^{\rho-n}D^n\Psi)(t),\ t \in [a, b], \tag{5}$$

where, in case $n = 0$ the operator ${}_aD^{\rho}$ defined in [12] as follows

$${}_aD^{\rho} = (1-\rho) + \rho D^1,\ \ (\text{here } \rho \in (0, 1]). \tag{6}$$

Above, D^n is the n-th order differential operator.

Remark 2.6. *Formula (5) can be written as follows*

$$({}_aD^{\rho}\Psi)(t) = (1-\rho+n)\Psi^{(n)}(t) + (\rho-n)\Psi^{(n+1)}(t). \tag{7}$$

Remark 2.7. *Following Example 5.1 in [13], if $\rho \in (1, 2)$, $\sigma = \rho - 1$ and $\gamma_\sigma = \frac{\sigma-1}{\sigma} = \frac{\rho-2}{\rho-1}$, we have*

$${}_aI^{\rho}\varphi(t) = \frac{1}{2-\rho}\int_a^t \left[1 - e^{\gamma_\sigma(t-s)}\right]\varphi(s)\,ds. \tag{8}$$

3. Main results

In this section we provide our main findings.

Theorem 3.1. *For any $\rho > 0$ with $n < \rho \leq n+1$, for an integrable function Ψ on $[a, t]$ we have*

$$\left({}_aI^{\rho}D^{\rho}\Psi\right)(t) = \Psi(t) - \sum_{k=0}^{k=n-1}\left(\frac{(D^k\Psi)(a)}{k!} - \frac{({}_aD^{\rho-n}\Psi)(a)}{\delta^{n-k}k!}\right)(t-a)^k - \frac{({}_aD^{\rho-n}\Psi)(a)}{\delta^n}e^{\delta(t-a)}, \tag{9}$$

where $\delta = \frac{\rho-n-1}{\rho-n}$.

Proof. From Definitions 2.3 and 2.5, we have

$$\begin{aligned}
\left({}_aI^{\rho}D^{\rho}\Psi\right)(t) &= \left(\mathbf{I}_a^n\, {}_aI^{\rho-n}\, {}_aD^{\rho-n}\Psi^{(n)}\right)(t),\\
&= \mathbf{I}_a^n\left[D^n\Psi(t) - e^{\delta(t-a)}({}_aD^{\rho-n}\Psi)(a)\right]\\
&= \Psi(t) - \sum_{k=0}^{n-1}\frac{(D^k\Psi)(a)}{k!}(t-a)^k - ({}_aD^{\rho-n}\Psi)(a)\mathbf{I}_a^n e^{\delta(t-a)}.
\end{aligned} \tag{10}$$

On other hand, we have

$$\begin{aligned}
\mathbf{I}_a^n e^{\delta(t-a)} &= \frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1} e^{\delta(s-a)} d\tau \\
&= -\frac{(t-a)^{n-1}}{\delta\Gamma(n)} + \frac{1}{\delta}\mathbf{I}_a^{n-1} e^{\delta(t-a)} \\
&= -\frac{(t-a)^{n-1}}{\delta\Gamma(n)} - \frac{(t-a)^{n-2}}{\delta^2\Gamma(n-1)} + \frac{1}{\delta^2}\mathbf{I}_a^{n-2} e^{\delta(t-a)} \\
&= -\frac{(t-a)^{n-1}}{\delta\Gamma(n)} - \frac{(t-a)^{n-2}}{\delta^2\Gamma(n-1)} - \frac{(t-a)^{n-3}}{\delta^3\Gamma(n-2)} + \frac{1}{\delta^3}\mathbf{I}_a^{n-3} e^{\delta(t-a)} \\
&= -\frac{(t-a)^{n-1}}{\delta\Gamma(n)} - \frac{(t-a)^{n-2}}{\delta^2\Gamma(n-1)} - \frac{(t-a)^{n-3}}{\delta^3\Gamma(n-2)} - \dots - \frac{(t-a)}{\delta^{n-1}\Gamma(2)} + \frac{1}{\delta^{n-1}}\mathbf{I}_a^{1} e^{\delta(t-a)} \\
&= -\sum_{k=0}^{n-1}\frac{(t-a)^k}{k!\delta^{n-k}} + \frac{1}{\delta^n}e^{\delta(t-a)}.
\end{aligned}$$

Substituting the value of $\mathbf{I}_a^n e^{\delta(t-a)}$ in the formula (10), we get

$$\begin{aligned}
({}_aI^\rho D^\rho \Psi)(t) &= \Psi(t) - \sum_{k=0}^{n-1}\frac{(D^k\Psi)(a)}{k!}(t-a)^k - ({}_aD^{\rho-n}\Psi)(a)\left(-\sum_{k=0}^{n-1}\frac{(t-a)^k}{k!\delta^{n-k}} + \frac{1}{\delta^n}e^{\delta(t-a)}\right) \\
&= \Psi(t) - \sum_{k=0}^{n-1}\left(\frac{(D^k\Psi)(a)}{k!} - \frac{({}_aD^{\rho-n}\Psi)(a)}{\delta^{n-k}k!}\right)(t-a)^k - \frac{({}_aD^{\rho-n}\Psi)(a)}{\delta^n}e^{\delta(t-a)}.
\end{aligned}$$

□

Lemma 3.2. *Let $\rho > 0$ with $n < \rho \leq n+1$. In view of Theorem 3.1 , we have the following property:*

$$({}_aI^\rho D^\rho \Psi)(t) = \Psi(t) - \sum_{k=0}^{n-1} c_k (t-a)^k - c_n e^{\delta(t-a)}, \tag{11}$$

where $c_k \in \mathbb{R}$, with $k = 0, 1, 2, ..., n$ and $\delta = \frac{\rho-n-1}{\rho-n}$.

Now, we study the given problems according to the operator $\mathfrak{D}$ (Case: $\mathfrak{D} = {}_aD^{\rho_1}\,{}_aD^{\rho_2}$ and Case: $\mathfrak{D} = {}_aD^{\rho}$). In each case, by using the Green's function and its properties for each problem, we obtain the uniqueness of the solution, Lyapunov-type and Hartman-Wintner-type inequalities, non-existence of the solutions and sharper lower bound of the eigenvalues associated.

3.1. Case: $\mathfrak{D} = {}_aD^{\rho_1}\,{}_aD^{\rho_2}$

Consider the sequential local fractional proportional boundary value problem:

$$\begin{cases} ({}_aD^{\rho_1}\,{}_aD^{\rho_2})y(t) + P(t) = 0,\ t \in [a,b],\ \ P \in C[a,b], \\ \qquad y(a) = A, y(b) = B,\ A, B \in \mathbb{R}. \end{cases} \tag{12}$$

The following result plays a key role in deriving the main results for the problems (1) and (2) with $\mathfrak{D} = {}_aD^{\rho_1}\,{}_aD^{\rho_2}$.

Lemma 3.3. *The linear boundary value problem (12) has the unique solution represented by the integral equation*

$$y(t) = A + \frac{(B-A)\left(e^{\gamma_1(t-a)} - e^{\gamma_2(t-a)}\right)}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}} + \int_a^b G(t,s)P(s)\,ds, \tag{13}$$

where

$$G(t,s)=\begin{cases}\frac{\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]}{\delta\left[e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right]}-\frac{1}{\delta}\left[e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}\right]:=g_1(t,s), & a\le s\le t\le b,\\ \frac{\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]}{\delta\left[e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right]}:=g_2(t,s), & a\le t\le s\le b,\end{cases} \tag{14}$$

with $\gamma_1=\frac{\rho_1-1}{\rho_1},\gamma_2=\frac{\rho_2-1}{\rho_2}$ *and*

$$\delta=\rho_1\rho_2(\gamma_1-\gamma_2)=\rho_1-\rho_2. \tag{15}$$

Proof. From Remark 5.1 of [13], we have

$$y(t)=c_1+c_2\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]-\left({}_aI^{\rho_2}\,{}_aI^{\rho_1}P\right)(t), \tag{16}$$

where $c_1,c_2\in\mathbb{R}$.

By the boundary conditions $y(a)=A$, we get $c_1=A$, so

$$\begin{aligned}y(t) &= A+c_2\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]-\left({}_aI^{\rho_2}\,{}_aI^{\rho_1}P\right)(t)\\ &= A+c_2\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]-\frac{1}{\rho_2}\int_a^t e^{\gamma_2(t-s)}\left(\frac{1}{\rho_1}\int_a^s e^{\gamma_1(s-\tau)}P(\tau)\,d\tau\right)ds\\ &= A+c_2\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]-\frac{1}{\rho_1\rho_2}\int_a^t\left(\int_a^s e^{\gamma_2(t-s)}e^{\gamma_1(s-\tau)}P(\tau)\,d\tau\right)ds,\end{aligned}$$

By Fubini's Theorem, we get

$$\begin{aligned}y(t) &= A+c_2\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]-\frac{1}{\rho_1\rho_2}\int_a^t\left(\int_s^t e^{\gamma_2(t-\tau)}e^{\gamma_1(\tau-s)}d\tau\right)P(s)\,ds\\ &= A+c_2\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]-\frac{1}{\rho_1\rho_2}\int_a^t\left(\int_s^t e^{\gamma_2 t-\gamma_1 s}e^{(\gamma_1-\gamma_2)\tau}d\tau\right)P(s)\,ds\\ &= A+c_2\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]-\frac{1}{\rho_1\rho_2}\int_a^t\left(\frac{e^{\gamma_2 t-\gamma_1 s}}{(\gamma_1-\gamma_2)}\left[e^{(\gamma_1-\gamma_2)t}-e^{(\gamma_1-\gamma_2)s}\right]\right)P(s)\,ds,\end{aligned}$$

which implies that

$$y(t)=A+c_2\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]-\frac{1}{\delta}\int_a^t\left[e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}\right]P(s)\,ds, \tag{17}$$

where δ is given by (15).

Using the boundary condition $y(b)=B$, we obtain

$$c_2=\frac{B-A}{e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}}+\frac{1}{\delta\left[e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right]}\int_a^b\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]P(s)\,ds. \tag{18}$$

Substituting the value of c_2 in (17) we get

$$\begin{aligned} y(t) &= A+\frac{(B-A)(e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)})}{e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}}+\frac{e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}}{\delta\left[e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right]}\int_a^b\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]P(s)\,ds \\ &\quad -\frac{1}{\delta}\int_a^t\left[e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}\right]P(s)ds \\ &= A+\frac{(B-A)(e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)})}{e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}} \\ &\quad +\frac{1}{\delta}\int_a^t\left(\frac{\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]}{e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}}-\left[e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}\right]\right)P(s)\,ds \\ &\quad +\frac{1}{\delta}\int_t^b\frac{\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]}{e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}}P(s)\,ds \\ &= A+\frac{(B-A)(e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)})}{e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}}+\int_a^b G(t,s)P(s)\,ds, \end{aligned}$$

where $G(t,s)$ is given by (14).

Conversely, if y is given by (13), then y satisfies boundary value problem presented in (12). This can be proved easily by applying ${}_aD^{\rho_1}\ {}_aD^{\rho_2}$ to both sides of (13) and substituting t by a and b afterwards. □

3.1.1. *On the Green's function*

Lemma 3.4. *Let* $\xi=\frac{\ln\frac{\gamma_2}{\gamma_1}}{(\gamma_1-\gamma_2)}, a,b\in\mathbb{R}(a<b)$, *and let*
$\varphi(t,s)=\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]$, $(t,s)\in[a,b]^2$. *Then*

$$\max_{t;s\in[a,b]}\left|\varphi(t,s)\right|=\varepsilon^2, \tag{19}$$

where

$$\varepsilon=\begin{cases}\max\left\{\left|e^{\gamma_1\xi}-e^{\gamma_2\xi}\right|,\left|e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right|\right\}, & \text{if } \xi\le b-a,\\ \left|e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right|, & \text{if } \xi\ge b-a.\end{cases} \tag{20}$$

And moreover $0<\varepsilon<1$.

Proof. Let $\varphi(t,s)=T(t)S(s)$, where $T(t)=e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}, t\in[a,b]$ and $S(s)=e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}, s\in[a,b]$.

Note that $T(t)$ is continuous function with $T(a)=0$ and $T(b)=e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}$.

On other hand, the function $T'(t)=\gamma_1e^{\gamma_1(t-a)}-\gamma_2e^{\gamma_2(t-a)}, t\in(a,b)$ has unique zero at the point

$$t^*=a+\frac{\ln\frac{\gamma_2}{\gamma_1}}{(\gamma_1-\gamma_2)},\ \text{(here } t^*>a). \tag{21}$$

If $t^*\in(a,b)$ then $T(t^*)=e^{\frac{\gamma_1}{(\gamma_1-\gamma_2)}\ln\frac{\gamma_2}{\gamma_1}}-e^{\frac{\gamma_2}{(\gamma_1-\gamma_2)}\ln\frac{\gamma_2}{\gamma_1}}$, we obtain $\max_{t\in[a,b]}|T(t)|=\varepsilon$ where ε is given by (20).

Also, we have $S(s)$ is continuous function with $S(a)=e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}$ and $S(b)=0$.

On other hand, the equation $S'(s)=0, s\in(a,b)$ has unique solution at the point

$$s^*=b-\frac{\ln\frac{\gamma_2}{\gamma_1}}{(\gamma_1-\gamma_2)},\ \text{(here } s^*<b). \tag{22}$$

If $s^*\in(a,b)$ then $S(s^*)=e^{\frac{\gamma_1}{(\gamma_1-\gamma_2)}\ln\frac{\gamma_2}{\gamma_1}}-e^{\frac{\gamma_2}{(\gamma_1-\gamma_2)}\ln\frac{\gamma_2}{\gamma_1}}$, we obtain $\max_{s\in[a,b]}|S(s)|=\varepsilon$ where ε is given by (20).

We conclude that $\max_{t;s\in[a,b]}\left|\varphi(t,s)\right|=\max_{t\in[a,b]}|T(t)|\max_{s\in[a,b]}|S(s)|=\varepsilon^2$.

Finally, the inequality $0<\varepsilon<1$ is immediate. Hence the proof is complete. □

Lemma 3.5. *The Green's function $G(t,s)$ defined in Lemma 3.3 satisfies the following properties:*
i). $G(t,s) \geq 0$, for all $s,t \in [a,b]$,
ii). $G(t,s) \leq \frac{\varepsilon\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]}{|\delta|\left[e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right]}$, for all $t,s \in [a,b]$,
iii). $G(t,s) \leq \frac{\varepsilon^2}{\delta\left[e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right]}$, for all $t,s \in [a,b]$,
where ε is given by (20).

Proof. Let $(i;j) \in \{(1;2),(2;1)\}$. If either $(i;j)=(1;2)$ or $(i;j)=(2;1)$, then, we have

$$\begin{aligned} 0<\rho_i<\rho_j<1 \quad &\Leftrightarrow \quad 1-\rho_i>1-\rho_j>0 \text{ and } \tfrac{1}{\rho_i}>\tfrac{1}{\rho_j}>1 \\ &\Leftrightarrow \quad \tfrac{1-\rho_i}{\rho_i}>\tfrac{1-\rho_j}{\rho_j}>0 \\ &\Leftrightarrow \quad \tfrac{\rho_i-1}{\rho_i}<\tfrac{\rho_j-1}{\rho_j}<0 \\ &\Leftrightarrow \quad \gamma_i<\gamma_j<0. \end{aligned}$$

Therefore, if either $\rho_1<\rho_2$(or $\rho_1>\rho_2$), then $\delta<(>)0$, $e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)} \leq (\geq)0$, $e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)} \leq (\geq)0$, and $e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}<(>)0$ respectively, for all $a\leq t\leq s\leq b, a\neq b$. (i.e., the value δ and the functions $e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}$, $e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}$ and $e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}$ they have the same sign).

We conclude that

$$g_2(t,s) \geq 0, \tag{23}$$

for all $a\leq t\leq s\leq b$,

Now, Let $g_1(t,s)=Mh(t,s)$, where $M=\frac{1}{\delta\left[e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right]}$, and

$$h(t,s)=\left[e^{\gamma_1(t-a)}-e^{\gamma_2(t-a)}\right]\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]-\left[e^{\gamma_1(b-a)}-e^{\gamma_2(b-a)}\right]\left[e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}\right], \quad a\leq s\leq t\leq b.$$

Note that, by the obove duscussion we have $M>0$.

Next, we have

$$\begin{aligned} h(t,s) &= e^{\gamma_1(t-a)}\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right]-e^{\gamma_2(t-a)}\left[e^{\gamma_1(b-s)}-e^{\gamma_2(b-s)}\right] \\ &\quad -e^{\gamma_1(b-a)}\left[e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}\right]+e^{\gamma_2(b-a)}\left[e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}\right] \\ &= e^{\gamma_1(b-s+t-a)}-e^{\gamma_2(b-s)+\gamma_1(t-a)}-e^{\gamma_1(b-s)+\gamma_2(t-a)}+e^{\gamma_2(b-s+t-a)} \\ &\quad -e^{\gamma_1(b-a+t-s)}+e^{\gamma_2(t-s)+\gamma_1(b-a)}+e^{\gamma_1(t-s)+\gamma_2(b-a)}-e^{\gamma_2(b-a+t-s)} \\ &= e^{\gamma_2(t-s)}e^{\gamma_1(b-a)}-e^{\gamma_2(b-s)}e^{\gamma_1(t-a)}+e^{\gamma_1(t-s)}e^{\gamma_2(b-a)}-e^{\gamma_1(b-s)}e^{\gamma_2(t-a)} \\ &= e^{\gamma_1(b-a)}e^{\gamma_2(b-s)}\left(\frac{e^{\gamma_2(t-s)}}{e^{\gamma_2(b-s)}}-\frac{e^{\gamma_1(t-a)}}{e^{\gamma_1(b-a)}}\right)+e^{\gamma_2(b-a)}e^{\gamma_1(b-s)}\left(\frac{e^{\gamma_1(t-s)}}{e^{\gamma_1(b-s)}}-\frac{e^{\gamma_2(t-a)}}{e^{\gamma_2(b-a)}}\right), \\ &= e^{\gamma_1(b-a)}e^{\gamma_2(b-s)}\left[e^{-\gamma_2(b-t)}-e^{-\gamma_1(b-t)}\right]+e^{\gamma_2(b-a)}e^{\gamma_1(b-s)}\left[e^{-\gamma_1(b-t)}-e^{-\gamma_2(b-t)}\right] \\ &= \left[e^{\gamma_1(b-a)}e^{\gamma_2(b-s)}-e^{\gamma_2(b-a)}e^{\gamma_1(b-s)}\right]\left[e^{-\gamma_2(b-t)}-e^{-\gamma_1(b-t)}\right] \\ &= e^{\gamma_1(b-a)}e^{\gamma_2(b-s)}\left[1-e^{(\gamma_2-\gamma_1)(s-a)}\right]\left[e^{-\gamma_2(b-t)}-e^{-\gamma_1(b-t)}\right], \end{aligned}$$

we obtain

$$g_1(t,s)=Me^{\gamma_1(b-a)}e^{\gamma_2(b-s)}\left[1-e^{(\gamma_2-\gamma_1)(s-a)}\right]\left[e^{-\gamma_2(b-t)}-e^{-\gamma_1(b-t)}\right], \ a\leq s\leq t\leq b.$$

Note that, if $\gamma_2<\gamma_1\leq 0$, then $e^{(\gamma_2-\gamma_1)(s-a)}\leq 1$ and $e^{-\gamma_2(b-t)}-e^{-\gamma_1(b-t)}\geq 0$, we get $g_1(t,s)\geq 0$.

On other hand, if $\gamma_1<\gamma_2\leq 0$, then $e^{(\gamma_2-\gamma_1)(s-a)}\geq 1$ and $e^{-\gamma_2(b-t)}-e^{-\gamma_1(b-t)}\leq 0$, we get $g_1(t,s)\geq 0$.

We Conclude that

$$g_1(t,s) \geq 0, \tag{24}$$

for all $a \leq s \leq t \leq b$.

Hence, by (23) and (24) we obtain the first property of above lemma.

Now, we prove the second and third properties.

From the first discussion above in this proof (above the inequality (23) we can conclude that, if either $\rho_1 < \rho_2$(or $\rho_1 > \rho_2$), then the value δ and the functions $e^{\gamma_1(t-a)} - e^{\gamma_2(t-a)}, e^{\gamma_1(b-s)} - e^{\gamma_2(b-s)}, e^{\gamma_1(t-s)} - e^{\gamma_2(t-s)}$ and $e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}$ they have the same sign, for all $a \leq s \leq t \leq b$. So, because $g_1(t,s) \geq 0$ we obtain

$$0 \leq g_1(t,s) \leq \frac{\left[e^{\gamma_1(t-a)} - e^{\gamma_2(t-a)}\right]\left[e^{\gamma_1(b-s)} - e^{\gamma_2(b-s)}\right]}{\delta\left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]}, \text{ for } a \leq s \leq t \leq b.$$

This yields

$$0 \leq G(t,s) \leq \frac{\left[e^{\gamma_1(t-a)} - e^{\gamma_2(t-a)}\right]\left[e^{\gamma_1(b-s)} - e^{\gamma_2(b-s)}\right]}{\delta\left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]}, \text{ for all } t,s \in [a,b]. \tag{25}$$

From Lemma 3.4 we have

$$0 \leq \frac{\varepsilon\left[e^{\gamma_1(t-a)} - e^{\gamma_2(t-a)}\right]}{\delta} \leq \frac{\varepsilon}{|\delta|}. \tag{26}$$

Therefore, by (25) and (26) we get

$$G(t,s) \leq \frac{\varepsilon\left[e^{\gamma_1(b-s)} - e^{\gamma_2(b-s)}\right]}{|\delta|\left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]}.$$

Now, we conclude that

$$G(t,s) \leq \frac{\varepsilon^2}{\delta\left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]}, \text{ for all } t,s \in [a,b].$$

The proof is complete. □

Proposition 3.6. *The Green's function G defined in Lemma 3.3, has the following property*

$$\max_{t\in[a,b]} \int_a^b |G(t,s)|\, ds = \frac{1}{(\rho_1 - 1)(\rho_2 - 1)}. \tag{27}$$

Proof. From the first proprety of Lemma 3.5, we have

$$\begin{aligned} \int_a^b |G(t,s)|\, ds &= \frac{e^{\gamma_1(t-a)} - e^{\gamma_2(t-a)}}{\delta\left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]} \int_a^b \left[e^{\gamma_1(b-s)} - e^{\gamma_2(b-s)}\right] ds \\ &\quad - \frac{1}{\delta} \int_a^t \left[e^{\gamma_1(t-s)} - e^{\gamma_2(t-s)}\right] ds. \end{aligned}$$

This yields

$$
\begin{aligned}
&\gamma_1\gamma_2\delta\int_a^b |G(t,s)|\,ds \\
= \;& \left(\frac{\gamma_2 e^{\gamma_1(b-a)} - \gamma_1 e^{\gamma_2(b-a)}}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}}\right)\left[e^{\gamma_1(t-a)} - e^{\gamma_2(t-a)}\right] - \left[\gamma_2 e^{\gamma_1(t-a)} - \gamma_1 e^{\gamma_2(t-a)}\right] \\
= \;& \left(\frac{\gamma_2 e^{\gamma_1(b-a)} - \gamma_1 e^{\gamma_2(b-a)}}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}} - \gamma_2\right)e^{\gamma_1(t-a)} - \left(\frac{\gamma_2 e^{\gamma_1(b-a)} - \gamma_1 e^{\gamma_2(b-a)}}{e^{\gamma_1 b-a)} - e^{\gamma_2(b-a)}} - \gamma_1\right)e^{\gamma_2(t-a)} \\
= \;& \left(\frac{\gamma_2 e^{\gamma_2(b-a)} - \gamma_1 e^{\gamma_2(b-a)}}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}}\right)e^{\gamma_1(t-a)} - \left(\frac{\gamma_2 e^{\gamma_1(b-a)} - \gamma_1 e^{\gamma_1(b-a)}}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}}\right)e^{\gamma_2(t-a)} \\
= \;& \left(\frac{\gamma_2 - \gamma_1}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}}\right)e^{\gamma_2(b-a)}e^{\gamma_1(t-a)} - \left(\frac{\gamma_2 - \gamma_1}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}}\right)e^{\gamma_1(b-a)}e^{\gamma_2(t-a)} \\
= \;& (\gamma_1 - \gamma_2)\left(\frac{e^{\gamma_1(b-a)}e^{\gamma_2(t-a)} - e^{\gamma_2(b-a)}e^{\gamma_1(t-a)}}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}}\right).
\end{aligned}
$$

Using the equalities $\gamma_1 = \frac{\rho_1-1}{\rho_1}, \gamma_2 = \frac{\rho_2-1}{\rho_2}$ and $\delta = \rho_1\rho_2\left(\gamma_1 - \gamma_2\right)$, we obtain

$$
\int_a^b |G(t,s)|\,ds = \frac{\phi(t)}{(\rho_1-1)(\rho_2-1)}, \tag{28}
$$

where

$$
\phi(t) = \frac{e^{\gamma_1(b-a)}e^{\gamma_2(t-a)} - e^{\gamma_2(b-a)}e^{\gamma_1(t-a)}}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}}, \quad t \in [a,b]. \tag{29}
$$

Obviously $\phi(a) = 1$ and $\phi(b) = 0$.

Now, for $t \in (a,b)$ we differentiate the function ϕ, we obtain

$$
\phi'(t) = \frac{\gamma_2 e^{\gamma_1(b-a)}e^{\gamma_2(t-a)} - \gamma_1 e^{\gamma_2(b-a)}e^{\gamma_1(t-a)}}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}}.
$$

The function $\phi'(t) = 0$ has unique solution at the point

$$
\hat{t} = b + \frac{\ln\frac{\gamma_2}{\gamma_1}}{(\gamma_1 - \gamma_2)} \text{ (here } \hat{t} > b\text{)}.
$$

Since ϕ is a continuous function on the interval $[a,b]$ and $\hat{t} > b$, which yields

$$
\max_{t\in[a,b]}\phi(t) = \phi(a) = 1. \tag{30}
$$

From (28) and (30) we conclude the formula (27). The proof is complete. □

3.1.2. Uniqueness result

In the result of uniqueness, we need the following assumption:
(H). Assume that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous function and satisfies a uniform Lipschitz condition with respect to the second variable on $[a,b] \times \mathbb{R}$ with Lipschitz constant K, that is,

$$
\left|f(t,\eta_1) - f(t,\eta_2)\right| \le K\left|\eta_1 - \eta_2\right|,
$$

for all $(t,\eta_1), (t,\eta_2) \in [a,b] \times \mathbb{R}$.

Theorem 3.7. *Let $\mathfrak{D} = {}_aD^{\rho_1}\ {}_aD^{\rho_2}$ and assume that* (*H*) *holds. If*

$$\frac{K}{(\rho_1 - 1)(\rho_2 - 1)} < 1, \tag{31}$$

then the fractional boundary value problem (1) has a unique solution on $[a, b]$ *for any values of A and B.*

Proof. Let $E = C([a, b], \mathbb{R})$ be the Banach space endowed with the norm $\|y\| = \sup_{t\in[a,b]} |y(t)|$, and we define the operator $N : E \to E$ by

$$Ny(t) = A + \frac{(B - A)\left(e^{\gamma_1(t-a)} - e^{\gamma_2(t-a)}\right)}{e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}} + \int_a^b G(t, s) f(s, y(s))ds, \tag{32}$$

where the function G is given by (14).

Notice that, the problem (1) (with $\mathfrak{D} = {}_a D^{\rho_1}\ {}_aD^{\rho_2}$) has a solution $y \in E$ if only if y is fixed point of the operator N.

Let $(t, y), (t, \overline{y}) \in [a, b] \times E$, we have

$$\begin{aligned} \left|Ny(t) - N\overline{y}(t)\right| &\leq \int_a^b G(t, s) \left|f(s, y(s)) - f(s, \overline{y}(s))\right| ds \\ &\leq \int_a^b KG(t, s) \left|y(s) - \overline{y}(s)\right| ds \\ &\leq K \int_a^b G(t, s)\, ds \left\|y - \overline{y}\right\|, \end{aligned}$$

using the Proposition 3.6, we get

$$\left\|Ny - N\overline{y}\right\| \leq \frac{K}{(\rho_1 - 1)(\rho_2 - 1)} \left\|y - \overline{y}\right\|. \tag{33}$$

The assumption (31) leads to principle of contraction mapping. Hence, N is contraction mapping, we conclude that the problem (1) with $\mathfrak{D} = {}_aD^{\rho_1}\ {}_aD^{\rho_2}$ has a unique solution. □

3.1.3. Lyapunov-type and Hartman-Wintner-type inequalities

We present the following Hartman-Wintner-type inequality for the fractional boundary value problem (2).

Theorem 3.8. *Let $\mathfrak{D} = {}_aD^{\rho_1}\ {}_aD^{\rho_2}$. If a nontrivial continuous solution to the problem (2) exists, then*

$$\int_a^b sign(\delta) \left[e^{\gamma_1(b-s)} - e^{\gamma_2(b-s)}\right] |q(s)| ds \geq \frac{\delta}{\varepsilon} \left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right], \tag{34}$$

where γ_1, γ_2 and δ are given in Lemma 3.3, and ε is difened by (20), and $sign(\delta) = \begin{cases} 1, & if\ \delta > 0, \\ -1, & if\ \delta < 0. \end{cases}$

Proof. From Lemma 3.3 (with $A = 0 = B$ and $P(t) = q(t)y(t)$), problem (2) is equivalent to the following integral equation

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds,$$

So, we have

$$|y(t)| \leq \int_a^b |G(t, s)||q(s)||y(s)|ds,$$

which yields

$$\|y\| \leq \|y\| \int_a^b \frac{\varepsilon \left[e^{\gamma_1(b-s)} - e^{\gamma_2(b-s)}\right]}{|\delta| \left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]} |q(s)| ds.$$

Since y is non trivial, with $|\delta| = \frac{\delta}{sign(\delta)}$, so

$$1 \leq \int_a^b \frac{\varepsilon sign(\delta) \left[e^{\gamma_1(b-s)} - e^{\gamma_2(b-s)}\right]}{\delta \left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]} |q(s)| ds,$$

from which the inequality in (34) follows. □

We have the following Lyapunov-type inequality.

Theorem 3.9. *Let* $\mathfrak{D} = {}_aD^{\rho_1}\ {}_aD^{\rho_2}$. *If a nontrivial continuous solution to the fractional boundary value problem (2) exists, then*

$$\int_a^b |q(s)| ds \geq \frac{\delta}{\varepsilon^2} \left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right], \tag{35}$$

where $\gamma_1 = \frac{\rho_1 - 1}{\rho_1}$ *and* $\gamma_2 = \frac{\rho_2 - 1}{\rho_2}$.

Proof. In the same way as above, by using the third property of Lemma 3.5, we get

$$1 \leq \int_a^b \frac{\varepsilon^2 |q(s)|}{\delta \left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]} ds, \tag{36}$$

from which the inequality in (35) follows. □

3.1.4. Lower bound for the eigenvalues

Consider the following sequential local fractional proportional eigenvalue problem involving two different orders $\rho_1, \rho_2 \in (0, 1)$

$$\begin{cases} \left({}_aD^{\rho_1}\ {}_aD^{\rho_2} y\right)(t) = \lambda y(t), \ t \in [a, b], \\ y(a) = 0 = y(b). \end{cases} \tag{37}$$

Then, we have the following resut:

Theorem 3.10. *If a nontrivial continuous solution to the fractional boundary value problem (37) exists, then*

$$|\lambda| \geq (\rho_1 - 1)(\rho_2 - 1). \tag{38}$$

Proof. From Lemma 3.3, the solution of problem (37) can be written as follows

$$y(t) = -\int_a^b \lambda G(t, s) y(s) ds.$$

Thus, for all $t \in [a, b]$, we have

$$\begin{aligned} |y(t)| &\leq |\lambda| \int_a^b |G(t, s)| |y(s)| ds \\ &\leq |\lambda| \|y\| \int_a^b |G(t, s)| ds \\ &\leq \frac{|\lambda| \|y\|}{(\rho_1 - 1)(\rho_2 - 1)}, \end{aligned}$$

which yields

$$\|y\| \leq \frac{|\lambda| \|y\|}{(\rho_1 - 1)(\rho_2 - 1)}.$$

Since y is non trivial, then $\|y\| \neq 0$, so

$$1 \leq \frac{|\lambda|}{(\rho_1 - 1)(\rho_2 - 1)}.$$

We obtain inequality (38). The proof is complete. □

3.1.5. Nonexistence results

We obtain the following result about the nonexistence for solutions of the fractional boundary value problems (2) with $\mathfrak{D} = {}_aD^{\rho_1}\, {}_aD^{\rho_2}$ and (37).

Theorem 3.11. *Let* $\mathfrak{D} = {}_aD^{\rho_1}\, {}_aD^{\rho_2}$. *If*

$$\int_a^b |q(s)| ds < \frac{\delta}{\varepsilon^2}\left[e^{\gamma_1(b-a)} - e^{\gamma_2(b-a)}\right]. \tag{39}$$

Then problem (2) has no nontrivial solution.

Proof. The proof follows from Theorem 3.9. □

Theorem 3.12. *If*

$$|\lambda| < (\rho_1 - 1)(\rho_2 - 1). \tag{40}$$

Then the boundary value problem (37) has no non-trivial solution.

Proof. The proof follows from Theorem 3.10. □

3.2. Case: $\mathfrak{D} = {}_aD^{\rho}$

Consider the following two-point local fractional proportional boundary value problem:

$$\begin{cases} ({}_aD^{\rho} y)(t) + P(t) = 0, \; t \in [a, b], \\ y(a) = A, y(b) = B, \; A, B \in \mathbb{R}, \end{cases} \tag{41}$$

where $1 < \rho < 2$ and $P : [a, b] \to \mathbb{R}$ is a continuous function.

The following result plays a key role in deriving the main results for the problems (1) and (2) with $\mathfrak{D} = {}_aD^{\rho}$.

Lemma 3.13. *Let* $\sigma = \frac{\rho-2}{\rho-1}$. *Problem (41) has the unique solution represented by the integral equation*

$$y(t) = A + \frac{(B - A)\left[e^{\sigma(t-a)} - 1\right]}{e^{\sigma(b-a)} - 1} + \int_a^b \widetilde{G}(t, s) P(s)\, ds, \tag{42}$$

where

$$\widetilde{G}(t, s) = \frac{1}{2 - \rho} \begin{cases} \frac{\left[1-e^{\sigma(t-a)}\right]\left[1-e^{\sigma(b-s)}\right]}{1-e^{\sigma(b-a)}} - \left[1 - e^{\sigma(t-s)}\right] := h_1(t, s), & a \leq s \leq t \leq b, \\ \frac{\left[1-e^{\sigma(t-a)}\right]\left[1-e^{\sigma(b-s)}\right]}{1-e^{\sigma(b-a)}} := h_2(t, s), & a \leq t \leq s \leq b. \end{cases} \tag{43}$$

Proof. Applying the operator ${}_aI^\rho$ on the differential equation $({}_aD^\rho y)(t) + P(t) = 0$, and using Lemma 3.2, we obtain

$$y(t) = c_0 + c_1 e^{\sigma(t-a)} - ({}_aI^\rho P)(t),$$

where $c_0, c_1 \in \mathbb{R}$.

By the formula (8) we get

$$y(t) = c_0 + c_1 e^{\sigma(t-a)} - \frac{1}{2-\rho}\int_a^t \left[1 - e^{\sigma(t-s)}\right] P(s)\,ds. \tag{44}$$

Using the boundary conditions $y(a) = A$ and $y(b) = B$, we obtain

$$c_1 = \frac{B-A}{e^{\sigma(b-a)} - 1} + \frac{1}{(2-\rho)\left[e^{\sigma(b-a)} - 1\right]}\int_a^b \left[1 - e^{\sigma(b-s)}\right] P(s)\,ds$$

and

$$c_0 = A - c_1$$

Substituting the values of c_1 and c_1 in (44) we get

$$\begin{aligned}
y(t) &= A + \frac{(B-A)\left[e^{\sigma(t-a)} - 1\right]}{e^{\sigma(b-a)} - 1} + \frac{\left[e^{\sigma(t-a)} - 1\right]}{(2-\rho)\left[e^{\sigma(b-a)} - 1\right]}\int_a^b \left[1 - e^{\sigma(b-s)}\right] P(s)\,ds \\
&\quad - \frac{1}{2-\rho}\int_a^t \left[1 - e^{\sigma(t-s)}\right] P(s)\,ds \\
&= A + \frac{(B-A)\left[e^{\sigma(t-a)} - 1\right]}{e^{\sigma(b-a)} - 1} + \frac{1}{(2-\rho)}\int_a^t \left(\frac{\left[1 - e^{\sigma(t-a)}\right]\left[1 - e^{\sigma(b-s)}\right]}{1 - e^{\sigma(b-a)}} - \left[1 - e^{\sigma(t-s)}\right]\right) P(s)\,ds \\
&\quad + \frac{1}{(2-\rho)}\int_t^b \frac{\left[1 - e^{\sigma(t-a)}\right]\left[1 - e^{\sigma(b-s)}\right]}{1 - e^{\sigma(b-a)}} P(s)\,ds \\
&= A + \frac{(B-A)\left[e^{\sigma(t-a)} - 1\right]}{e^{\sigma(b-a)} - 1} + \int_a^b \widetilde{G}(t,s) P(s)\,ds,
\end{aligned}$$

where $\widetilde{G}(t,s)$ is given by (43).

Conversely, any function y given by (42) is a solution of the boundary value problem shewed in (41). One can show this just by implementing D^ρ to both sides of (42) and finding the values of y at a and b. □

3.2.1. On the Green's function

Lemma 3.14. *The Green's function $\widetilde{G}$ defined in Lemma 3.13 satisfies the following properties:*
i). $\widetilde{G}(t,s) \geq 0$, for all $(s,t) \in [a,b] \times [a,b]$,
ii). $\max_{t\in[a,b]} |\widetilde{G}(t,s)| = \widetilde{G}(s,s)$, for all $s \in [a,b]$,
iii). $\max_{s\in[a,b]} |\widetilde{G}(s,s)| = \widetilde{G}(\frac{b+a}{2}, \frac{b+a}{2}) = \frac{1}{(2-\rho)\left[1-e^{\sigma(b-a)}\right]}\left(1 - e^{\sigma(\frac{b-a}{2})}\right)^2$.

Proof. For $a \leq t \leq s \leq b$, it can be easily checked that

$$0 \leq h_2(t,s) \leq h_2(s,s). \tag{45}$$

Next, for $a \le s \le t \le b$, we start by fixing an arbitrary $s \in [a,b]$. Differentiating the function $h_1(t,s)$ with respect to t , we get

$$\begin{aligned}\frac{\partial}{\partial t}h_1(t,s) &= \frac{-\sigma e^{\sigma(t-a)}\left[1-e^{\sigma(b-s)}\right]}{1-e^{\sigma(b-a)}}+\sigma e^{\sigma(t-s)}\\ &= -\sigma e^{\sigma(t-a)}\left(\frac{1-e^{\sigma(b-s)}}{1-e^{\sigma(b-a)}}-\frac{e^{\sigma(t-s)}}{e^{\sigma(t-a)}}\right)\\ &= -\sigma e^{\sigma(t-a)}\left(\frac{1-e^{\sigma(b-s)}}{1-e^{\sigma(b-a)}}-\frac{e^{-\sigma s}}{e^{-\sigma a}}\right).\end{aligned}$$

Because $\sigma < 0$, then we get $-\sigma e^{\sigma(t-a)} > 0$, $0 < \frac{1-e^{\sigma(b-s)}}{1-e^{\sigma(b-a)}} \le 1$ and $\frac{e^{-\sigma s}}{e^{-\sigma a}} \ge 1$. So, we obtain

$$\frac{\partial}{\partial t}h_1 \le 0, \tag{46}$$

i.e., h_1 is decreasing function with respect to t when $t \in [s,b]$, we get

$$0 = h_1(b,s) \le h_1(t,s) \le h_1(s,s). \tag{47}$$

By (45) and (47), we obtain

$$0 \le \widetilde{G}(t,s) \le \widetilde{G}(s,s). \tag{48}$$

Hence, the inequality (48) gives us the first and second properties.

Now, we prove the third property. We have

$$|\widetilde{G}(s,s)| = \frac{\Psi(s)}{(2-\rho)\left[1-e^{\sigma(b-a)}\right]}, \tag{49}$$

where

$$\Psi(s) = \left[1-e^{\sigma(s-a)}\right]\left[1-e^{\sigma(b-s)}\right],\ s \in [a,b].$$

It follows that we only need to get the maximum value of the function Ψ.
Obviously, $\Psi(a) = \Psi(b) = 0$. So, for $s \in (a,b)$, differentiate $\Psi(s)$.

$$\Psi'(s) = -\sigma e^{\sigma(s-a)}\left[1-e^{\sigma(b-s)}\right] + \sigma e^{\sigma(b-s)}\left[1-e^{\sigma(s-a)}\right].$$

On other hand we have

$$\begin{aligned}\Psi'(s) = 0 \quad &\Leftrightarrow e^{\sigma(s-a)}\left[1-e^{\sigma(b-s)}\right] = e^{\sigma(b-s)}\left[1-e^{\sigma(s-a)}\right]\\ &\Leftrightarrow e^{\sigma(s-a)} - e^{\sigma(b-a)} = e^{\sigma(b-s)} - e^{\sigma(b-a)}\\ &\Leftrightarrow s = \tfrac{b+a}{2}.\end{aligned}$$

Since Ψ is continuous function, we conclude that

$$\max_{s\in[a,b]}|\Psi(s)| = \Psi\left(\frac{b+a}{2}\right) = \left(1-e^{\sigma\left(\frac{b-a}{2}\right)}\right)^2. \tag{50}$$

From (49) and (50) we obtain the third property. □

Proposition 3.15. *The Green's function $\widetilde{G}$ defined in Lemma 3.13 has the following property*

$$\max_{t\in[a,b]}\int_a^b |\widetilde{G}(t,s)|ds = \frac{\widehat{\phi}}{2-\rho}, \tag{51}$$

where

$$\widehat{\phi} = \frac{1}{\sigma} - \frac{b-a}{e^{\sigma(b-a)}-1} - \frac{1}{\sigma}\ln\frac{\left(e^{\sigma(b-a)}-1\right)}{\sigma(b-a)}. \tag{52}$$

Proof. From the first proprety of Lemma 3.14 we get

$$\begin{aligned}\int_a^b |\widetilde{G}(t,s)|ds &= \frac{e^{\sigma(t-a)}-1}{(2-\rho)\left[e^{\sigma(b-a)}-1\right]}\int_a^b\left[1-e^{\sigma(b-s)}\right]ds \\ &\quad -\frac{1}{2-\rho}\int_a^t\left[1-e^{\sigma(t-s)}\right]ds \\ &= \frac{\left[e^{\sigma(t-a)}-1\right]}{(2-\rho)\left[e^{\sigma(b-a)}-1\right]}\left[(b-a)+\frac{1}{\sigma}\left(1-e^{\sigma(b-a)}\right)\right] \\ &\quad -\frac{1}{2-\rho}\left[t-a+\frac{1}{\sigma}\left(1-e^{\sigma(t-a)}\right)\right]\end{aligned}$$

Then we obtain after simplifications

$$\int_a^b |\widetilde{G}(t,s)|ds = \frac{\phi(t)}{2-\rho}, \tag{53}$$

where

$$\phi(t) = \frac{(b-a)}{e^{\sigma(b-a)}-1}e^{\sigma(t-a)} - t + a - \frac{b-a}{e^{\sigma(b-a)}-1}, \quad t\in[a,b]. \tag{54}$$

It follows that we only need to get the maximum value of the function $\phi(t)$. Obviously $\phi(a) = 0$ and $\phi(b) = 0$. Now, for $t \in (a,b)$ we differentiate the function ϕ, we obtain

$$\phi'(t) = \frac{\sigma(b-a)}{e^{\sigma(b-a)}-1}e^{\sigma(t-a)} - 1.$$

The function $\phi'(t) = 0$ has unique solution at the point

$$\hat{t} = a + \frac{1}{\sigma}\ln\frac{\left(e^{\sigma(b-a)}-1\right)}{\sigma(b-a)}.$$

Since ϕ is a continuous function on the interval $[a,b]$ with $\phi(a) = 0$ and $\phi(b) = 0$, which yields $\max_{t\in[a,b]}\phi(\hat{t}) = \phi(\hat{t})$. By substituting the value $\hat{t}$ in (54) we obtain

$$\max_{t\in[a,b]}\phi(t) = \widehat{\phi} = \frac{1}{\sigma} - \frac{b-a}{e^{\sigma(b-a)}-1} - \frac{1}{\sigma}\ln\frac{\left(e^{\sigma(b-a)}-1\right)}{\sigma(b-a)}. \tag{55}$$

From (53) and (55) we conclude the formula (51). The proof is complete. □

3.2.2. Uniqueness result

Theorem 3.16. *Let $\mathfrak{D} = {}_aD^\rho$ and assume that (H) holds. If*

$$\frac{\widehat{\phi}K}{2-\rho} < 1. \tag{56}$$

Then, the fractional boundary value problem (1) has a unique solution on $[a,b]$ for any values of A and B.

Proof. We define the operator $\widetilde{N} : E \to E$ by

$$\widetilde{N}y(t) = A + \frac{(B-A)\left[e^{\sigma(t-a)}-1\right]}{e^{\sigma(b-a)}-1} + \int_a^b \widetilde{G}(t,s)f(s,y(s))ds, \tag{57}$$

where the function $\widetilde{G}$ is given by (43).

Notice that problem (1) with $\mathfrak{D} = {}_aD^\rho$ has a solution $y \in E$ if only if y is fixed point of the operator $\widetilde{N}$. Let $(t,y),(t,\overline{y}) \in [a,b] \times E$, we have

$$\begin{aligned} \left|\widetilde{N}y(t) - \widetilde{N}\overline{y}(t)\right| &\leq \int_a^b \widetilde{G}(t,s)\left|f(s,y(s)) - f(s,\overline{y}(s))\right| ds \\ &\leq \int_a^b K\widetilde{G}(t,s)\left|y(s) - \overline{y}(s)\right| ds \\ &\leq K\int_a^b \widetilde{G}(t,s)\,ds\left\|y-\overline{y}\right\|, \end{aligned}$$

using Proposition 3.15, we get

$$\left\|\widetilde{N}y - \widetilde{N}\overline{y}\right\| \leq \frac{\widehat{\phi}K}{2-\rho}\left\|y-\overline{y}\right\|. \tag{58}$$

The condition (56) leads to principle of contraction mapping. Hence, $\widetilde{N}$ is contraction mapping, we conclude that the problem (1) with $\mathfrak{D} = {}_aD^\rho$ has a unique solution. □

3.2.3. Lyapunov-type and Hartman-Wintner-type inequalities

We present the following Hartman-Wintner-type inequality for the fractional boundary value problem (2).

Theorem 3.17. *Let $\mathfrak{D} = {}_aD^\rho$. If a nontrivial continuous solution to the problem (2) exists, then*

$$\int_a^b \left[1-e^{\sigma(s-a)}\right]\left[1-e^{\sigma(b-s)}\right]|q(s)|ds \geq (2-\rho)\left[1-e^{\sigma(b-a)}\right], \tag{59}$$

where $\sigma = \frac{\rho-2}{\rho-1}$.

Proof. From Lemma 3.13 (with $A = B = 0$ and $P(t) = q(t)y(t)$), problem (2) is equivalent to the following fractional integral equation

$$y(t) = \int_a^b \widetilde{G}(t,s)q(s)y(s)ds,$$

Then, the rest of the poof is similar to the poof Theorem 3.8 above. □

We have the following Lyapunov-type inequality.

Theorem 3.18. *Let $\mathfrak{D} = {}_aD^\rho$. If a nontrivial continuous solution to the fractional boundary value problem (2) exists, then*

$$\int_a^b \left|q(s)\right| ds \geq \frac{(2-\rho)\left[1-e^{\sigma(b-a)}\right]}{\left(1-e^{\sigma(\frac{b-a}{2})}\right)^2}. \tag{60}$$

where $\sigma = \frac{\rho-2}{\rho-1}$.

Proof. The proof follows by using the inequality (59) with the formula (50). □

3.2.4. Lower bound for the eigenvalues

Consider the following local fractional proportional eigenvalue problem

$$\begin{cases} ({}_aD^\rho y)(t) = \lambda y(t), 1 < \rho < 2, \quad t \in [a,b], \\ y(a) = 0 = y(b). \end{cases} \tag{61}$$

Then we have the following resut:

Theorem 3.19. *If a nontrivial continuous solution to the fractional boundary value problem (61) exists, then*

$$|\lambda| \geq \frac{2-\rho}{\widehat{\phi}}, \tag{62}$$

where $\widehat{\phi}$ *is given by (52).*

Proof. From Lemma 3.13, the solution of the problem (61) can be written as follows

$$y(t) = -\int_a^b \lambda \widetilde{G}(t,s)\, y(s)\, ds.$$

Then, the proof can be executed imititating the proof of Theorem 3.10 and using the formula (51). □

3.2.5. Nonexistence results

We obtain the following result about the nonexistence for solutions of the boundary value problems (61) and (2) with $\mathfrak{D} = {}_aD^\rho$.

Theorem 3.20. *Let $\mathfrak{D} = {}_aD^\rho$ and assume that*

$$\int_a^b |q(s)| ds < \frac{(2-\rho)\left[1-e^{\sigma(b-a)}\right]}{\left(1-e^{\sigma(\frac{b-a}{2})}\right)^2}. \tag{63}$$

Then problem (2) has no nontrivial solution.

Proof. The proof follows from Theorem 3.18. □

Theorem 3.21. *If*

$$|\lambda| < \frac{2-\rho}{\widehat{\phi}}, \tag{64}$$

then, the boundary value problem (61) has no non-trivial solution.

Proof. The proof follows from Theorem 3.19. □

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