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On fractional evolution equations with an extended $\Psi-$fractional derivative

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Abstract. This manuscript aims to highlight the existence and uniqueness results for a class of fuzzy nonlinear fractional evolution equations. Our approach is based on the application of an extended $\Psi-$Caputo fractional derivative of order $q \in (0, 1)$ valid on fuzzy functions paired with Banach contraction principle. As an example of application, we provide one at the end of this paper to show how the results can be used.

1. Introduction

The subject of fuzzy fractional calculus and its potential applications has received more attention in recent years, in large part because it has developed into a potent tool with accurate and successful results in simulating a number of complex phenomena in a variety of seemingly disparate and widespread fields of science and engineering (see [2, 11, 15, 24, 28, 31, 34, 35]). This theory has been proposed to handle uncertainty due to incomplete information that appears in many mathematical or computer models of some deterministic real-world phenomena. We begin by presenting an overview of fuzzy fractional differential equations in this paper.
Agarwal et al. [3] pioneered the introduction of fuzzy fractional calculus in one of the earliest papers to handle fractional-order systems with uncertain initial values or ambiguous connections between parameters. In order to establish the existence and uniqueness of fractional differential equations with uncertainty, Arshad and Lupulescu [8] used the results published in [3]. Then, Allahviranloo et al. [5] used the Riemann-Liouville extended H-differentiability to solve the fuzzy fractional differential equations (FFDEs) and reported some novel results in support of this idea. Salahshour et al. [36] solve several types of FFDEs based on the Riemann-Liouville fuzzy derivative and use the fuzzy Laplace transform approach. Fard et al. in [21] defined stability criteria for hybrid fuzzy systems on time scales in the Lyapunov sense based on the delta-Hukuhara derivative for fuzzy valued functions. In [30], Khastan et al. look at whether there are any solutions to a particular class of uncertain differential equations involving nonlocal derivatives. The method is founded on the use of an expanded Krasnoselskii fixed point theorem applicable to fuzzy metric spaces. This theorem leads them to conclude that the relevant problem has a fuzzy solution that is specified on a particular fuzzy interval. We direct readers to the articles for a variety of fundamental publications

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associated with fuzzy fractional differential equations and the theory of fractional differential equations [1, 17–20, 37, 40–42] and references therein.

Motivated by the above works, in the present paper, we investigate the existence and uniqueness of solutions for the following fuzzy fractional evolution problems

$$\begin{cases} {}^{c}D_{0^+}^{q} u(t) = \mathcal{A}(t)u(t), \quad t \in J = [0,T], \\ \\ u(0) = u_0, \end{cases} \tag{1}$$

and

$$\begin{cases} {}^{c}D_{0^+}^{q} u(t) = \mathcal{A}(t)u(t) + h(t,u(t)), \quad t \in J = [0,T], \\ \\ u(0) = u_0, \end{cases} \tag{2}$$

where ${}^{C}D_{0^+,gH}^{q,\Psi}$ is the extended $\Psi-$Caputo-Fuzzy fractional derivative of $u(t)$ at order $q \in (0,1)$, $T > 0$, h is a fuzzy continuous function and $\mathcal{A}(t)$ is a bounded linear operator. Our approach have not yet been studied, as far as the authors are knowledgeable.

The paper is structured as shown below. Section 2 provides some fundamental characteristics of fuzzy sets, fuzzy number operations, and precise definitions of fuzzy $\Psi-$fractional integral and $\Psi-$fractional derivative that will be used in the follow-up. We present the existence and uniqueness results of the solutions to the fractional evolution problems (1) and (2) in Sections 3 and 4. In Section 5, an interesting example is provided, and Section 6 concludes.

2. Basic definitions and notations

This section will provide a quick overview of some of the notations, terminology, and findings from the literature on fuzzy sets theory and fuzzy fractional calculus that will be used throughout the remainder of this work. For details, we refer to [9, 42]

Definition 2.1. *[42] A fuzzy number is mapping $u : \mathbb{R} \to [0,1]$ such that*

1. *u is upper semi-continuous,*
2. *u is normal,that is, there exist $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,*
3. *u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$,*
4. *$\overline{\{x \in \mathbb{R},\ u(x) > 0\}}$ is compact.*

The $\alpha - Cut$ of a fuzzy number u is defined as follows:

$$[u]^{\alpha} = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}.$$

The $\alpha - cut$ of fuzzy number u by $[u]^{\alpha} = [u_l(\alpha), u_r(\alpha)]$.
We denote by E^1 the collection of all fuzzy numbers and $\tilde{0}$ the fuzzy zero defined by

$$\tilde{0}(x) = \begin{cases} 1 & if \quad x = 0, \\ 0 & \quad elswhere. \end{cases}$$

• We denote by $C(J, E^1)$ the space of all fuzzy-valued functions which are continuous on J.
• We denote by $\mathbf{P}_c(\mathbb{R})$ the set of all bounded and closed intervals of $\mathbb{R}$.

Definition 2.2. *[23] Let $\alpha \in [0,1]$ and $u \in E^1$ such that $[u]^\alpha = [u_l(\alpha), u_r(\alpha)]$.*
We define the diameter of $\alpha-$ level set $[u]^\alpha$ of the fuzzy set u as follows:

$$d([u]^\alpha) = u_r(\alpha) - u_l(\alpha).$$

Definition 2.3. *[23] The generalized Hukuhara difference of two fuzzy numbers $u, v \in E^1$ is defined as follows:*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} i) \quad u = v + w, \\ or \\ ii) \quad v = u + (-w). \end{cases}$$

Proposition 2.4. *[10] If $u \in E^1$ and $v \in E^1$, then the following properties hold.*

1) *If $u \ominus_{gH} v$ exists then it is unique.*
2) *$u \ominus_{gH} u = \tilde{0}$.*
3) *$(u + v) \ominus_{gH} v = u$.*
4) *$u \ominus_{gH} v = \tilde{0} \Leftrightarrow u = v$.*

Definition 2.5. *[42] According to the Zadeh's extension principle, the addition on E^1 is defined by:*

$$(u \oplus v)(z) = \sup_{z=x+y} min\{u(x), v(y)\}.$$

And scalar multiplication of a fuzzy number is given by:

$$(k \odot u)(x) = \begin{cases} u(x/k) & ,k \in \mathbb{R} - \{0\}, \\ \tilde{0} & ,k = 0. \end{cases}$$

Remark 2.6. *[33] Let $u, v \in E^1$ and $\alpha \in [0,1]$, then we have*

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha.$$

$$[u - v]^\alpha = [u_1^\alpha - v_2^\alpha, u_2^\alpha - v_1^\alpha].$$

$$[ku]^\alpha = k[u]^\alpha = \begin{cases} \left[ku_1^\alpha, ku_2^\alpha\right] & if \quad k \geq 0, \\ \left[ku_2^\alpha, ku_1^\alpha\right] & if \quad k \leq 0. \end{cases}$$

$$[uv]^\alpha = [minu_1^\alpha v_1^\alpha, u_1^\alpha v_2^\alpha, u_2^\alpha v_1^\alpha, u_2^\alpha v_2^\alpha, maxu_1^\alpha v_1^\alpha, u_1^\alpha v_2^\alpha, u_2^\alpha v_1^\alpha, u_2^\alpha v_2^\alpha].$$

Definition 2.7. *[9] Let $u, v \in E^1$ and $\alpha \in [0,1]$, then the Hausdorf distance between u and v is given by:*

$$D(u, v) = \sup_{\alpha \in [0,1]} max\{| u_l(\alpha) - v_l(\alpha) |, | u_r(\alpha) - v_r(\alpha) |\}.$$

Proposition 2.8. *[25] D is a metric on E^1 and has the following properties:*

1. *(E^1, D) is a complete metric space.*
2. *$D(u + w, v + w) = D(u, v)$, $\forall u, v, w \in E^1$.*
3. *$D(ku, kv) = |k|D(u, v)$, $\forall u, v \in E^1$ and $k \in R$.*
4. *$D(u + w, v + z) \leq D(u, v) + D(w, z)$, $\forall u, v, w, z \in E^1$.*

Remark 2.9. *Let $x, y \in C([J, E^1)$. It is easy to see that the space $\left(C(J, E^1), D_s\right)$ is a Banach space where*

$$D_c(x, y) = \sup_{s \in J} D(u(s), v(s)).$$

Definition 2.10. *[14] Let $f : J \to E^1$ and $t_0 \in J$. We say that f is Hukuhara differentiable at t_0 if there exists $f'_{gH}(t_0) \in E^1$ such that:*

$$f'_{gH}(t_0) = \lim_{h \to 0^+} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h} = \lim_{h \to 0^-} \frac{f(t_0) \ominus_{gH} f(t_0 - h)}{h}.$$

Definition 2.11. *[25]*

1) *A function $F : J \longrightarrow E^1$ is strongly measurable if $\forall \alpha \in [0, 1]$, the set-valued mapping $F_\alpha : J \longrightarrow \mathcal{P}_c(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.*
2) *A function $F : J \longrightarrow E^1$ is called integrably bounded, if there exists an integrable function h such that, $|x| < h(t)$ $\forall x \in F_0(t)$.*

Definition 2.12. *Let $F : J \longrightarrow E^1$. The integral of F on J denoted by $\int_J F(t)dt$ is defined by*

$$\left[\int_J F(t)dt\right]^\alpha = \int_J F_\alpha(t)dt = \left\{\int_J f(t)dt \mid f : J \longrightarrow \mathbb{R} \text{ is a measurable selection for } F_\alpha\right\}.$$

for all $\alpha \in [0, 1]$.

Proposition 2.13. *[25] Let $F : J \longrightarrow E^1$ be a fuzzy function. If F is strongly measurable and integrably bounded then it is integrable.*

Definition 2.14. *[23] A fuzzy function $u : J \to E^1$ is colled d-increasing ,(d-decreasing)on J if for every $\alpha \in [0, 1]$ the real function $t \to d([u(t)]^\alpha)$ is nondecreasing ,(nonincreasing)respectively.*

Remark 2.15. *If $u : J \to E^1$ is d-increasing or d-decreasing on J,then we say that $u(t)$ is d-monotone on J.*

2.1. Fuzzy extended $\Psi-$fractional integrals and $\Psi-$fractional derivatives

Proposition 2.16. *[25]If $u \in E^1$ then the following properties hold*

(1) $[u]^\beta \subset [u]^\alpha \quad if \quad 0 \le \alpha \le \beta \le 1.$
(2) *If $\alpha_n \subset [0, 1]$ is a nondecreasing sequence which converges to α, then*

$$[u]^\alpha = \bigcap_{n \ge 1} [u]^{\alpha_n}.$$

Conversely, if $A^\alpha = \{[u_1^\alpha, u_2^\alpha]; \alpha \in [0, 1]\}$ is a family of closed real intervals verifying (1) and (2), then A^α defined a fuzzy number $u \in E^1$ such that $[u]^\alpha = A^\alpha$.

Let $0 < q < 1$ and $\Psi \in C^1(J, \mathbb{R}^+)$ such that $\Psi'(t) > 0$ for all $t \in J$. The left-sided Ψ-Riemann-Liouville fractional integral at order q of a real function $g : J \to \mathbb{R}$ is given by

$$I_{0^+}^{q,\Psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{q-1} g(s)ds.$$

See [7, 16] for more details on $\Psi-$fractional calculus.
Let $f(t) \in L\left(J, E^1\right)$ such that $f(t) = [f_1^\alpha(t), f_2^\alpha(t)]$.
Suppose that $f_1^\alpha, f_2^\alpha \in L(J, \mathbb{R})$ for all $\alpha \in [0, 1]$ and let

$$A^\alpha = \left[\frac{1}{\Gamma(q)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{q-1} f_1^\alpha(s)ds, \frac{1}{\Gamma(q)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{q-1} f_2^\alpha(s)ds\right], \tag{3}$$

where $\Gamma(.)$ is the Euler gamma function.

Lemma 2.17. *[8] The family $\{A^\alpha; \alpha \in [0,1]\}$ given by* (3)*, defined a fuzzy number $u \in E^1$ such that $[u]^\alpha = A^\alpha$.*

Definition 2.18. *Let $\alpha \in (0,1]$ and $h(t) \in L\left(J, E^1\right)$. The fuzzy $\Psi-$fractional integral at order $q \in (0,1)$ of h denoted by*

$$I^{q,\Psi}_{0^+,gH}h(t) = \frac{1}{\Gamma(q)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{q-1} \odot h(s)ds$$

is defined by

$$\left[I^{q,\Psi}_{0^+,gH}h(t)\right]^\alpha = \left[I^{q,\Psi}_{0^+}h^\alpha_1(t), I^{q,\Psi}_{0^+}h^\alpha_2(t)\right].$$

Proposition 2.19. *Let $f, g \in L(J, E^1)$ and $b \in E^1$, then we have:*

1. $I^{q,\Psi}_{0^+,gH}(bf)(t) = bI^{q,\Psi}_{0^+,gH}f(t)$.
2. $I^{q,\Psi}_{0^+,gH}(f + g)(t) = I^{q,\Psi}_{0^+,gH}f(t) + I^{q,\Psi}_{0^+,gH}g(t)$.
3. $I^{q,\Psi}_{0^+,gH}f(t) = I^{q_1+q_2,\Psi}_{0^+,gH}f(t)$,*where* $(q_1, q_2) \in [0,1]^2$.

Proof. See [7] □

Definition 2.20. *[36]Let $f \in C(J, E^1) \cap L(J, E^1)$. The function f is called fuzzy Caputo fractional differentiable of order $0 < q < 1$ at t if there exists an element ${}^CD^{q,\Psi}_{0^+,gH}g(t) \in E^1$ such that*

$${}^CD^{q,\Psi}_{0^+,gH}g(t) = \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} \odot g'_{gH}(s)ds.$$

Remark 2.21. *Let $[f(t)]^\alpha = \left[f^\alpha_1(t), f^\alpha_2(t)\right]$ for each $\alpha \in [0,1]$, then*

$$\left[{}^CD^{q,\Psi}_{0^+,gH}f(t)\right]^\alpha = \left[{}^cD^q_{0^+}f^\alpha_1(t), {}^cD^q_{0^+}f^\alpha_2(t)\right],$$

where

$${}^cD^q_{0^+}f^\alpha_1(t) := \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1}(f^\alpha_1)'(s)ds,$$

and

$${}^cD^q_{0^+}f^\alpha_2(t) := \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1}(f^\alpha_2)'(s)ds.$$

3. Fuzzy linear fractional evolution problem

Definition 3.1. *A fuzzy function $u(t)$ is a solution of the problem* (1) *if and only if*

1. *$u(t)$ is continuous and $u(t) \in D(A(t))$ for all $t \in J$,*
2. *${}^CD^{q,\Psi}_{0^+,gH}u(t)$ exists and continuous on J, where $0 < q < 1$,*
3. *$u(t)$ satisfies* (1).

Lemma 3.2. *A d-monotone fuzzy function $u(t)$ is a solution of the problem* (1) *if and only if*

1. *$u(t)$ is continuous and $u(t) \in D(\mathcal{A}(t))$ for all $t \in J$.*
2. *$u(t)$ satisfies the following integral equation*

$$u(t) \ominus_{gH} u_0 = \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} \odot \mathcal{A}(s)u(s))ds.$$

3. *The function* $t \to I^{q,\Psi}_{0^+,gH}\mathcal{A}(t)u(t)$ *is d-increasing on* J.

Proof. See the proof of Theorem 3 in [23]. □

We assume the following assumptions throughout the rest of this paper.

(H_1) $\mathcal{A}(t)$ is a bounded linear operator on E^1 for each $t \in J$.

(H_2) The function $t \to \mathcal{A}(t)$ is continuous.

(H_3) The function h is continuous and there exists a positive constant K such that

$$D_c(h(t,u),h(t,v)) \leq KD_c((u,v) \quad for \quad all \quad u,v \in C(J,E^1).$$

Theorem 3.3. *Assume that the hypotheses* $(H_1) - (H_2)$ *are verified, then* (1) *has a unique solution on* J.

Proof. By using Picard's iteration method [29] we prove that the problem (1) has a unique solution defined on J.
Let

$$A_0 = \sup_{s\in J} D(\mathcal{A}(s),\tilde{0}).$$

and let $\mathcal{T}_\Psi : C(J,E^1) \to C(J,E^1)$ be the operator defined as follows:

$$\mathcal{T}_\Psi u(t) \ominus_{gH} u_0 = \frac{1}{\Gamma(q)}\int_0^t (\Psi(t)-\Psi(s))^{q-1} \odot \mathcal{A}(s)u(s))ds.$$

Let $u,v \in C(J,E^1)$, then we have

$$D(\mathcal{T}_\Psi u(t),\mathcal{T}_\Psi v(t)) \leq \frac{A_0(\Psi(T)-\Psi(0))^q}{\Gamma(q+1)}D_c(\mathcal{T}_\Psi u,\mathcal{T}_\Psi v),$$

and by induction we can write

$$D(\mathcal{T}^n_\Psi u(t),\mathcal{T}^n_\Psi v(t)) \leq \frac{\left(\frac{A_0(\Psi(T)-\Psi(0))^q}{\Gamma(q+1)}\right)^n}{n!}D_c(\mathcal{T}^n_\Psi u,\mathcal{T}^n_\Psi v),$$

it follows that

$$D_c(\mathcal{T}^n_\Psi u,\mathcal{T}^n_\Psi v) \leq \frac{\left(\frac{A_0(\Psi(T)-\Psi(0))^q}{\Gamma(q+1)}\right)^n}{n!}D_c(\mathcal{T}^n_\Psi u,\mathcal{T}^n_\Psi v).$$

Finally, since $\frac{\left(\frac{A_0(\Psi(T)-\Psi(0))^q}{\Gamma(q+1)}\right)^n}{n!} < 1$ for large n. It follows from the Banach fixed point theorem that the operator $\mathcal{T}_\Psi$ has a unique fixed point u wich is the solution of the problem (1). □

4. Fuzzy quasilinear fractional evolution problem

Definition 4.1. *A fuzzy function* $u(t)$ *is said te be a solution of the problem* (2) *if and only if*

1. $u(t)$ *is continuous and* $u(t) \in D(\mathcal{A}(t))$ *for all* $t \in J$.
2. $u(t)$ *satisfies* (2).

Lemma 4.2. *A d-monotone fuzzy function* $u(t)$ *is a solution of* (2) *if and only if*

1. $u(t)$ *is continuous and* $u(t) \in D(\mathcal{A}(t))$ *for all* $t \in J$.

2. *$u(t)$ satisfies the following integral equation*

$$
\begin{aligned}
u(t) \ominus_{gH} u_0 = & \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} \odot \mathcal{A}(s)u(s))ds \\
& + \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} \odot h(s, u(s))ds.
\end{aligned}
$$

3. *The function $t \to I^q_{0^+}(\mathcal{A}(t)u(t) + h(t, u(t)))$is d-increasing on J.*

Proof. See the proof of Theorem 3 in [23]. □

Theorem 4.3. *The problem* (2) *has a unique solution on J if the assumptions $(H_1)-(H_3)$ are satisfied and the following inequality holds*

$$
\frac{(A_0 + K)(\Psi(T) - \Psi(0))^q}{\Gamma(q+1)} < \frac{1}{2}. \tag{4}
$$

Proof. Let $\mathcal{L}_\Psi : C(J, E^1) \to C(J, E^1)$ be the operator given by

$$
\begin{aligned}
\mathcal{L}_\Psi u(t) \ominus_{gH} u_0 = & \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} \odot \mathcal{A}(s)u(s))ds \\
& + \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} \odot h(s, u(s))ds.
\end{aligned}
$$

Let η be a positive real number such that $\eta \geq 2\left(D(u_0, \tilde{0}) + \dfrac{h_0(\Psi(T) - \Psi(0))^q}{\Gamma(q+1)}\right)$. where $h_0 = D_c(h, \tilde{0})$.

Let us show that $\mathcal{L}_\Psi B_\eta(\tilde{0}, \eta) \subset B_\eta(\tilde{0}, \eta)$ where

$$
B_\eta(\tilde{0}, \eta) = \{v \in C([J, E^1) : D(\tilde{0}, v) \leq \eta\}.
$$

Let $u \in B_\eta$ then we have

$$
\begin{aligned}
D(\mathcal{L}_\Psi u(t), \tilde{0}) \leq & D(x_0, \tilde{0}) + \frac{A_0}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} D(u(s), \tilde{0})ds \\
& + \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} D(h(s, u(s)), \tilde{0})ds \\
& \leq D(x_0, \tilde{0}) + \frac{\eta A_0(\Psi(T) - \Psi(0))^q}{\Gamma(q+1)} + \frac{(\eta K + h_0)(\Psi(T) - \Psi(0))^q}{\Gamma(q+1)}, \\
& \leq \eta.
\end{aligned}
$$

Thus $\mathcal{L}_\Psi$ maps B_η into itself.

Let $u, v \in C(J, E^1)$ we have

$$
\begin{aligned}
D(\mathcal{L}_\Psi v(t), \mathcal{L}_\Psi u(t)) &\leq \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} D(A(s)v(s), A(s)u(s)) ds, \\
&+ \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} D(h(s, v(s)), h(s, u(s))) ds, \\
&\leq \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} D(A(s)v(s), A(s)u(s)) ds, \\
&+ \frac{K}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} D(v(s), u(s)) ds, \\
&\leq \frac{(A_0 + K)(\Psi(T) - \Psi(0))^q}{\Gamma(q+1)} D_c(u, v), \\
&\leq \frac{(A_0 + K)(\Psi(T) - \Psi(0))^q}{\Gamma(q+1)} D_c(u, v),
\end{aligned}
$$

hence

$$
D_c(\mathcal{L}_\Psi u, \mathcal{L}_\Psi v) \leq \frac{1}{2} D_c(u, v),
$$

Thus, $\mathcal{L}_\Psi$ is a contraction mapping and hence te operator $\mathcal{L}_\Psi$ has a unique fixed point u wich is the solution of the problem (2). □

5. An illustrative example

Consider the following fuzzy fractional evolution problem

$$
\begin{cases}
{}^c D_{0^+}^{\frac{1}{2}, \sqrt{1+t}} u(t) = \dfrac{1}{10} sin(t) u(t) + \dfrac{1}{9+t} u(t), \qquad t \in [0, 1], \\
\\
u(0) = u_0.
\end{cases}
\tag{5}
$$

Here, we consider $\alpha = \frac{1}{2}, \Psi = \sqrt{1+t}, T = 1$ and $\mathcal{A}$ given by
$\mathcal{A}(t) = \frac{1}{10} sin(t)\ id$, where id is the identity mapping defined on E^1, and $h(t, u) = \dfrac{1}{9+t} u(t)$.
It's clear that $\mathcal{A}(t)$ is a bounded linear operator and since the function $t \to sin(t)$ is continuous $\mathcal{A}(t)$ is continuous and therefore (H_1) and (H_2) hold.
On the other hand, we have

$$
\begin{aligned}
D(h(t, u(t)), h(t, v(t))) &= D\left(\frac{1}{9+t} u(t), \frac{1}{9+t} v(t)\right), \\
D(h(t, u), h(t, v)) &\leq \frac{1}{10} D(u, v),
\end{aligned}
$$

it follows that (H_3) is verified and the Theorem 4.3 holds with $K = \dfrac{1}{10}$. It remains to check (4) in Theorem 4.3 is also satisfied.
For this purpose we have $K = A_0 = \frac{1}{10}$, and

$$
\left(\frac{(K + A_0)(\Psi(T) - \Psi(0))^\alpha}{\Gamma(\alpha + 1)} \right) = \left(\frac{(\frac{1}{10} + \frac{1}{10}) \times (\sqrt{2} - 1)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \right) = 0.14 < \frac{1}{2}.
$$

Finally, all the conditions of Theorem 4.3 are satisfied for the problem (5), then it has a unique solution on $[0, 1]$.

6. Conclusion

In this curent paper, we studied the existence and uniqueness of solutions for uncertain fractional evolution equations involving the extended $\Psi-$Caputo fractional derivative of an arbitrary order $q \in (0,1)$. The results are obtained by using fuzzy $\Psi-$Caputo fractional calculus and the well-known Banach fixed point theorem. We also provide an example to make our results clear.

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