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Stability properties of fractional second linear multistep methods in the implicit form: Theory and applications

Safar Irandoust-Pakchin[a,*], Somaiyeh Abdi-Mazraeh[a], Shahram Rezapour[b,c,*]

[a]*Department of Applied Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, University of Tabriz, Tabriz, Iran*
[b]*Department of Mathematics, Azarbaijan Shahid Madani, , Tabriz, Iran*
[c]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*

Abstract. The main purpose of this paper is to numerically solve the fractional differential equations (FDE)s with the fractional order in $(1, 2)$ using the implicit forms of the special case of fractional second linear multistep methods (FSLMM)s. The studies are focused on the stability properties and proving that the proposed methods are $A(\alpha)-$stable. For this purpose, after introducing the FSLMMs, the implicit family of FSLMMs based on fractional backward difference formula 1 (FBDF1) are constructed which have the first, and second order of convergence.The stability regions of the proposed methods are thoroughly studied. Furthermore, in order to show the validity of the proposed theories, some numerical examples are reported. Finally, the application of proposed method for solving the Bagley-Torvik (B-T) equation is also presented.

1. Introduction

The ability of FDEs for modeling many phenomena in the real world has made them appear strongly in life applications, and in the basic sciences such as mathematical, physical, computer, and chemistry sciences, biology, also engineering such as mechanics, electricity, and chemical engineering [8, 18, 21, 22]. Since the analytical solution of many FDEs is unavailable, the construction of effective numerical methods for them is an interesting topic for researchers in the fractional calculus [1–3, 7, 10–12, 14, 19, 20].

One of the main efficient methods for solving FDEs is the fractional linear multistep method (FLMMs) based on FBDFs which have high accuracy and high stability regions. These methods were firstly introduced by Lubich [20] for fractional order in $(0, 1)$. Garrappa et al. [10, 11] extended FLMMs as p-FLMMs in the explicit and implicit forms using FBDF1 of the first and second orders of consistency and convergence which have higher accuracy and larger stability regions. Farzaneh Bonab et.al [7] extended p-FLMMs in the explicit forms in the first, second, third, and four orders using FBDF3. Because of the more complexity of the method in the implicit forms, they did not try to study the stability properties and convergence analysis of their methods for the fractional order in $(0, 1)$. Note that these methods is only defined for the fractional

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* Corresponding authors: Safar Irandoust-Pakchin and Shahram Rezapour
Email addresses: s.irandoust@tabrizu.ac.ir (Safar Irandoust-Pakchin), s.abdi_m@tabrizu.ac.ir (Somaiyeh Abdi-Mazraeh), sh.rezapour@azaruniv.ac.ir (Shahram Rezapour)

order in $(0, 1)$. For the order of fractional differentiation in $(1, 2)$, there are a few numerical methods for solving FDEs such as shifted Grünwald-Letnikov and Adams methods [18], modified FBDF methods [9, 17]. Most recently, Irandoust et al. have designed the explicit p-FSLMMs based on FBDFs (FBDF1, FBDF2, and FBDF3), to numerically solve FDEs for fractional order in (1, 2) [15]. It is worth noting that as the FLMMs with the order in $(0, 1)$ is an extension of traditional LMMs, the p-FSLMMs with the order in $(1, 2)$ are an extension of traditional the SLMMs.

In this work, the new classes of implicit p-FSLMMs based on FBDF1 are constructed with the first and second orders of convergence for the fractional differentiation of order in $(1, 2)$. Moreover, the stability regions of the new methods are thoroughly studied and it results that the proposed methods are $A(\alpha)-$stable. Consequently, the proposed methods have the larger stability regions comparing previous methods.

This paper is organized as follows: In section 2, the SLMMs and their relations with p-FSLMMs are described. In Section 3, the implicit p–FSLMMs are constructed based on theories of pervious section and the stability properties of the proposed method are investigated. In Section 4, the application of proposed methods to solve FDEs, especially the Bagley-Torvik equation, confirms the proposed theories. Finally, a brief conclusion of the study is presented in the last section.

2. SLMMs and their relations with FSLMMs

The special case of the $n-$step SLMMs is as [13, 16]

$$\sum_{j=0}^{n} \alpha_j y_{n-j} = h^2 \sum_{j=0}^{n} \beta_j V_{n-j}, \tag{2.1}$$

for solving

$$\begin{cases} y''(t) = V(t, y), & t \in [t_0, T], \\ y(t_0) = y_{t_0},\ y'(t_0) = y'_{t_0}, \end{cases} \tag{2.2}$$

where $V : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ is a smooth function, $\alpha_n = 1$, α_0 and β_0 do not both vanish and $V_{n-j} = V(t_{n-j}, y(t_{n-j})),\ \ j = 0,\ 1, \ldots, n.$

Irandoust et al. [15] have extended the mentioned method and they called it the FSLMMs as

$$\sum_{j=0}^{n} \alpha_j^{(\upsilon)} y_{n-j} = h^{\upsilon} \sum_{j=0}^{n} \gamma_j^{(\upsilon)} V_{n-j}. \tag{2.3}$$

for solving FDEs in the Caputo fractional derivative

$$\begin{cases} {}_C\mathrm{D}^{\upsilon}_{t_0,t} y(t) = V(t, y), & 1 < \upsilon < 2,\ t \in (t_0, T], \\ y(t_0) = y_0,\ y'(t_0) = y'_0. \end{cases} \tag{2.4}$$

Also, similar p–FLMMs for $0 < \upsilon \le 1$ [7], they have utilized p–FSLMMs using FBDF1 for numerical solution of (2.4) in the following form

$$\sum_{j=0}^{n-1} w_j^{(\upsilon)} y_{n-j} = b_n y_0 + k_n y'_0 + h^{\upsilon} \sum_{j=0}^{p} \gamma_j^{(\upsilon)} V_{n-j}, \tag{2.5}$$

where $b_n = \sum\limits_{j=0}^{n-1} w_j^{(\upsilon)},\ k_n = h \sum\limits_{j=0}^{n} (n-j) w_j^{(\upsilon)},$ p is an integer and $1 < \upsilon < 2$. Moreover, the FBDF1 is applied as

$$W^{(\upsilon)}(x) = (1 - x)^{\upsilon} = \sum_{j=0}^{\infty} w_j^{(\upsilon)} x^j, \quad |x| < 1, \tag{2.6}$$

where $w_j^{(\upsilon)}$, $j = 0, 1, \dots$ can be obtained as follows [24]

$$\begin{aligned} & w_0^{(\upsilon)} = 1, \\ & w_j^{(\upsilon)} = (1 - \frac{1+\upsilon}{j}) w_{j-1}^{(\upsilon)}, \quad j = 1, 2, \dots \end{aligned} \tag{2.7}$$

2.1. Consistency and stability properties

The following theorem expresses the required conditions for consistency of p–FSLMMs (2.5).

Theorem 2.1. *([15]) The p–FSLMMs (2.5) with the coefficients of FBDF1 is consistent of order one when*

$$\sum_{j=0}^{p} \gamma_j^{(\upsilon)} = 1 \tag{2.8}$$

and if (2.8) holds, it is consistent of order two when

$$\sum_{j=0}^{p} j\gamma_j^{(\upsilon)} = \frac{\upsilon}{2}. \tag{2.9}$$

In order to study the linear stability of p–FSLMMs (2.5) , the following linear test problem can be considered

$$\begin{cases} {}_C D_{t_0,t}^{\upsilon} y(t) = \lambda y, & 1 < \upsilon \leq 2, \ t \in [t_0, T], \ \lambda \in \mathbb{C} \\ y(t_0) = y_0, \ y'(t_0) = y'_0. \end{cases} . \tag{2.10}$$

Theorem 2.2. *([15]) Let $1 < \upsilon \leq 2$, $\bar{h} = h^{\upsilon}\lambda$, $\gamma(x) = \sum_{j=0}^{p} \gamma_j^{(\upsilon)} x^j$ and $\{y_n\}$ is the numerical solution of (2.10) respected to defined p-FSLMMs (2.5). If*

$$W_1^{(\upsilon)}(\frac{1}{\hat{y}}) - \bar{h}\gamma(\frac{1}{\hat{y}}) \neq 0, \quad \hat{y} \in \mathbb{C}, \ |\hat{y}| > 1, \tag{2.11}$$

then $\lim_{n\to\infty} y_n = 0$.

Lemma 2.3. *([15]) For the coefficients (2.6) of the FBDF1, the following relationships hold when $1 < \upsilon < 2$:*

$$\begin{aligned} & 1)\ w_j^{(\upsilon)} > 0 \text{ and } w_{j+1}^{(\upsilon)} < w_j^{(\upsilon)} < \dots < w_0^{(\upsilon)} = 1, \ j = 2, 3, \dots \\ & 2)\ \sum_{j=0}^{\infty} w_j^{(\upsilon)} = 0. \end{aligned} \tag{2.12}$$

Theorem 2.4. *([20]) A convolution quadrature $w^{(\upsilon)}$ is convergent of first and second order if and only if it is stable and is consistent of first and second order, respectively.*

In accordance with the mentioned theorems and lemma in this section, it is derived that the p-FSLMMs (2.5) are convergent.

Definition 2.5. *let $\mathbb{S}$ is the stability domain of a numerical method. The method is called $A(\alpha)$-stable if $\mathbb{S}(\alpha) \subseteq \mathbb{S}$ wherein $\mathbb{S}(\alpha) = \{y \in \mathbb{C} : |\arg(y) - \pi| < \alpha\}$.*

3. Implicit p-FSLMMs based on FBDF1

In this section, some implicit p-FSLMMs with the coefficients of FBDF1 are constructed. Also, the stability properties of the proposed methods are considered. Using Theorem (2.1), one can construct some implicit p-FSLMMs of order one, two as follows

$$\sum_{j=0}^{n-1} w_j^{(\upsilon)} y_{n-j} = b_n y_0 + k_n y_0' + h^{\upsilon} V(t_n, y_n), \tag{3.13}$$

$$\sum_{j=0}^{n-1} w_j^{(\upsilon)} y_{n-j} = b_n y_0 + k_n y_0' + h^{\upsilon}\Big[(1-\frac{\upsilon}{4})V(t_n, y_n) + \frac{\upsilon}{4} V(t_{n-2}, y_{n-2})\Big]. \tag{3.14}$$

The stability regions of the constructed implicit methods have been plotted in Figures 1-2. Moreover, in Figure 3, the stability regions of proposed methods are compared with explicit FSLMMs [15] for $\upsilon = 1.3$. In this figure, the left and right columns show the mentioned first and second order methods, respectively. Note that the stability regions are inside of the closed regions for explicit FSLMMs [15] and outside of the closed regions for implicit proposed methods. Seeing Figure 3, it is clear that the proposed methods have the wider stability regions than the explicit FSLMMs and this is one of the most important strengths of the proposed methods compared to the mentioned methods which is fully illustrated in Example 1 of section 4.

Lemma 3.1. *Let $1 < \upsilon < 2$ and $G(x)$ is as $G(x) = \dfrac{(1-x)^{\upsilon}}{\gamma(x)}$ where*

$$\gamma(x) = \begin{cases} 1 & \text{if } p = 0 \\ (1-\frac{\upsilon}{4}) + \frac{\upsilon}{4}x^2 & \text{if } p = 2 \end{cases}, \quad x \in \mathbb{C} \text{ such that } |x| \le 1, \tag{3.15}$$

then

$$\Im\mathrm{m}(x) = 0,\ \Re\mathrm{e}(x) \in [-1, 1] \ \text{ if and only if } \ \Im\mathrm{m}(G(x)) = 0,\ \Re\mathrm{e}(G(x)) \ge 0.$$

Proof. The direct conditional proposition is quite obvious. The proof by contradiction is used to confirm the correctness of the inverse conditional proposition. Suppose $|x| \le 1$ such that $\Re\mathrm{e}(x) \in [-1, 1]$ and $\Im\mathrm{m}(x) > 0$. The fuction $G(x)$ is rewritten as follows;

$$G(x) = \frac{(1-x)^{\upsilon}}{\gamma(x)} = (1-x)^{\upsilon-1}\frac{(1-x)}{\gamma(x)}.$$

Therefore, one can conclude

$$\arg(G(x)) = (\upsilon - 1)\arg(1-x) - \arg(\frac{\gamma(x)}{1-x}). \tag{3.16}$$

Now, suppose $x = e^{i\psi}$. Then one can derive

$$1 - x = \rho e^{i\mu}, \ \text{ such that } \ \rho = 2\sin(\frac{\psi}{2}), \ \ \mu = \frac{\psi}{2} - \frac{\pi}{2}. \tag{3.17}$$

Since $\Im\mathrm{m}(x) > 0$, it is resulted $\arg(x) = \psi \in (0, \pi)$ as $\mu \in (\frac{-\pi}{2}, 0)$. On the other hand, according to the assumption it is concluded that $\arg(G(x)) = 0$. Therefore, from Eq.(3.16) it can be concluded that

$$\arg(\frac{\gamma(x)}{1-x}) = \Big((1-\upsilon)\frac{\pi}{2}, 0\Big). \tag{3.18}$$

Assuming $f(x) = \frac{\gamma(x)}{1-x} = (1-x)^{-1}\gamma(x)$ and substituting $x = e^{i\psi}$, it can be written

$$f(x) = \rho^{-1} e^{-i\mu}\gamma(x). \tag{3.19}$$

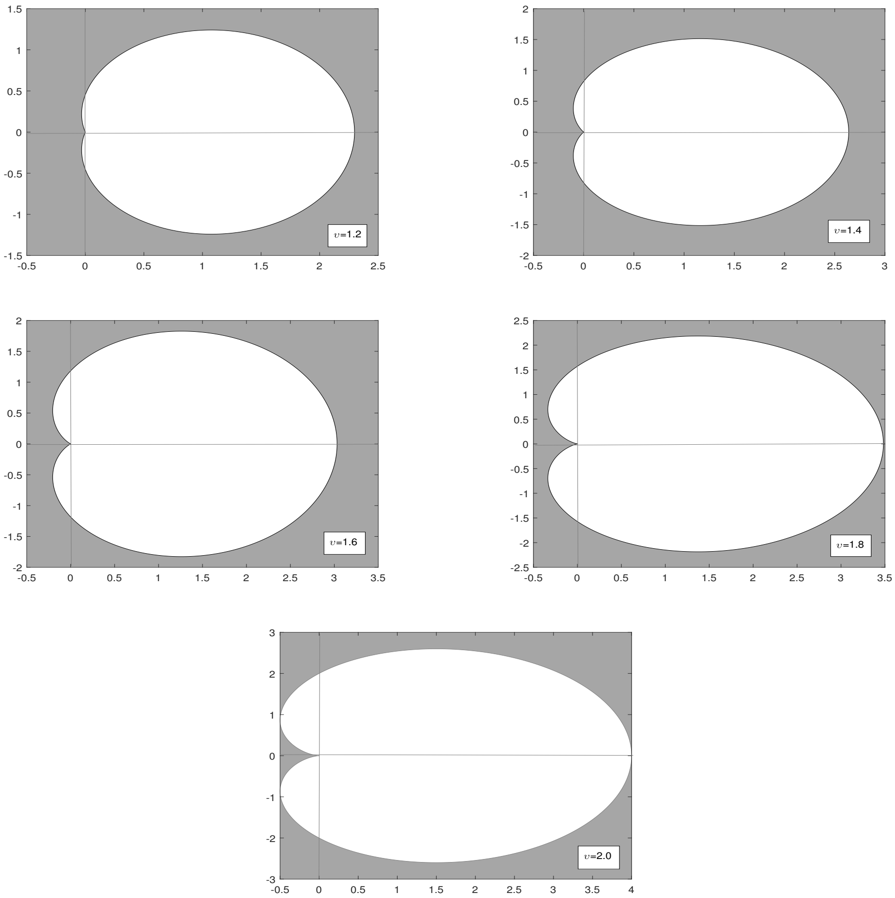

Figure 1: Stability regions (the grey areas) for the proposed method in Eq.(3.13) for $\upsilon = 1.2,\ 1.4,\ 1.6,\ 1.8,\ 2$.

Now, consider the following two cases;
case 1: *In this case put* $p = 0$ *in Eq.(3.15). So, Eq.(3.19) is written as*

$$\begin{aligned} & f(x) = \rho^{-1}e^{-i\mu} \times 1 = \rho^{-1}[\cos(-\mu) + i\sin(-\mu)] \\ & \Rightarrow \mathfrak{Im}(f(x)) = -\rho^{-1}\sin(\mu) = 2^{-1}\cot(\frac{\psi}{2}) > 0. \end{aligned} \tag{3.20}$$

From Eq.(3.20), it can be derived $\arg(f(x)) \in (0, \pi)$ *which is the contradiction with Eq.(3.18).*
case 2: *In this case put* $p = 2$ *in Eq.(3.15). So, Eq.(3.19) is written as*

$$\begin{aligned} f(x) &= \rho^{-1}e^{-i\mu}\Big[(1 - \frac{\upsilon}{4}) + \frac{\upsilon}{4}e^{2i\psi}\Big], \\ \Rightarrow \mathfrak{Im}(f(x)) &= \rho^{-1}\Big[(1 - \frac{\upsilon}{4})\sin(-\mu) + \frac{\upsilon}{4}\sin(2\psi - \mu)\Big] \\ &= 2^{-1}\sin(\frac{\psi}{2})\Big[(1 - \frac{\upsilon}{4})\cos(\frac{\psi}{2}) + \frac{\upsilon}{4}\cos(\frac{3\psi}{2}) \\ &\geq 2^{-1}(-1)\Big[(1 - \frac{\upsilon}{4})(-1) + \frac{\upsilon}{4}(-1)\Big] \geq 0 \end{aligned} \tag{3.21}$$

From Eq.(3.21) we obtain $\arg(f(x)) \in (0, \pi)$ *which is the contradiction with Eq.(3.18).* □

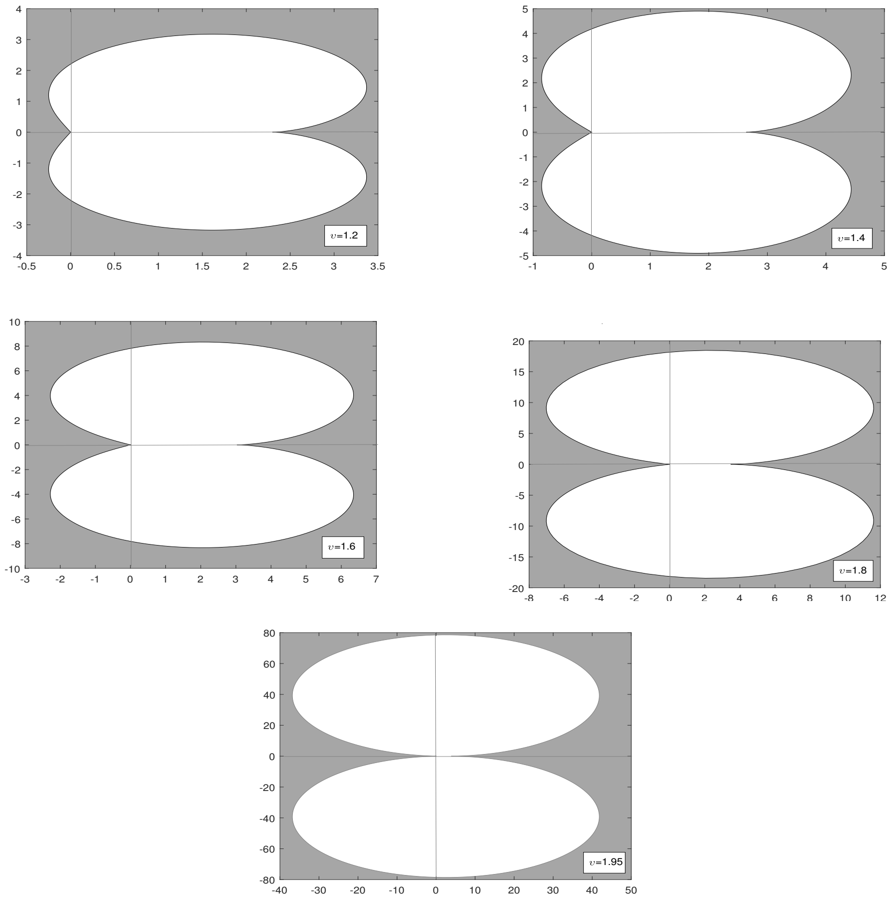

Figure 2: Stability regions (the grey areas) for the proposed method in Eq.(3.14) for $\upsilon = 1.2,\ 1.4,\ 1.6,\ 1.8,\ 1.95$.

Remark 3.2. *Let $x = x_0 + iq$ and $|x_0| \leq 1$. Since $q = 0$, according to Lemma 3.1 it can be derived $G(x) \in \mathbb{R}$ and $G(x) \geq 0$. Therefore, considering $G(x) = G(x_0) = \frac{(1-x_0)}{\gamma(x_0)}$, it can be concluded that*

$$G(x_0) \geq 0 \quad \textit{if and only if} \quad \gamma(x_0) > 0, \quad \textit{for} \quad |x_0| \leq 1. \tag{3.22}$$

For example, using Theorem 3.4, one can construct the other implicit p-FSLMMs such as

$$\sum_{j=0}^{n-1} w_j^{(\upsilon)} y_{n-j} = b_n y_0 + k_n y_0' + h^{\upsilon}\Big[(1-\frac{\upsilon}{2})V(t_n, y_n) + \frac{\upsilon}{2}V(t_{n-1}, y_{n-1})\Big]. \tag{3.23}$$

In the method (3.23), it can be easily checked that

$$\gamma(x_0) = (1-\frac{\upsilon}{2}) + \frac{\upsilon}{2}x_0 \not> 0 \quad \text{for} \quad |x_0| \leq 1.$$

Figure 4 shows that the proposed method (3.23) has no stability regions for various υ because of no satisfying in Remark 3.2.

Lemma 3.1 shows that if $\Im\mathfrak{m}(x) = 0$ then $\Im\mathfrak{m}(G(x)) = 0$. Now, to show the behavior of the $\Im\mathfrak{m}(G(x))$ when $\Im\mathfrak{m}(x) \neq 0$, that plays a very important role in order to obtain the stability regions of the proposed methods, the following lemma is presented.

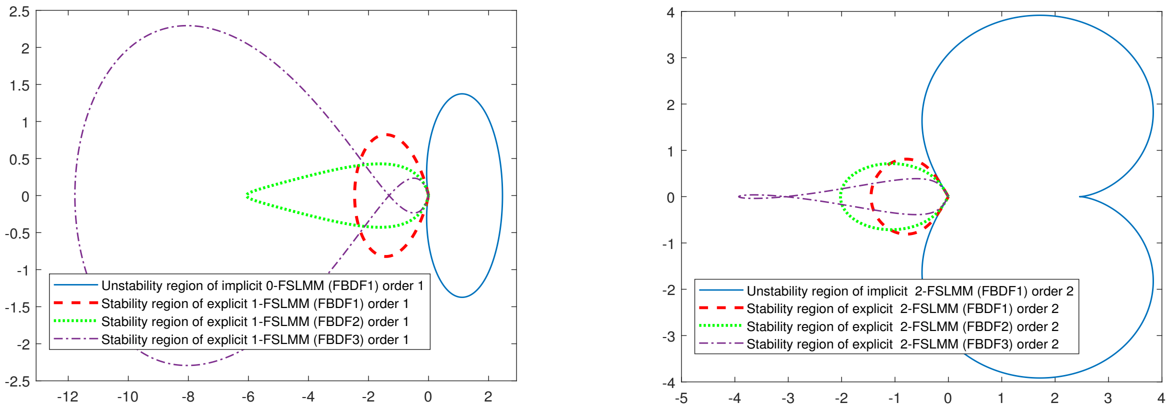

Figure 3: Comparison the stability regions of implicit proposed methods (outside of the closed regions) and explicit methods [15] (inside of the closed regions) for $\upsilon = 1.3$.

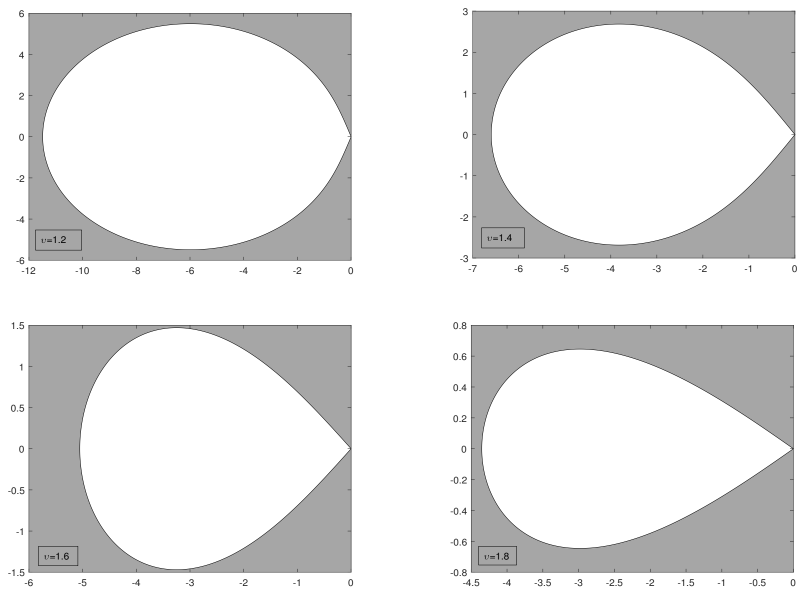

Figure 4: Stability regions (the grey areas) for the proposed method in Eq.(3.23) for $\upsilon = 1.2,\ 1.4,\ 1.6,\ 1.8$.

Lemma 3.3. *For the p-FSLMMs (3.13)-(3.14) it is:* $\Im\mathfrak{m}(G(x)) \leq 0$ *when* $\arg(x) \in [0, \pi]$ *and* $\Im\mathfrak{m}(G(x)) \geq 0$ *when* $\arg(x) \in [\pi, 2\pi]$.

Proof. *Assuming* $x = e^{i\psi}$ *and using Eq.(3.17), consider the following two cases;*
case1: *Put* $p = 0$ *in Eq.(3.15). Therefore, it can be written that*

$$G(x) = \frac{(1-x)^{\upsilon}}{1} = \rho^{\upsilon} e^{i\upsilon\mu}, \tag{3.24}$$
$$\Rightarrow \Im\mathfrak{m}(G(x)) = (2\sin(\frac{\psi}{2}))^{\upsilon} \sin(\upsilon\mu).$$

Since $\psi = \arg(x) \in [0,\pi]$, *it can be concluded* $\upsilon\mu \in [-\frac{\upsilon\pi}{2}, 0]$ *and* $\Im m G(x) \leq 0$.

case2: *Put* $p = 2$ *in Eq.(3.15). Therefore, one can write that*

$$\begin{aligned} G(x) &= \frac{(1-x)^{\upsilon}}{(1-\frac{\upsilon}{4}) + \frac{\upsilon}{4}x^2} = \frac{\rho^{\upsilon}\cos(\upsilon\mu) + i\rho^{\upsilon}\sin(\upsilon\mu)}{(1-\frac{\upsilon}{4}) + \frac{\upsilon}{4}\cos(2\psi) + i\frac{\upsilon}{4}\sin(2\psi)} \\ &= \frac{ac+bd}{c^2+d^2} + i\frac{bc+ad}{c^2+d^2}, \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} &a = \rho^{\upsilon}\cos(\upsilon\mu), \ \ b = \rho^{\upsilon}\sin(\upsilon\mu), \\ &c = (1-\frac{\upsilon}{4}) + \frac{\upsilon}{4}\cos(2\psi), \ \ d = \frac{\upsilon}{4}\sin(2\psi). \end{aligned}$$

From Eq.(3.25), it is clear that

$$\Im m(G(x)) = \frac{bc+ad}{c^2+d^2} = \frac{\rho^{\upsilon}(1-\frac{\upsilon}{4})\sin(\upsilon\mu) - \frac{\upsilon}{4}\rho^{\upsilon}\sin(2\psi-\upsilon\mu)}{c^2+d^2}. \tag{3.26}$$

Because $\psi \in [0,\pi]$ *then* $\sin(\upsilon\mu) < 0$ *and* $\sin(2\psi - \upsilon\mu) > 0$. *Therefore, considering Eq.(3.26), one can derive that* $\Im m(G(x)) < 0$. *Similarly, the lemma can be proved for* $\psi \in [\pi, 2\pi]$. □

Theorem 3.4. *Let* $\upsilon \in (1,2)$ *and* $|x| \leq 1$. *For the p-FSLMMs (3.13)-(3.14) it results*

$$\begin{cases} \arg(G(x)) \leq \pi - \alpha, & when \ \ \Im m(G(x)) \geq 0 \\ \\ \arg(G(x)) \geq \alpha - \pi, & when \ \ \Im m(G(x)) \leq 0, \end{cases} \tag{3.27}$$

where

$$\alpha \leq \pi - \max_{\psi\in[\pi,2\pi]} \arg(G(e^{i\psi})). \tag{3.28}$$

Proof. *Suppose* $\arg(x) \in [\pi, 2\pi]$. *According to Lemma 3.3, it is clear that* $\Im m(G(x)) \geq 0$. *In p-FSLMM (3.13) the function* $G(x)$ *is as* $G(x) = (1-x)^{\upsilon}$ *that substituting* $x = e^{i\psi}$ *for it (see Eq.(3.24)), one can result that*

$$F(\psi) = \arg(G(e^{i\psi})) = \arctan\left(\frac{\sin(\upsilon(\frac{\psi}{2}-\frac{\pi}{2}))}{\cos(\upsilon(\frac{\psi}{2}-\frac{\pi}{2}))}\right) = \upsilon(\frac{\psi}{2} - \frac{\pi}{2}).$$

The function $F(\psi)$ *is a monotonically increasing function in the interval* $[\pi, 2\pi]$. *As a result*

$$\max_{\psi\in[\pi,2\pi]} F(\psi) = F(2\pi) = \frac{\upsilon\pi}{2} \ \Rightarrow \max_{\psi\in[\pi,2\pi]} \arg(G(e^{i\psi})) = \frac{\upsilon\pi}{2}. \tag{3.29}$$

In p-FSLMM (3.14) the function $G(x)$ *is as* $\frac{(1-x)^{\upsilon}}{(1-\frac{\upsilon}{4})+\frac{\upsilon}{4}x^2}$ *that substituting* $x = e^{i\psi}$ *for it (see Eq.(3.25)), it can be concluded that*

$$\begin{aligned} f(\psi) = \arg(G(e^{i\psi})) &= \arctan\left(\frac{bc+ad}{ac+bd}\right) \\ &= \arctan\left(\frac{(1-\frac{\upsilon}{4})\sin(\upsilon\mu) - \frac{\upsilon}{4}\sin(2\psi-\upsilon\mu)}{(1-\frac{\upsilon}{4})\cos(\upsilon\mu) + \frac{\upsilon}{4}\cos(2\psi-\upsilon\mu)}\right) \\ &= \arctan(U(\psi)), \\ \Rightarrow f'(\psi) &= \frac{U'(\psi)}{1+U^2(\psi)}. \end{aligned} \tag{3.30}$$

Considering Eq.(3.30), it is clear that to determine the sign of $f'(\psi)$, it is enough to know the sign of $U'(\psi)$. Therefore, it can be written that

$$\begin{aligned}
&U(\psi) = \frac{(1-\frac{\upsilon}{4})\sin(\upsilon\mu) - \frac{\upsilon}{4}\sin(2\psi-\upsilon\mu)}{(1-\frac{\upsilon}{4})\cos(\upsilon\mu) + \frac{\upsilon}{4}\cos(2\psi-\upsilon\mu)} = \frac{V(\psi)}{M(\psi)},\\
&\Rightarrow U'(\psi) = \frac{1}{M^2(\psi)}\Big[\Big((1-\frac{\upsilon}{4})\frac{\upsilon}{2}\cos(\upsilon\mu) - \frac{\upsilon}{4}(2-\frac{\upsilon}{2})\cos(2\psi-\upsilon\mu)\Big)\times\Big((1-\frac{\upsilon}{4})\cos(\upsilon\mu)\\
&+\frac{\upsilon}{4}\cos(2\psi-\upsilon\mu)\Big) - \Big(-(1-\frac{\upsilon}{4})\frac{\upsilon}{2}\sin(\upsilon\mu) - \frac{\upsilon}{4}(2-\frac{\upsilon}{2})\sin(2\psi-\upsilon\mu)\Big)\times\Big((1-\frac{\upsilon}{4})\\
&\times\sin(\upsilon\mu) - \frac{\upsilon}{4}\sin(2\psi-\upsilon\mu)\Big)\Big] = \frac{1}{M^2(\psi)}\Big[(1-\frac{\upsilon}{4})^2\frac{\upsilon}{2} + (1-\frac{\upsilon}{4})\frac{\upsilon^2}{8}\Big(\cos(\upsilon\mu)\cos(2\psi-\upsilon\mu)\\
&-\sin(\upsilon\mu)\sin(2\psi-\upsilon\mu)\Big) - \frac{\upsilon}{4}(2-\frac{\upsilon}{2})(1-\frac{\upsilon}{4})\Big(\cos(2\psi-\upsilon\mu)\cos(\upsilon\mu) - \sin(2\psi-\upsilon\mu)\\
&\times\sin(\upsilon\mu)\Big)\Big] = \frac{(1-\frac{\upsilon}{4})^2\frac{\upsilon}{2} - (\frac{\upsilon}{4})^2(2-\frac{\upsilon}{2}) + \cos(2\psi)[(1-\frac{\upsilon}{4})\frac{\upsilon^2}{8} - \frac{\upsilon}{4}(2-\frac{\upsilon}{2})(1-\frac{\upsilon}{4})]}{M^2(\psi)}\\
&= \frac{(\frac{\upsilon}{2} - \frac{3}{8}\upsilon^2 + \frac{1}{16}\upsilon^3)(1-\cos 2\psi)}{M^2(\psi)} = \frac{(\upsilon - \frac{3}{4}\upsilon^2 + \frac{1}{8}\upsilon^3)sin^2(\psi)}{M^2(\psi)} \geq 0, \ \ when \ \ \upsilon \in (1,2).
\end{aligned} \tag{3.31}$$

Now, from Eq.(3.30) and Eq.(3.31), it can be derived $f'(\psi) \geq 0$ for $\psi \in [\pi, 2\pi]$. Therefore, one can write

$$\max_{\psi\in[\pi,2\pi]} f(\psi) = f(2\pi) = \frac{\upsilon\pi}{2} \ \ \Rightarrow \max_{\psi\in[\pi,2\pi]} \arg(G(e^{i\psi})) = \frac{\upsilon\pi}{2}. \tag{3.32}$$

Finally, by using Eqs.(3.28) and (3.32), when $\Im\mathrm{m}(G(x)) \geq 0$ one can write

$$\arg(G(x)) \leq \frac{\upsilon\pi}{2} \Rightarrow \pi - \arg(G(x)) \geq \pi - \frac{\upsilon\pi}{2} \geq \alpha \Rightarrow \arg(G(x)) \leq \pi - \alpha \tag{3.33}$$

Similarly, the theorem can be proved for $\Im\mathrm{m}(G(x)) \leq 0$. □

Theorem 3.5. *let $\alpha \in [0, \frac{\pi}{2}]$. The p-FSLMMs (3.13)-(3.14) are $A(\alpha)$-stable for any α such that*

$$\alpha \leq \pi - \max_{\psi\in[\pi,2\pi]} \arg(G(e^{i\psi})) = (2-\upsilon)\frac{\pi}{2}.$$

Proof. *The proof is easily obtained by using Theorem 3.4.* □

In Table 1, the values of α of A(α)−stability for the p-FSLMMs (3.13)-(3.14) are listed with the different values of υ. This table confirms Theorem 3.5. Considering Table 1, it is clear that the stability regions of the method (3.13) are larger than the stability regions of the method (3.14).

Table 1: Angles α of $A(\alpha)$–stability for p-FSLMMs (3.13) − (3.14).

υ	α for (3.13)	α for (3.14)	$(2-\upsilon)\frac{\pi}{2}$
1.0	90.00	90.00	90.00
1.1	77.41	76.05	80.99
1.2	65.33	63.95	72.00
1.3	55.02	53.53	62.99
1.4	45.50	44.17	54.00
1.5	36.76	35.33	45.00
1.6	29.01	27.55	35.99
1.7	21.64	20.18	27.00
1.8	14.85	13.30	17.99
1.9	07.97	06.67	09.00
2.0	00.18	00.00	00.00

4. Numerical examples

In this section, some numerical tests are provided to confirm the accuracy and efficiency of the proposed methods. Also, the application of the proposed methods for solving the Bagley-Torvik (B-T) equation is presented.

Example 1: Consider the FDE as

$$\begin{aligned}
&{}_C\mathrm{D}^{\upsilon}_{0,t}y(t) = \lambda y(t), \;\; 1 < \upsilon < 2, \quad t \in (0,1], \\
&y(0) = 1, \;\; y'(0) = 0.
\end{aligned} \tag{4.34}$$

The problem (4.34) have the exact solution as $y(t) = E_{\upsilon,1}(\lambda t^{\upsilon}) = \sum_{k=0}^{\infty} \frac{(\lambda t^{\upsilon})^k}{\Gamma(\upsilon k+1)}$. In Tables 2 and 3, the proposed methods are compared with the explicit FSLMMs of same order at $t = 1$ for $\upsilon = 1.05$ and various values of h. Considering these tables, it results that due to being the $A(\alpha)$-stable, the proposed methods are more effective than explicit FSLMMs [15].

Table 2: Comparison of error between the proposed methods and explicit FSLMMs [15] with order 1 for $\lambda = -100$ and $\upsilon = 1.05$.

h	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{50}$
Explicit 1-FSLMM(FBDF1) order 1 [15]	6.42e+1	6.83e+5	3.08e+5	2.80e-5
Explicit 1-FSLMM(FBDF2) order 1 [15]	1.19e-0	3.07e-0	2.77e-6	1.83e-6
Explicit 1-FSLMM(FBDF3) order 1 [15]	1.45e-0	7.06e-4	8.44e-4	1.23e-5
Implicit 0-FSLMM(FBDF1) order 1	5.39e-5	2.20e-5	1.33e-5	7.83e-6

Table 3: Comparison of error between the proposed methods and explicit FSLMM [15] with order 2 for $\lambda = -30$ and $\upsilon = 1.05$.

h	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{25}$	$\frac{1}{35}$
Explicit 2-FSLMM(FBDF1) order 2 [15]	6.61e-0	3.06e+1	6.76e-1	3.48e-6
Explicit 2-FSLMM(FBDF2) order 2 [15]	2.33e-0	1.01e-0	3.66e-5	2.52e-6
Explicit 2-FSLMM(FBDF3) order 2 [15]	3.70e-1	2.26e-6	5.31e-6	3.73e-6
Implicit 2-FSLMM(FBDF1) order 2	9.20e-4	2.33e-5	1.68e-5	9.85e-6

Example 2: Consider the FDE as

$$\begin{aligned}
&{}_C\mathrm{D}^{\upsilon}_{0,t}y(t) = t^4 - y(t) + \frac{4!t^{4-\upsilon}}{\Gamma(5-\upsilon)}, \;\; 1 < \upsilon < 2, \quad t \in (0,1] \\
&y(0) = y'(0) = 0,
\end{aligned} \tag{4.35}$$

where the analytical solution is $y(t) = t^4$. In Figure 5, $log_{10}(|y(t_N) - y_N|)$ versus $log_{10}(h)$ is plotted for $\upsilon = 1.6$. Moreover, the order of convergence ($EOC(N) = \log_2 \frac{\mathbf{E}(N)}{\mathbf{E}(N/2)}$) of proposed methods is shown for each execution in the same figure.

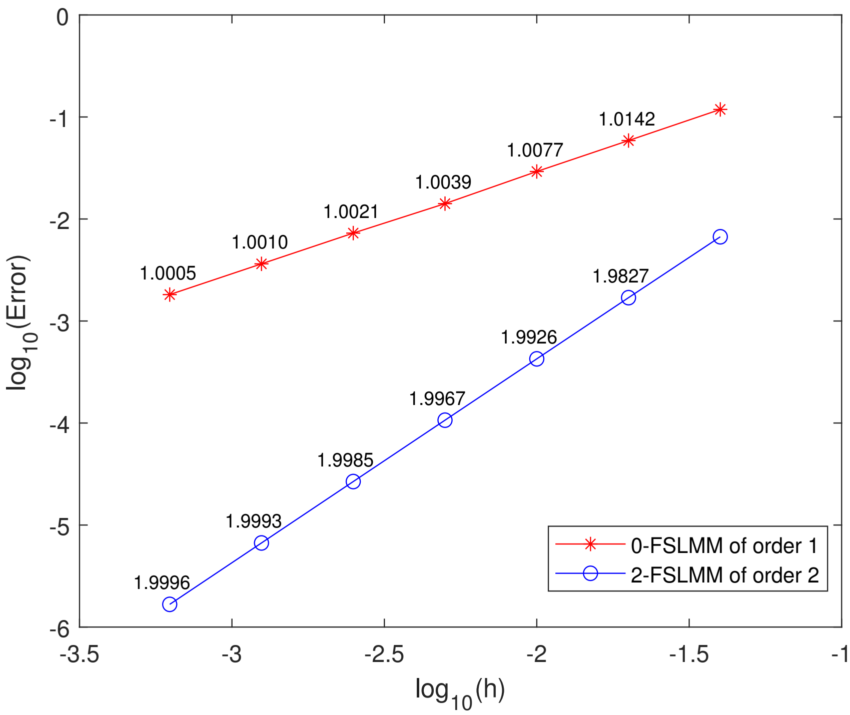

Figure 5: Errors and order of convergence for Example 2 at $t = 1$ for $\upsilon = 1.6$ and $h = \frac{1}{25}, \frac{1}{50}, \ldots, \frac{1}{1600}$.

Example 3. Consider the nonlinear FDE [18]

$$\begin{aligned} &{}_C\mathrm{D}_{0,t}^{\upsilon} y(t) + y^2(t) = g(t), \quad 1 < \upsilon < 2, \quad t \in (0,1], \\ &y(0) = 0, \quad y'(0) = 0, \end{aligned} \tag{4.36}$$

where $g(t) = \frac{\Gamma(6)}{\Gamma(6-\upsilon)} t^{5-\upsilon} - \frac{3\Gamma(5)}{\Gamma(5-\upsilon)} t^{4-\upsilon} + \frac{2\Gamma(4)}{\Gamma(4-\upsilon)} t^{3-\upsilon} + (t^5 - 3t^4 + 2t^3)^2$. The exact solution of the problem (4.36) is $y(t) = t^5 - 3t^4 + 2t^3$. In Table 4, the absolute errors of the proposed methods are listed at $t = 1$ for $\upsilon = 1.35, 1.80$ and different values of h. According to this table, it is shown that the errors are convergent to zeros when the step size reduces.

Table 4: Errors in Example 3 for different values of h and υ at $t = 1$.

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	υ
0-FSLMM(FBDF1) of order 1	2.2e-2	1.0e-2	5.3e-3	2.6e-3	1.3e-3	1.35
2-FSLMM(FBDF1) of order 2	2.0e-3	4.5e-4	1.0e-4	2.5e-5	6.1e-6	1.35
0-FSLMM(FBDF1) of order 1	3.4e-2	1.5e-2	7.4e-3	3.6e-3	1.7e-3	1.80
2-FSLMM(FBDF1) of order 2	4.6e-3	1.0e-3	2.5e-4	5.9e-5	1.3e-5	1.80

Example 4. Consider the following FDE as

$$\begin{aligned} &B_1 \, {}_C D_{0,t}^{\frac{3}{2}} y(t) + B_2 y''(t) + B_3 y(t) = g(t), \quad t \in (0,T], \\ &y(0) = d_1, \quad y'(0) = d_2, \end{aligned} \tag{4.37}$$

Table 5: Comparison of absolute errors for Example 4 in $t = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$.

t	0	0.2	0.4	0.6	0.8	1.0
FNNs method [23]	7.63e-6	1.12e-5	8.19e-6	1.01e-5	2.03e-5	1.24e-5
Explicit 1-FSLMM(FBDF1) order 1 [15]	0.00	2.99e-6	4.82e-6	6.01e-6	6.71e-6	7.02e-6
Explicit 2-FSLMM(FBDF1) order 2 [15]	0.00	5.41e-6	9.59e-6	1.30e-5	1.57e-5	1.79e-5
Explicit 1-FSLMM(FBDF2) order 1 [15]	0.00	1.17e-6	1.41e-7	1.92e-6	4.61e-6	7.68e-6
Explicit 2-FSLMM(FBDF2) order 2 [15]	0.00	1.08e-6	1.92e-6	2.60e-6	3.15e-5	3.59e-6
Explicit 1-FSLMM(FBDF3) order 1 [15]	0.00	1.22e-6	2.22e-7	1.81e-6	4.47e-6	7.75e-6
Explicit 2-FSLMM(FBDF3) order 2 [15]	0.00	1.08e-6	1.92e-7	2.61e-6	3.17e-6	3.30e-6
Implicit 0-FSLMM(FBDF1) order 1	0.00	1.76e-6	1.68e-6	6.94e-7	8.68e-7	2.81e-6
Implicit 2-FSLMM(FBDF1) order 2	0.00	4.07e-6	7.16e-6	9.71e-6	1.17e-5	1.33e-5

where $B_1 = 2B_2\sqrt{\xi\rho}$, $B_2 = m$, $B_3 = k$ and where ξ is the viscosity, ρ is the fluid density [4, 5]. This equation is known as the Bagley-Torvik (B-T) equation which appears in the motion of an plate with mass and area equal to m and A immersed and connected to a spring of stiffness k to a fixed point in a Newtonian fluid (see Figure 6).

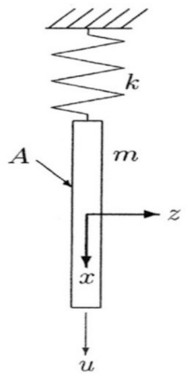

Figure 6: The immersed plate into a Newtonian fluid .

The problem (4.37) has got the exact solution as $y(t) = t^2$ for case $B1 = B2 = B3 = 1, g(t) = 2 + 4\sqrt{t/\pi} + t^2, T = 1$ and $d_1 = d_2 = 0$. This problem is studied in [15] by using the explicit forms of the fractional second linear multistep methods. These explicit FSLMMs are constructed based on fractional backward difference formulas 1, 2, and 3 (FBDF1, FBDF2, and FBDF3) with the first, second, third, and fourth orders of convergence. Moreover, the problem (4.37) is solved in [23] by using fractional neural networks (FNNs) optimized with interior point algorithms. In order to demonstrate the efficiency of the proposed method, the results of the mentioned methods and the proposed methods are reported in Table 5 for $h = \frac{1}{40960}$. It should be noted that the authors have applied the following approximation to obtain the results of the above table

$$y_i'' = \frac{2y_i - 5y_{i-1} + 4y_{i-2} - y_{i-3}}{h^2} + o(h^2). \tag{4.38}$$

5. Conclusion

In this work, the new implicit class of fractional second linear multistep methods based on fractional backward difference formula 1 (FBDF1) was presented while fractional derivative lies in the interval $(1, 2)$. Accordingly, the methods with the first and second orders of convergence were obtained. Moreover, the stability regions of the new methods were thoroughly investigated and it resulted that the proposed methods were $A(\alpha)-$stable. The reported results clearly showed that the proposed methods are very effective in solving FDEs.

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