



Cohomology and deformations of twisted \mathcal{O} -operators on 3-Lie algebras

T. Chtioui^a, A. Hajjaji^{b,d}, S. Mabrouk^c, A. Makhlouf^d

^aUniversity of Gabes, Faculty of Sciences Gabes, City Riadh 6072 Zrig, Gabes, Tunisia.

^bUniversity of Sfax, Faculty of Sciences Sfax, BP 1171, 3038 Sfax, Tunisia

^cUniversity of Gafsa, Faculty of Sciences Gafsa, 2112 Gafsa, Tunisia

^dUniversité de Haute Alsace IRIMAS - Département de Mathématiques, Mulhouse, France

Abstract. The purpose of this paper is to introduce and study twisted \mathcal{O} -operators on 3-Lie algebras. We construct an L_∞ -algebra whose Maurer-Cartan elements are twisted \mathcal{O} -operators and define a cohomology of a twisted \mathcal{O} -operator T as the Chevalley-Eilenberg cohomology of a certain 3-Lie algebra induced by T with coefficients in a suitable representation. Then we consider infinitesimal and formal deformations of twisted \mathcal{O} -operators.

1. Introduction

A natural generalization of binary operations appeared first when Cayley studied cubic matrices which are ternary operations. Furthermore one may consider in general n -ary operations of associative type or Lie type. In particular, 3-Lie algebras and more generally, n -Lie algebras [21] are generalizations of Lie algebras to ternary and n -ary cases.

The first instances of ternary Lie algebras are related to Nambu Mechanics [32], which was formulated algebraically by Takhtajan in [39]. The first complete algebraic study of n -Lie algebras is due to Filippov, see [21]. We refer to [7, 9, 12] for the realizations and classifications of 3-Lie algebras and n -Lie algebras. Ternary operations turn to be useful in many mathematics and physics domains, like string theory. The quantization of the Nambu brackets in [8] was a motivation to present a general construction of $(n + 1)$ -Lie algebras induced by n -Lie algebras using the n -ary brackets and trace-like linear forms, see also [5–7, 26]. The structure of 3-Lie (super)algebras induced by Lie (super)algebras, classification of 3-Lie algebras and application to constructions of B.R.S. algebras have been considered in [1–4].

A deformation theory based on one-parameter formal power series was introduced first by Gerstenhaber in [22] for associative algebras and then extended to Lie algebras by Nijenhuis and Richardson in [33]. It is shown that deformations are controlled by suitable cohomologies, Hochschild cohomology in associative case and Chevalley-Eilenberg cohomology in Lie case. The same approach was used for various algebraic structures. Deformation theory of 3-Lie algebras was studied and investigated in [20].

In [11], the authors studied the solutions of 3-Lie classical Yang-Baxter equation, that lead to introduce the notion of \mathcal{O} -operator on 3-Lie algebras with respect to a representation. In particular, Rota-Baxter

2020 Mathematics Subject Classification. 17B70, 17A40, 17B60, 17B56, 17B38.

Keywords. Twisted \mathcal{O} -operator, Maurer-Cartan element, L_∞ -algebra, cohomology, deformation.

Received: 01 April 2022; Revised: 29 August 2022; Accepted: 15 March 2023

Communicated by Dragan S. Djordjević

Email addresses: chtioui.taoufik@yahoo.fr (T. Chtioui), atefhajjaji100@gmail.com (A. Hajjaji), mabrouksami00@yahoo.fr (S. Mabrouk), abdenacer.makhlouf@uha.fr (A. Makhlouf)

operators on 3-Lie algebras, introduced in [10], are \mathcal{O} -operators on a 3-Lie algebra with respect to the adjoint representation.

Twisted Rota-Baxter operators introduced by Uchino in the context of associative algebras [42, 43] are algebraic analogue of twisted Poisson structure introduced and studied in [27, 36]. They are also related to NS-algebras considered by Leroux in [31], see also [24]. Twisted Rota-Baxter operators on Lie algebras and Leibniz algebras were studied in [17, 19]. A cohomology of twisted Rota-Baxter operators was derived, in [17, 18], from a suitable L_∞ -algebra whose Maurer-Cartan elements are given by twisted Rota-Baxter operators. Such a cohomology can be seen as the Hochschild (resp. Chevalley-Eilenberg) cohomology of a certain Lie algebra with coefficients in a suitable representation. A cohomology of a twisted relative Rota-Baxter operator as a Loday-Pirashvili cohomology of a certain Leibniz algebra was constructed in [19]. One may see [13, 30] for Hom-type version of Nijenhuis Bracket and cohomologies of relative Rota-Baxter Lie algebras.

The main purpose of this paper is to study twisted \mathcal{O} -operators on 3-Lie algebras. We provide some characterizations and key constructions. We construct an L_∞ -algebra whose Maurer-Cartan elements are twisted \mathcal{O} -operators and define a cohomology of a twisted \mathcal{O} -operator T as the Chevalley-Eilenberg cohomology of a certain 3-Lie algebra induced by T with coefficients in a suitable representation. Furthermore, we define a cohomology of twisted \mathcal{O} -operators that controls their deformations.

The paper is organized as follows. In Section 2, we briefly recall basics about representations and cohomology of 3-Lie algebras. Then in Section 3, we introduce Θ -twisted \mathcal{O} -operators on 3-Lie algebras, provide some examples and characterization results. Section 4 is devoted to constructing an L_∞ -algebra whose Maurer-Cartan elements are twisted \mathcal{O} -operators and defining the cohomology of a twisted \mathcal{O} -operator on a 3-Lie algebra with coefficients in a suitable representation. It is also compared to a cohomology of twisted \mathcal{O} -operators as Chevalley-Eilenberg cohomology. In Section 5, we study deformations of twisted \mathcal{O} -operators and show that they are controlled by the cohomology theory established in Section 4.

In this paper, we work over an algebraically closed field \mathbb{K} of characteristic 0 and all the vector spaces are over \mathbb{K} and finite-dimensional.

2. Preliminaries

In this section, we recall some basic results about 3-Lie algebras and their representations. Our main references are [14, 21, 25, 39]. A 3-Lie algebra \mathfrak{g} is a vector space together with a skew-symmetric 3-linear map $[\cdot, \cdot, \cdot]_{\mathfrak{g}} : \wedge^3 \mathfrak{g} \rightarrow \mathfrak{g}$, such that for all $x_i \in \mathfrak{g}, 1 \leq i \leq 5$, the following Filippov-Jacobi Identity (sometimes called fundamental identity or Nambu identity) holds

$$[x_1, x_2, [x_3, x_4, x_5]_{\mathfrak{g}}]_{\mathfrak{g}} = [[x_1, x_2, x_3]_{\mathfrak{g}}, x_4, x_5]_{\mathfrak{g}} + [x_3, [x_1, x_2, x_4]_{\mathfrak{g}}, x_5]_{\mathfrak{g}} + [x_3, x_4, [x_1, x_2, x_5]_{\mathfrak{g}}]_{\mathfrak{g}}. \tag{1}$$

For $x_1, x_2 \in \mathfrak{g}$, define $ad_{x_1, x_2} \in \mathfrak{gl}(\mathfrak{g})$ by

$$ad_{x_1, x_2} x = [x_1, x_2, x]_{\mathfrak{g}}, \quad \forall x \in \mathfrak{g}. \tag{2}$$

Then Filippov-Jacobi identity may be expressed as ad_{x_1, x_2} is a derivation, i.e.

$$ad_{x_1, x_2} [x_3, x_4, x_5]_{\mathfrak{g}} = [ad_{x_1, x_2} x_3, x_4, x_5]_{\mathfrak{g}} + [x_3, ad_{x_1, x_2} x_4, x_5]_{\mathfrak{g}} + [x_3, x_4, ad_{x_1, x_2} x_5]_{\mathfrak{g}}.$$

Definition 2.1. Let $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ be a 3-Lie algebra, V be a vector space and $\rho : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a linear map. The pair (V, ρ) is called a representation of \mathfrak{g} (or V is a \mathfrak{g} -module) if ρ satisfies, for all $x_1, x_2, x_3, x_4 \in \mathfrak{g}$,

$$\rho(x_1, x_2)\rho(x_3, x_4) = \rho([x_1, x_2, x_3]_{\mathfrak{g}}, x_4) + \rho(x_3, [x_1, x_2, x_4]_{\mathfrak{g}}) + \rho(x_3, x_4)\rho(x_1, x_2), \tag{3}$$

$$\rho([x_1, x_2, x_3]_{\mathfrak{g}}, x_4) = \rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_2, x_3)\rho(x_1, x_4) + \rho(x_3, x_1)\rho(x_2, x_4). \tag{4}$$

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ be a 3-Lie algebra. The linear map $ad : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ defines a representation of $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ on itself, which is called the adjoint representation.

Let (V, ρ) be a representation of a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$. Denote by

$$\mathfrak{C}_{3Lie}^n(\mathfrak{g}; V) = \text{Hom}(\underbrace{\wedge^2 \mathfrak{g} \otimes \cdots \otimes \wedge^2 \mathfrak{g}}_{n-1} \wedge \mathfrak{g}, V), \quad (n \geq 1),$$

which is the space of n -cochains. Consider the differential $\partial : \mathfrak{C}_{3Lie}^n(\mathfrak{g}; V) \rightarrow \mathfrak{C}_{3Lie}^{n+1}(\mathfrak{g}; V)$ defined by

$$\begin{aligned} & (\partial f)(\mathfrak{x}_1, \dots, \mathfrak{x}_n, x_{n+1}) \\ &= \sum_{1 \leq j < k \leq n} (-1)^j f(\mathfrak{x}_1, \dots, \widehat{\mathfrak{x}}_j, \dots, \mathfrak{x}_{k-1}, [x_j, y_j, x_k]_{\mathfrak{g}} \wedge y_k + x_k \wedge [x_j, y_j, y_k]_{\mathfrak{g}}, \mathfrak{x}_{k+1}, \dots, \mathfrak{x}_n, x_{n+1}) \\ &+ \sum_{j=1}^n (-1)^j f(\mathfrak{x}_1, \dots, \widehat{\mathfrak{x}}_j, \dots, \mathfrak{x}_n, [x_j, y_j, x_{n+1}]_{\mathfrak{g}}) + \sum_{j=1}^n (-1)^{j+1} \rho(x_j, y_j) f(\mathfrak{x}_1, \dots, \widehat{\mathfrak{x}}_j, \dots, \mathfrak{x}_n, x_{n+1}) \\ &+ (-1)^{n+1} (\rho(y_n, x_{n+1}) f(\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1}, x_n) + \rho(x_{n+1}, x_n) f(\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1}, y_n)), \end{aligned} \tag{5}$$

for all $\mathfrak{x}_i = x_i \wedge y_i \in \wedge^2 \mathfrak{g}$, $i = 1, 2, \dots, n$ and $x_{n+1} \in \mathfrak{g}$. It was proved in [14, 39] that $\partial \circ \partial = 0$. Thus $(\oplus_{n=1}^{+\infty} \mathfrak{C}_{3Lie}^n(\mathfrak{g}; V), \partial)$ is a cochain complex which is called Chevalley-Eilenberg cochain complex of 3-Lie algebras. The quotient space $H_{3Lie}^n(\mathfrak{g}; V) = Z_{3Lie}^n(\mathfrak{g}; V) / B_{3Lie}^n(\mathfrak{g}; V)$, where $Z_{3Lie}^n(\mathfrak{g}; V) = \{f \in \mathfrak{C}_{3Lie}^n(\mathfrak{g}; V) \mid \partial f = 0\}$ is the space of n -cocycles and $B_{3Lie}^n(\mathfrak{g}; V) = \{f = \partial g \mid g \in \mathfrak{C}_{3Lie}^{n-1}(\mathfrak{g}; V)\}$ is the space of n -coboundaries, is called the n^{th} cohomology group of the 3-Lie algebra \mathfrak{g} with coefficients in V .

Let $\Theta \in \mathfrak{C}_{3Lie}^2(\mathfrak{g}; V)$ be a 2-cocycle in the Chevalley-Eilenberg cochain complex, that is $\Theta : \wedge^3 \mathfrak{g} \rightarrow V$ is a trilinear map satisfying, for all $x_1, x_2, y_1, y_2, y_3 \in \mathfrak{g}$,

$$\begin{aligned} & \Theta(x_1, x_2, [y_1, y_2, y_3]_{\mathfrak{g}}) + \rho(x_1, x_2) \Theta(y_1, y_2, y_3) - \Theta([x_1, x_2, y_1]_{\mathfrak{g}}, y_2, y_3) - \Theta(y_1, [x_1, x_2, y_2]_{\mathfrak{g}}, y_3) \\ & - \Theta(y_1, y_2, [x_1, x_2, y_3]_{\mathfrak{g}}) - \rho(y_2, y_3) \Theta(x_1, x_2, y_1) - \rho(y_3, y_1) \Theta(x_1, x_2, y_2) - \rho(y_1, y_2) \Theta(x_1, x_2, y_3) = 0. \end{aligned} \tag{6}$$

The direct sum $\mathfrak{g} \oplus V$ carries a 3-Lie algebra structure given by

$$[(x, u), (y, v), (z, w)]_{\Theta} = ([x, y, z]_{\mathfrak{g}}, \rho(x, y)w + \rho(z, x)v + \rho(y, z)u + \Theta(x, y, z)), \tag{7}$$

which is called the Θ -twisted semi-direct product and denoted by $\mathfrak{g} \ltimes_{\Theta} V$.

3. Twisted \mathcal{O} -operators on 3-Lie algebras

In this section, we introduce twisted \mathcal{O} -operators on 3-Lie algebras. We give some constructions and provide examples. Let $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ be a 3-Lie algebra, (V, ρ) be a representation and $\Theta \in \mathfrak{C}_{3Lie}^2(\mathfrak{g}; V)$ be a 2-cocycle in the Chevalley-Eilenberg cochain complex.

Definition 3.1. A linear map $T : V \rightarrow \mathfrak{g}$ is said to be a Θ -twisted \mathcal{O} -operator if T satisfies

$$[Tu, Tv, Tw]_{\mathfrak{g}} = T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \Theta(Tu, Tv, Tw)), \tag{8}$$

for all $u, v, w \in V$.

Using the Θ -twisted semi-direct product, one can characterize twisted \mathcal{O} -operators by their graphs.

Proposition 3.1. A linear map $T : V \rightarrow \mathfrak{g}$ is a Θ -twisted \mathcal{O} -operator if and only if the graph $Gr(T) = \{(Tu, u) \mid u \in V\}$ is a subalgebra of the Θ -twisted semi-direct product $\mathfrak{g} \ltimes_{\Theta} V$.

Proof. Let $(Tu, u), (Tv, v)$ and $(Tw, w) \in Gr(T)$. Then we have

$$\begin{aligned} & [(Tu, u), (Tv, v), (Tw, w)]_{\Theta} \\ &= ([Tu, Tv, Tw]_{\mathfrak{g}}, \rho(Tu, Tv)w + \rho(Tw, Tu)v + \rho(Tv, Tw)u + \Theta(Tu, Tv, Tw)). \end{aligned}$$

Assume that $Gr(T)$ is a subalgebra of the Θ -twisted semi-direct product $\mathfrak{g} \ltimes_{\Theta} V$, then we have

$$[Tu, Tv, Tw]_{\mathfrak{g}} = T(\rho(Tu, Tv)w + \rho(Tw, Tu)v + \rho(Tv, Tw)u + \Theta(Tu, Tv, Tw)).$$

On the other hand, if T is a Θ -twisted \mathcal{O} -operator, then we obtain

$$\begin{aligned} & [(Tu, u), (Tv, v), (Tw, w)]_{\Theta} \\ &= (T(\rho(Tu, Tv)w + \rho(Tw, Tu)v + \rho(Tv, Tw)u + \Theta(Tu, Tv, Tw)), \\ & \quad \rho(Tu, Tv)w + \rho(Tw, Tu)v + \rho(Tv, Tw)u + \Theta(Tu, Tv, Tw)) \in Gr(T). \end{aligned}$$

Hence, $Gr(T)$ is a subalgebra of the Θ -twisted semi-direct product $\mathfrak{g} \ltimes_{\Theta} V$. \square

The set $Gr(T)$ is isomorphic to V as a vector space by the identification $(Tu, u) \cong u$. A Θ -twisted \mathcal{O} -operator T induces a 3-Lie algebra structure on V , where the bracket is given by

$$[u, v, w]_T = \rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \Theta(Tu, Tv, Tw). \tag{9}$$

It is obvious that T is a 3-Lie algebra morphism, that is $T([u, v, w]_T) = [Tu, Tv, Tw]_{\mathfrak{g}}$.

Definition 3.2. Let $T : V \rightarrow \mathfrak{g}$ and $T' : V' \rightarrow \mathfrak{g}'$ be Θ -twisted and Θ' -twisted \mathcal{O} -operators. A morphism of twisted \mathcal{O} -operators from T to T' consists of a pair (ϕ, ψ) of a 3-Lie algebra morphism $\phi : \mathfrak{g} \rightarrow \mathfrak{g}'$ and a linear map $\psi : V \rightarrow V'$ satisfying

$$\psi(\rho(x, y)u) = \rho'(\phi(x), \phi(y))\psi(u), \quad \forall x, y \in \mathfrak{g}, u \in V, \tag{10}$$

$$\psi \circ \Theta = \Theta' \circ (\phi \otimes \phi \otimes \phi), \tag{11}$$

$$\phi \circ T = T' \circ \psi. \tag{12}$$

Example 3.2. Any \mathcal{O} -operator (in particular, Rota-Baxter operator of weight 0) on a 3-Lie algebra is a Θ -twisted \mathcal{O} -operator with $\Theta = 0$.

Example 3.3. Let \mathfrak{g} be a 3-Lie algebra and V be a \mathfrak{g} -module. Suppose $\theta : \mathfrak{g} \rightarrow V$ is an invertible 1-cochain in the Chevalley-Eilenberg cochain complex of \mathfrak{g} with coefficients in V . Then $T = \theta^{-1} : V \rightarrow \mathfrak{g}$ is a Θ -twisted \mathcal{O} -operator with $\Theta = -\partial\theta$. The proof follows from the fact that

$$\begin{aligned} \Theta(Tu, Tv, Tw) &= -(\partial\theta)(Tu, Tv, Tw) \\ &= -\rho(Tu, Tv)\theta(Tw) - \rho(Tv, Tw)\theta(Tu) - \rho(Tw, Tu)\theta(Tv) + \theta([Tu, Tv, Tw]_{\mathfrak{g}}). \end{aligned} \tag{13}$$

By applying T to both sides of (13), we get the identity (8).

Example 3.4. Let $N : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Nijenhuis operator on a 3-Lie algebra \mathfrak{g} , i.e. N satisfies the identity

$$\begin{aligned} [Nx, Ny, Nz]_{\mathfrak{g}} &= N([Nx, Ny, z]_{\mathfrak{g}} + [Nx, y, Nz]_{\mathfrak{g}} + [x, Ny, Nz]_{\mathfrak{g}} \\ & \quad - N([Nx, y, z]_{\mathfrak{g}} + [x, Ny, z]_{\mathfrak{g}} + [x, y, Nz]_{\mathfrak{g}}) + N^2[x, y, z]_{\mathfrak{g}}), \quad \forall x, y, z \in \mathfrak{g}. \end{aligned} \tag{14}$$

In this case, \mathfrak{g} carries a new 3-Lie algebra structure given by the following bracket

$$[x, y, z]_N = [Nx, Ny, z]_{\mathfrak{g}} + [Nx, y, Nz]_{\mathfrak{g}} + [x, Ny, Nz]_{\mathfrak{g}} - N([Nx, y, z]_{\mathfrak{g}})$$

$$+ [x, Ny, z]_{\mathfrak{g}} + [x, y, Nz]_{\mathfrak{g}} - N[x, y, z]_{\mathfrak{g}}.$$

We denote this 3-Lie algebra by \mathfrak{g}_N . Moreover, the 3-Lie algebra \mathfrak{g}_N has a representation on \mathfrak{g} given by $\rho(x, y)z = [Nx, Ny, z]_{\mathfrak{g}}$, for all $x, y, z \in \mathfrak{g}$. With this representation, the map $\Theta : \wedge^3 \mathfrak{g}_N \rightarrow \mathfrak{g}$ defined by

$$\Theta(x, y, z) = -N([Nx, y, z]_{\mathfrak{g}} + [x, Ny, z]_{\mathfrak{g}} + [x, y, Nz]_{\mathfrak{g}} - N[x, y, z]_{\mathfrak{g}})$$

is a 2-cocycle in the Chevalley-Eilenberg cohomology of \mathfrak{g}_N with coefficients in \mathfrak{g} . Then it is easy to observe that the identity map $id : \mathfrak{g} \rightarrow \mathfrak{g}_N$ is a Θ -twisted \mathcal{O} -operator.

Given a Θ -twisted \mathcal{O} -operator T and a 1-cochain θ , we construct a $(\Theta + \partial\theta)$ -twisted \mathcal{O} -operator under certain condition. First we have the following observation.

Proposition 3.5. *Let \mathfrak{g} be a 3-Lie algebra and V be a \mathfrak{g} -module. For any 2-cocycle $\Theta \in \mathfrak{C}_{3Lie}^2(\mathfrak{g}; V)$ and 1-cochain $\theta \in \mathfrak{C}_{3Lie}^1(\mathfrak{g}; V)$, we have a 3-Lie algebra isomorphism*

$$\mathfrak{g} \ltimes_{\Theta} V \cong \mathfrak{g} \ltimes_{\Theta + \partial\theta} V.$$

Proof. Define $\psi_{\theta} : \mathfrak{g} \ltimes_{\Theta} V \rightarrow \mathfrak{g} \ltimes_{\Theta + \partial\theta} V$ by $\psi_{\theta}(x, u) = (x, u - \theta(x))$, for all $(x, u) \in \mathfrak{g} \oplus V$. Then we have,

$$\begin{aligned} & \psi_{\theta}([(x, u), (y, v), (z, w)]_{\Theta}) \\ &= ([x, y, z]_{\mathfrak{g}}, \rho(x, y)w + \rho(z, x)v + \rho(y, z)u + \Theta(x, y, z) - \theta([x, y, z]_{\mathfrak{g}})) \\ &= ([x, y, z]_{\mathfrak{g}}, \rho(x, y)w + \rho(z, x)v + \rho(y, z)u + \Theta(x, y, z) \\ &\quad - \rho(x, y)\theta(z) - \rho(z, x)\theta(y) - \rho(y, z)\theta(x) + (\partial\theta)(x, y, z)) \\ &= [(x, u - \theta(x)), (y, v - \theta(y)), (z, w - \theta(z))]_{\Theta + \partial\theta}, \end{aligned}$$

for all $(x, u), (y, v), (z, w) \in \mathfrak{g} \oplus V$. This proves the result. \square

Proposition 3.6. *Let $T : V \rightarrow \mathfrak{g}$ be a Θ -twisted \mathcal{O} -operator. For any 1-cochain $\theta \in \mathfrak{C}_{3Lie}^1(\mathfrak{g}; V)$, if the linear map $(Id_V - \theta \circ T) : V \rightarrow V$ is invertible, then the linear map $T \circ (Id_V - \theta \circ T)^{-1} : V \rightarrow \mathfrak{g}$ is a $(\Theta + \partial\theta)$ -twisted \mathcal{O} -operator.*

Proof. Consider the subalgebra $Gr(T) \subset \mathfrak{g} \ltimes_{\Theta} V$ of the Θ -twisted semi-direct product. Thus by Proposition 3.5, we get that

$$\psi_{\theta}(Gr(T)) = \{(Tu, u - (\theta \circ T)(u)) \mid u \in V\} \subset \mathfrak{g} \ltimes_{\Theta + \partial\theta} V$$

is a subalgebra. Since the map $(Id_V - \theta \circ T) : V \rightarrow V$ is invertible, we have $\psi_{\theta}(Gr(T))$ is the graph of the linear map $T \circ (Id_V - \theta \circ T)^{-1}$. In this case, it follows from Proposition 3.1 that $T \circ (Id_V - \theta \circ T)^{-1}$ is a $(\Theta + \partial\theta)$ -twisted \mathcal{O} -operator. \square

Next, we give a construction of a new Θ -twisted \mathcal{O} -operator out of an old one and a suitable 1-cocycle. Let $T : V \rightarrow \mathfrak{g}$ be a Θ -twisted \mathcal{O} -operator. Suppose $\theta \in \mathfrak{C}_{3Lie}^1(\mathfrak{g}; V)$ is a 1-cocycle in the Chevalley-Eilenberg cochain complex of \mathfrak{g} with coefficients in V . Then θ is said to be T -admissible if the linear map $(Id_V + \theta \circ T) : V \rightarrow V$ is invertible.

Proposition 3.7. *Let $\theta \in \mathfrak{C}_{3Lie}^1(\mathfrak{g}; V)$ be a T -admissible 1-cocycle. Then $T \circ (Id_V + \theta \circ T)^{-1} : V \rightarrow \mathfrak{g}$ is a Θ -twisted \mathcal{O} -operator.*

Proof. Consider the deformed subspace

$$\tau_{\theta}(Gr(T)) = \{(Tu, u + (\theta \circ T)(u)) \mid u \in V\} \subset \mathfrak{g} \ltimes_{\Theta} V.$$

Since θ is a 1-cocycle, $\tau_{\theta}(Gr(T)) \subset \mathfrak{g} \ltimes_{\Theta} V$ turns out to be a subalgebra. Furthermore, since the map $(Id_V + \theta \circ T)$ is invertible, then $\tau_{\theta}(Gr(T))$ is the graph of the map $T \circ (Id_V + \theta \circ T)^{-1}$. Hence the result follows from Proposition 3.1. \square

The Θ -twisted \mathcal{O} -operator in the above proposition is called the gauge transformation of T associated with θ . We denote this Θ -twisted \mathcal{O} -operator simply by T_θ .

Proposition 3.8. *Let T be a Θ -twisted \mathcal{O} -operator and θ be a T -admissible 1-cocycle. Then the 3-Lie algebra structures on V induced from the Θ -twisted \mathcal{O} -operators T and T_θ are isomorphic.*

Proof. Consider the linear isomorphism $(Id_V + \theta \circ T) : V \rightarrow V$. For any $u, v, w \in V$, we have

$$\begin{aligned} & [(Id_V + \theta \circ T)(u), (Id_V + \theta \circ T)(v), (Id_V + \theta \circ T)(w)]_{T_\theta} \\ &= \rho(Tv, Tw)(Id_V + \theta \circ T)(u) + \rho(Tw, Tu)(Id_V + \theta \circ T)(v) \\ &+ \rho(Tu, Tv)(Id_V + \theta \circ T) + \Theta(Tu, Tv, Tw) \\ &= \rho(Tv, Tw)u + \rho(Tw, Tu)v + \rho(Tu, Tv)w + \Theta(Tu, Tv, Tw) \\ &+ \rho(Tv, Tw)(\theta \circ T)(u) + \rho(Tw, Tu)(\theta \circ T)(v) + \rho(Tu, Tv)(\theta \circ T)(w) \\ &= [u, v, w]_T + \theta([Tu, Tv, Tw]_\mathfrak{g}) \\ &= [u, v, w]_T + \theta \circ T([u, v, w]_T) = (Id_V + \theta \circ T)([u, v, w]_T). \end{aligned}$$

This shows that $(Id_V + \theta \circ T) : (V, [\cdot, \cdot, \cdot]_T) \rightarrow (V, [\cdot, \cdot, \cdot]_{T_\theta})$ is a 3-Lie algebra isomorphism. \square

4. Cohomology of twisted \mathcal{O} -operators

In this section, we construct an L_∞ -algebra whose Maurer-Cartan elements are Θ -twisted \mathcal{O} -operators on 3-Lie algebras. Such characterization of Θ -twisted \mathcal{O} -operator T allows us to introduce a cohomology of T . Next, we show that the cohomology of T is equivalently described by the Chevalley-Eilenberg cohomology of V with coefficients in a suitable representation on \mathfrak{g} .

4.1. Maurer-Cartan characterization and cohomology

Let \mathfrak{g} be a vector space. Consider the graded vector space

$$C^*(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n \geq 0} C^n(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n \geq 0} \underbrace{Hom(\wedge^2 \mathfrak{g} \otimes \cdots \otimes \wedge^2 \mathfrak{g} \wedge \mathfrak{g}, \mathfrak{g})}_n.$$

The degree of elements in $C^n(\mathfrak{g}, \mathfrak{g})$ is defined to be n . Then the graded vector space $C^*(\mathfrak{g}, \mathfrak{g})$ equipped with the graded commutator bracket

$$[P, Q]_{3Lie} = P \circ Q - (-1)^{pq} Q \circ P, \quad \forall P \in C^p(\mathfrak{g}, \mathfrak{g}), Q \in C^q(\mathfrak{g}, \mathfrak{g}), \tag{15}$$

is a graded Lie algebra, with $P \circ Q \in C^{p+q}(\mathfrak{g}, \mathfrak{g})$ defined by

$$\begin{aligned} & (P \circ Q)(\mathfrak{x}_1, \dots, \mathfrak{x}_{p+q}, x) \\ &= \sum_{k=1}^p (-1)^{(k-1)q} \sum_{\sigma \in \mathcal{S}(k-1, q)} (-1)^\sigma P(\mathfrak{x}_{\sigma(1)}, \dots, \mathfrak{x}_{\sigma(k-1)}, Q(\mathfrak{x}_{\sigma(k)}, \dots, \mathfrak{x}_{\sigma(k+q-1)}, \mathfrak{x}_{k+q}) \wedge y_{k+q}, \mathfrak{x}_{k+q+1}, \dots, \mathfrak{x}_{p+q}, x) \\ &+ \sum_{k=1}^p (-1)^{(k-1)q} \sum_{\sigma \in \mathcal{S}(k-1, q)} (-1)^\sigma P(\mathfrak{x}_{\sigma(1)}, \dots, \mathfrak{x}_{\sigma(k-1)}, \mathfrak{x}_{k+q} \wedge Q(\mathfrak{x}_{\sigma(k)}, \dots, \mathfrak{x}_{\sigma(k+q-1)}, y_{k+q}), \mathfrak{x}_{k+q+1}, \dots, \mathfrak{x}_{p+q}, x) \\ &+ \sum_{\sigma \in \mathcal{S}(p, q)} (-1)^{pq} (-1)^\sigma P(\mathfrak{x}_{\sigma(1)}, \dots, \mathfrak{x}_{\sigma(p)}, Q(\mathfrak{x}_{\sigma(p+1)}, \dots, \mathfrak{x}_{\sigma(p+q-1)}, \mathfrak{x}_{\sigma(p+q)}, x)), \end{aligned}$$

where $\mathfrak{x}_i = x_i \wedge y_i \in \wedge^2 \mathfrak{g}$, $i = 1, 2, \dots, p + q$ and $x \in \mathfrak{g}$. See [35] for more details.

We recall from [35] the following result.

Proposition 4.1. *Let \mathfrak{g} be a vector space. Then $\pi \in C^1(\mathfrak{g}, \mathfrak{g}) = Hom(\wedge^3 \mathfrak{g}, \mathfrak{g})$ defines a 3-Lie algebra structure on \mathfrak{g} if and only if π is a Maurer-Cartan element of the graded Lie algebra $(C^*(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{3Lie})$, i.e. it satisfies the Maurer-Cartan equation $[\pi, \pi]_{3Lie} = 0$. Moreover, $(C^*(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{3Lie}, d_\pi)$ is a differential graded Lie algebra, where d_π is defined by*

$$d_\pi := [\pi, \cdot]_{3Lie}. \tag{16}$$

The notion of an L_∞ -algebra was introduced by Schlessinger and Stasheff in [37, 38]. See [28, 29] for more details.

Definition 4.1. An L_∞ -algebra is a \mathbb{Z} -graded vector space $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k$ equipped with a collection of linear maps $l_k : \otimes^k \mathfrak{g} \rightarrow \mathfrak{g}$ of degree 1 ($k \geq 1$) with the property that, for any homogeneous elements $x_1, \dots, x_n \in \mathfrak{g}$, we have

(i) (graded symmetry) for every $\sigma \in \mathfrak{S}_n$,

$$l_n(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}, x_{\sigma(n)}) = \varepsilon(\sigma)l_n(x_1, \dots, x_{n-1}, x_n),$$

(ii) (generalized Jacobi identity) for all $n \geq 1$,

$$\sum_{i=1}^n \sum_{\sigma \in \mathfrak{S}_{(i, n-i)}} \varepsilon(\sigma)l_{n-i+1}(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0.$$

Definition 4.2. A Maurer-Cartan element of an L_∞ -algebra $(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k, \{l_i\}_{i=1}^{+\infty})$ is an element $\alpha \in \mathfrak{g}^0$ satisfying the Maurer-Cartan equation

$$\sum_{n=1}^{+\infty} \frac{1}{n!} l_n(\alpha, \dots, \alpha) = 0. \tag{17}$$

Let α be a Maurer-Cartan element of an L_∞ -algebra $(\mathfrak{g}, \{l_i\}_{i=1}^{+\infty})$. For all $k \geq 1$ and $x_1, \dots, x_n \in \mathfrak{g}$, define a series of linear maps $l_k^\alpha : \otimes^k \mathfrak{g} \rightarrow \mathfrak{g}$ of degree 1 by

$$l_k^\alpha(x_1, \dots, x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{n+k}(\underbrace{\alpha, \dots, \alpha}_n, x_1, \dots, x_k). \tag{18}$$

Theorem 4.2. [23] With the above notations, $(\mathfrak{g}, \{l_i^\alpha\}_{i=1}^{+\infty})$ is an L_∞ -algebra, obtained from the L_∞ -algebra $(\mathfrak{g}, \{l_i\}_{i=1}^{+\infty})$ by twisting with the Maurer-Cartan element α . Moreover, $\alpha + \alpha'$ is a Maurer-Cartan element of $(\mathfrak{g}, \{l_i^\alpha\}_{i=1}^{+\infty})$ if and only if α' is a Maurer-Cartan element of the twisted L_∞ -algebra $(\mathfrak{g}, \{l_i^\alpha\}_{i=1}^{+\infty})$.

Let (V, ρ) be a representation of a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\mathfrak{g})$ and let Θ be a 2-cocycle in the cohomology of \mathfrak{g} with coefficients in V . For convenience, we use $\pi : \wedge^3 \mathfrak{g} \rightarrow \mathfrak{g}$ to indicate the 3-Lie bracket on \mathfrak{g} . Then $\pi + \rho + \Theta$ corresponds to the semi-direct product 3-Lie algebra structure on $\mathfrak{g} \oplus V$ given by

$$[x + u, y + v, z + w]_\Theta = [x, y, z]_\mathfrak{g} + \rho(x, y)w + \rho(z, x)v + \rho(y, z)u + \Theta(x, y, z). \tag{19}$$

Therefore, we have

$$[\pi + \rho + \Theta, \pi + \rho + \Theta]_{3Lie} = 0.$$

Consider the graded vector space

$$C^*(V, \mathfrak{g}) = \bigoplus_{n \geq 0} C^n(V, \mathfrak{g}) = \bigoplus_{n \geq 0} \text{Hom}(\underbrace{\wedge^2 V \otimes \dots \otimes \wedge^2 V}_n \wedge V, \mathfrak{g}).$$

Define

$$l_3 : C^m(V, \mathfrak{g}) \times C^n(V, \mathfrak{g}) \times C^p(V, \mathfrak{g}) \rightarrow C^{m+n+p+1}(V, \mathfrak{g}),$$

$$l_4 : C^m(V, \mathfrak{g}) \times C^n(V, \mathfrak{g}) \times C^p(V, \mathfrak{g}) \times C^q(V, \mathfrak{g}) \rightarrow C^{m+n+p+q+1}(V, \mathfrak{g})$$

by

$$l_3(P, Q, R) = [[[\pi + \rho, P]_{3Lie}, Q]_{3Lie}, R]_{3Lie},$$

$$l_4(P, Q, R, S) = [[[[\Theta, P]_{3Lie}, Q]_{3Lie}, R]_{3Lie}, S]_{3Lie}.$$

One method for constructing explicit L_∞ -algebras is given by Voronov’s higher derived brackets [44]. Moreover, using the above method, the ternary bracket l_3 and the 4-ary bracket l_4 are compatible in the sense of L_∞ -algebra. This follows since Θ is a 2-cocycle. In summary, we obtain the following result.

Theorem 4.3. *Let (V, ρ) be a representation of a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ and Θ be a 2-cocycle in the cohomology of \mathfrak{g} with coefficients in V . Then the graded vector space $C^*(V, \mathfrak{g})$ is an L_∞ -algebra with*

$$l_1 = l_2 = 0, \quad l_3(\cdot, \cdot, \cdot), \quad l_4(\cdot, \cdot, \cdot, \cdot), \tag{20}$$

and higher brackets are trivial. A linear map $T : V \rightarrow \mathfrak{g}$ is a Θ -twisted \mathcal{O} -operator if and only if T is a solution of the Maurer-Cartan equation of the L_∞ -algebra $(C^*(V, \mathfrak{g}), l_3, l_4)$, i.e.

$$\frac{1}{3!}l_3(T, T, T) + \frac{1}{4!}l_4(T, T, T, T) = 0.$$

Proof. Using the above discussion, the first part follows. For the second part, we have that for any $T \in Hom(V, \mathfrak{g})$,

$$l_4(T, T, T, T)(u, v, w) = -24T(\Theta(Tu, Tv, Tw)). \tag{21}$$

Next, as in [40, Theorem 3.4] proof we have

$$l_3(T, T, T)(u, v, w) = 6([Tu, Tv, Tw]_{\mathfrak{g}} - T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v)). \tag{22}$$

Hence from Eqs. (21) and (22), we get

$$\begin{aligned} & \left(\frac{1}{3!}l_3(T, T, T) + \frac{1}{4!}l_4(T, T, T, T) \right)(u, v, w) \\ &= [Tu, Tv, Tw]_{\mathfrak{g}} - T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v) - T(\Theta(Tu, Tv, Tw)). \end{aligned}$$

Thus, a linear map $T \in Hom(V, \mathfrak{g})$ is a Θ -twisted \mathcal{O} -operator of a 3-Lie algebra \mathfrak{g} with respect to a representation ρ if and only if T is a Maurer-Cartan element of the L_∞ -algebra $(C^*(V, \mathfrak{g}), l_3, l_4)$. \square

Proposition 4.4. *Let T be a Θ -twisted \mathcal{O} -operator of a 3-Lie algebra \mathfrak{g} with respect to a representation ρ . Then $C^*(V, \mathfrak{g})$ carries a twisted L_∞ -algebra structure given by*

$$l_1^T(P) = \frac{1}{2}l_3(T, T, P) + \frac{1}{6}l_4(T, T, T, P), \tag{23}$$

$$l_2^T(P, Q) = l_3(T, P, Q) + \frac{1}{2}l_4(T, T, P, Q), \tag{24}$$

$$l_3^T(P, Q, R) = l_3(P, Q, R) + l_4(T, P, Q, R), \tag{25}$$

$$l_4^T(P, Q, R, S) = l_4(P, Q, R, S), \tag{26}$$

$$l_k^T = 0, \quad k \geq 5, \tag{27}$$

where $P \in C^p(V, \mathfrak{g}), Q \in C^q(V, \mathfrak{g}), R \in C^r(V, \mathfrak{g})$ and $S \in C^s(V, \mathfrak{g})$. Moreover, for any linear map $T' : V \rightarrow \mathfrak{g}$, the sum $T + T'$ is a Θ -twisted \mathcal{O} -operator if and only if T' is a Maurer-Cartan element in the twisted L_∞ -algebra $(C^*(V, \mathfrak{g}), l_1^T, l_2^T, l_3^T, l_4^T)$, that is T' satisfies

$$l_1^T(T') + \frac{1}{2!}l_2^T(T', T') + \frac{1}{3!}l_3^T(T', T', T') + \frac{1}{4!}l_4^T(T', T', T', T') = 0.$$

Proof. For the first part, since T is a Maurer-Cartan element of the L_∞ -algebra $(C^*(V, \mathfrak{g}), l_3, l_4)$, by Theorem 4.2, we have that $C^*(V, \mathfrak{g})$ carries a twisted L_∞ -algebra structure. For the second part, by Theorem 4.3, $T + T'$ is a Θ -twisted \mathcal{O} -operator if and only if

$$\frac{1}{3!}l_3(T + T', T + T', T + T') + \frac{1}{4!}l_4(T + T', T + T', T + T', T + T') = 0. \tag{28}$$

Applying $\frac{1}{3!}l_3(T, T, T) + \frac{1}{4!}l_4(T, T, T, T) = 0$, the above condition is equivalent to

$$\begin{aligned} & \frac{1}{3!}(3l_3(T, T, T') + 3l_3(T, T', T') + l_3(T', T', T')) \\ & + \frac{1}{4!}(4l_4(T, T, T, T') + 6l_4(T, T, T', T') + 4l_4(T, T', T', T') + l_4(T', T', T', T')) = 0. \end{aligned}$$

That is, $l_1^T(T') + \frac{1}{2!}l_2^T(T', T') + \frac{1}{3!}l_3^T(T', T', T') + \frac{1}{4!}l_4^T(T', T', T', T') = 0$, which implies that T' is a Maurer-Cartan element of the twisted L_∞ -algebra $(C^*(V, \mathfrak{g}), l_1^T, l_2^T, l_3^T, l_4^T)$. \square

The above characterization of a Θ -twisted \mathcal{O} -operator T allows us to define a cohomology associated to T . More precisely, we define $C_T^n(V, \mathfrak{g}) = \text{Hom}(\underbrace{\wedge^2 V \otimes \dots \otimes \wedge^2 V}_{n \geq 0} \wedge V, \mathfrak{g})$, for $n \geq 0$ and the differential operator

$d_T : C_T^n(V, \mathfrak{g}) \rightarrow C_T^{n+1}(V, \mathfrak{g})$ by

$$d_T(f) = \frac{1}{2}l_3(T, T, f) + \frac{1}{6}l_4(T, T, T, f), \quad f \in C_T^n(V, \mathfrak{g}). \tag{29}$$

The corresponding cohomology groups are

$$H_T^n(V, \mathfrak{g}) = \frac{Z_T^n(V, \mathfrak{g})}{B_T^n(V, \mathfrak{g})} = \frac{\{f \in C_T^n(V, \mathfrak{g}) \mid d_T(f) = 0\}}{\{d_T(g) \mid g \in C_T^{n-1}(V, \mathfrak{g})\}}.$$

4.2. Cohomology of twisted \mathcal{O} -operators as Chevalley-Eilenberg cohomology

In this subsection, we define a cohomology of a Θ -twisted \mathcal{O} -operator as the Chevalley-Eilenberg cohomology of the 3-Lie algebra $(V, [\cdot, \cdot, \cdot]_T)$ given by Eq. (9) with coefficients in a suitable representation on \mathfrak{g} . This cohomology will be used in Section 5 to study formal deformations of T .

Proposition 4.5. *Let T be a Θ -twisted \mathcal{O} -operator on a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ) . Define $\rho_\Theta : \wedge^2 V \rightarrow \mathfrak{gl}(\mathfrak{g})$ by*

$$\rho_\Theta(u, v)x = [Tu, Tv, x]_{\mathfrak{g}} - T(\rho(Tv, x)u + \rho(x, Tu)v + \Theta(x, Tu, Tv)), \quad \forall u, v \in V, x \in \mathfrak{g}. \tag{30}$$

Then $(\mathfrak{g}, \rho_\Theta)$ is a representation of the 3-Lie algebra $(V, [\cdot, \cdot, \cdot]_T)$ on the vector space \mathfrak{g} .

Proof. By a direct calculation using the definition of ρ_Θ , we get

$$\begin{aligned} & \rho_\Theta(u_1, u_2)\rho_\Theta(u_3, u_4)x - \rho_\Theta([u_1, u_2, u_3]_T, u_4)x \\ & - \rho_\Theta(u_3, [u_1, u_2, u_4]_T)x - \rho_\Theta(u_3, u_4)\rho_T(u_1, u_2)x \\ & = [Tu_1, Tu_2, [Tu_3, Tu_4, x]_{\mathfrak{g}}]_{\mathfrak{g}} + [Tu_1, Tu_2, T\rho(Tu_3, x)u_4]_{\mathfrak{g}} - [Tu_1, Tu_2, T\rho(Tu_4, x)u_3]_{\mathfrak{g}} \\ & - [Tu_1, Tu_2, T\Theta(x, Tu_3, Tu_4)]_{\mathfrak{g}} + T\rho(Tu_1, [Tu_3, Tu_4, x]_{\mathfrak{g}})u_2 + T\rho(Tu_1, T\rho(Tu_3, x)u_4)u_2 \\ & - T\rho(Tu_1, T\rho(Tu_4, x)u_3)u_2 - T\rho(Tu_1, T\Theta(x, Tu_3, Tu_4))u_2 - T\rho(Tu_2, [Tu_3, Tu_4, x]_{\mathfrak{g}})u_1 \\ & - T\rho(Tu_2, T\rho(Tu_3, x)u_4)u_1 + T\rho(Tu_2, T\rho(Tu_4, x)u_3)u_1 + T\rho(Tu_2, T\Theta(x, Tu_3, Tu_4))u_1 \\ & - T\Theta(T\rho(Tu_3, x)u_4, Tu_1, Tu_2) + T\Theta(T\rho(Tu_4, x)u_3, Tu_1, Tu_2) \\ & + T\Theta(T\Theta(x, Tu_3, Tu_4), Tu_1, Tu_2) - T\Theta([Tu_3, Tu_4, x]_{\mathfrak{g}}, Tu_1, Tu_2) \end{aligned}$$

$$\begin{aligned}
 & - [[Tu_1, Tu_2, Tu_3]_{\mathfrak{g}}, Tu_4, x]_{\mathfrak{g}} - T(\rho([Tu_1, Tu_2, Tu_3]_{\mathfrak{g}}, x)u_4 + T\rho(Tu_4, x)\rho(Tu_1, Tu_2)u_3 \\
 & + T\rho(Tu_4, x)\rho(Tu_2, Tu_3)u_1 + T\rho(Tu_4, x)\rho(Tu_3, Tu_1)u_2 + T\rho(Tu_4, x)\Theta(Tu_1, Tu_2, Tu_3) \\
 & + T\Theta(x, [Tu_1, Tu_2, Tu_3]_{\mathfrak{g}}, Tu_4) - [Tu_3, [Tu_1, Tu_2, Tu_4]_{\mathfrak{g}}, x]_{\mathfrak{g}} - T\rho(Tu_3, x)\rho(Tu_1, Tu_2)u_4 \\
 & - T\rho(Tu_3, x)\rho(Tu_2, Tu_4)u_1 - T\rho(Tu_3, x)\rho(Tu_4, Tu_1)u_2 - T\rho(Tu_3, x)\Theta(Tu_1, Tu_2, Tu_4) \\
 & + T(\rho([Tu_1, Tu_2, Tu_4]_{\mathfrak{g}}, x)u_3 + T\Theta(x, Tu_3, [Tu_1, Tu_2, Tu_4]_{\mathfrak{g}}) \\
 & - [Tu_3, Tu_4, [Tu_1, Tu_2, x]_{\mathfrak{g}}]_{\mathfrak{g}} - [Tu_3, Tu_4, T\rho(Tu_1, x)u_2]_{\mathfrak{g}} + [Tu_3, Tu_4, T\rho(Tu_2, x)u_1]_{\mathfrak{g}} \\
 & + [Tu_3, Tu_4, T\Theta(x, Tu_1, Tu_2)]_{\mathfrak{g}} - T\rho(Tu_3, [Tu_1, Tu_2, x]_{\mathfrak{g}})u_4 - T\rho(Tu_3, T\rho(Tu_1, x)u_2)u_4 \\
 & + T\rho(Tu_3, T\rho(Tu_2, x)u_1)u_4 + T\rho(Tu_3, T\Theta(x, Tu_1, Tu_2))u_4 + T\rho(Tu_4, [Tu_1, Tu_2, x]_{\mathfrak{g}})u_3 \\
 & + T\rho(Tu_4, T\rho(Tu_1, x)u_2)u_3 - T\rho(Tu_4, T\rho(Tu_2, x)u_1)u_3 - T\rho(Tu_4, T\Theta(x, Tu_1, Tu_2))u_3 \\
 & + T\Theta(T\rho(Tu_1, x)u_2, Tu_3, Tu_4) - T\Theta(T\rho(Tu_2, x)u_1, Tu_3, Tu_4) \\
 & - T\Theta(T\Theta(x, Tu_1, Tu_2), Tu_3, Tu_4) + T\Theta([Tu_1, Tu_2, x]_{\mathfrak{g}}, Tu_3, Tu_4) \\
 (1)+(3)+(8) & = -T(\Theta(Tu_1, Tu_2, [Tu_3, Tu_4, x]_{\mathfrak{g}}) + \rho(Tu_1, Tu_2)\Theta(Tu_3, Tu_4, x) \\
 & - \rho(Tu_4, x)\Theta(Tu_1, Tu_2, Tu_3) - \Theta([Tu_1, Tu_2, Tu_3]_{\mathfrak{g}}, Tu_4, x) \\
 & - \rho(x, Tu_3)\Theta(Tu_1, Tu_2, Tu_4) - \Theta(Tu_3, [Tu_1, Tu_2, Tu_4]_{\mathfrak{g}}, x) \\
 & - \rho(Tu_3, Tu_4)\Theta(Tu_1, Tu_2, x) - \Theta(Tu_3, Tu_4, [Tu_1, Tu_2, x]_{\mathfrak{g}})) \\
 (6) & = -T((\partial\Theta)(Tu_1, Tu_2, Tu_3, Tu_4, x)) = 0.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & \rho_{\Theta}([u_1, u_2, u_3]_T, u_4)x - \rho_{\Theta}(u_1, u_2)\rho_{\Theta}(u_3, u_4)x \\
 & - \rho_{\Theta}(u_2, u_3)\rho_{\Theta}(u_1, u_4)x - \rho_{\Theta}(u_3, u_1)\rho_{\Theta}(u_2, u_4)x = 0.
 \end{aligned}$$

Hence the result follows. \square

Let $\partial_{\Theta} : \mathfrak{C}_{3Lie}^n(V; \mathfrak{g}) \rightarrow \mathfrak{C}_{3Lie}^{n+1}(V; \mathfrak{g})$, ($n \geq 1$) be the corresponding coboundary operator of the 3-Lie algebra $(V, [\cdot, \cdot, \cdot]_T)$ with coefficients in the representation $(\mathfrak{g}, \rho_{\Theta})$. More precisely, $\partial_{\Theta} : \mathfrak{C}_{3Lie}^n(V; \mathfrak{g}) \rightarrow \mathfrak{C}_{3Lie}^{n+1}(V; \mathfrak{g})$ is given by

$$\begin{aligned}
 & (\partial_{\Theta} f)(\mathfrak{u}_1, \dots, \mathfrak{u}_n, u_{n+1}) \\
 & = \sum_{1 \leq j < k \leq n} (-1)^j f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_{k-1}, [u_j, v_j, u_k]_T \wedge v_k + u_k \wedge [u_j, v_j, v_k]_T, \mathfrak{u}_{k+1}, \dots, \mathfrak{u}_n, u_{n+1}) \\
 & + \sum_{j=1}^n (-1)^j f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, [u_j, v_j, u_{n+1}]_T) + \sum_{j=1}^n (-1)^{j+1} \rho_{\Theta}(u_j, v_j) f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, u_{n+1}) \\
 & + (-1)^{n+1} (\rho_{\Theta}(v_n, u_{n+1}) f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, u_n) + \rho_{\Theta}(u_{n+1}, u_n) f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n)), \tag{31}
 \end{aligned}$$

for all $\mathfrak{u}_i = u_i \wedge v_i \in \wedge^2 V$, $i = 1, 2, \dots, n$ and $u_{n+1} \in V$. It is obvious that $f \in \mathfrak{C}_{3Lie}^1(V; \mathfrak{g})$ is closed if and only if

$$\begin{aligned}
 & [Tu, Tv, f(w)]_{\mathfrak{g}} + [f(u), Tv, Tw]_{\mathfrak{g}} + [Tu, f(v), Tw]_{\mathfrak{g}} \\
 & - f(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \Theta(Tu, Tv, Tw)) \\
 & - T(\rho(Tv, f(w))u + \rho(f(w), Tu)v + \Theta(f(w), Tu, Tv)) \\
 & - T(\rho(Tw, f(u))v + \rho(f(u), Tv)w + \Theta(f(u), Tv, Tw)) \\
 & - T(\rho(Tu, f(v))w + \rho(f(v), Tw)u + \Theta(f(v), Tw, Tu)) = 0.
 \end{aligned}$$

For all $\mathfrak{X} \in \mathfrak{g} \wedge \mathfrak{g}$, we define $\delta(\mathfrak{X}) : V \rightarrow \mathfrak{g}$ by

$$\delta(\mathfrak{X})(v) = T(\rho(\mathfrak{X})v + \Theta(\mathfrak{X}, Tv)) - [\mathfrak{X}, Tv]_{\mathfrak{g}}, \quad \forall v \in V. \tag{32}$$

Proposition 4.6. *Let T be a Θ -twisted O -operator on a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ) . Then $\delta(\mathfrak{X})$ is a 1-cocycle on the 3-Lie algebra $(V, [\cdot, \cdot, \cdot]_T)$ with coefficients in $(\mathfrak{g}, \rho_{\Theta})$.*

Proof. For all $u, v, w \in V$, we have

$$\begin{aligned} & (\partial_{\Theta}\delta(\mathfrak{X}))(u, v, w) \\ &= [Tu, Tv, \delta(\mathfrak{X})(w)]_{\mathfrak{g}} + [\delta(\mathfrak{X})(u), Tv, Tw]_{\mathfrak{g}} + [Tu, \delta(\mathfrak{X})(v), Tw]_{\mathfrak{g}} \\ & \quad - \delta(\mathfrak{X})(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \Theta(Tu, Tv, Tw)) \\ & \quad - T(\rho(Tv, \delta(\mathfrak{X})(w))u + \rho(\delta(\mathfrak{X})(w), Tu)v + \Theta(\delta(\mathfrak{X})(w), Tu, Tv)) \\ & \quad - T(\rho(Tw, \delta(\mathfrak{X})(u))v + \rho(\delta(\mathfrak{X})(u), Tv)w + \Theta(\delta(\mathfrak{X})(u), Tv, Tw)) \\ & \quad - T(\rho(Tu, \delta(\mathfrak{X})(v))w + \rho(\delta(\mathfrak{X})(v), Tw)u + \Theta(\delta(\mathfrak{X})(v), Tw, Tu)) \\ &= [Tu, Tv, T\rho(\mathfrak{X})w]_{\mathfrak{g}} + [Tu, Tv, T\Theta(\mathfrak{X}, w)]_{\mathfrak{g}} - [Tu, Tv, [\mathfrak{X}, Tw]_{\mathfrak{g}}]_{\mathfrak{g}} \\ & \quad + [T\rho(\mathfrak{X})u, Tv, Tw]_{\mathfrak{g}} + [T\Theta(\mathfrak{X}, Tu), Tv, Tw]_{\mathfrak{g}} - [[\mathfrak{X}, Tu]_{\mathfrak{g}}, Tv, Tw]_{\mathfrak{g}} \\ & \quad + [Tu, T\rho(\mathfrak{X})v, Tw]_{\mathfrak{g}} + [Tu, T\Theta(\mathfrak{X}, Tv), Tw]_{\mathfrak{g}} - [Tu, [\mathfrak{X}, Tv]_{\mathfrak{g}}, Tw]_{\mathfrak{g}} \\ & \quad - T\rho(\mathfrak{X})(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \Theta(Tu, Tv, Tw)) \\ & \quad - T\Theta(\mathfrak{X}, [Tu, Tv, Tw]_{\mathfrak{g}}) + [\mathfrak{X}, [Tu, Tv, Tw]_{\mathfrak{g}}]_{\mathfrak{g}} \\ & \quad - T(\rho(Tv, T\rho(\mathfrak{X})w)u + \rho(Tv, T\Theta(\mathfrak{X}, Tw))u - \rho(Tv, [\mathfrak{X}, Tw]_{\mathfrak{g}})u \\ & \quad + \rho(T\rho(\mathfrak{X})w, Tu)v + \rho(T\Theta(\mathfrak{X}, Tw), Tu)v - \rho([\mathfrak{X}, Tw]_{\mathfrak{g}}, Tu)v \\ & \quad + \Theta(T\rho(\mathfrak{X})w, Tu, Tv) + \Theta(T\Theta(\mathfrak{X}, Tw), Tu, Tv) - \Theta([\mathfrak{X}, Tw]_{\mathfrak{g}}, Tu, Tv)) \\ & \quad - T(\rho(Tw, T\rho(\mathfrak{X})u)v + \rho(Tw, T\Theta(\mathfrak{X}, Tu))v - \rho(Tw, [\mathfrak{X}, Tu]_{\mathfrak{g}})v \\ & \quad + \rho(T\rho(\mathfrak{X})u, Tv)w + \rho(T\Theta(\mathfrak{X}, Tu), Tv)w - \rho([\mathfrak{X}, Tu]_{\mathfrak{g}}, Tv)w \\ & \quad + \Theta(T\rho(\mathfrak{X})u, Tv, Tw) + \Theta(T\Theta(\mathfrak{X}, Tu), Tv, Tw) - \Theta([\mathfrak{X}, Tu]_{\mathfrak{g}}, Tv, Tw)) \\ & \quad - T(\rho(Tu, T\rho(\mathfrak{X})v)w + \rho(Tu, T\Theta(\mathfrak{X}, Tv))w - \rho(Tu, [\mathfrak{X}, Tv]_{\mathfrak{g}})w \\ & \quad + \rho(T\rho(\mathfrak{X})v, Tw)u + \rho(T\Theta(\mathfrak{X}, Tv), Tw)u - \rho([\mathfrak{X}, Tv]_{\mathfrak{g}}, Tw)u \\ & \quad + \Theta(T\rho(\mathfrak{X})v, Tw, Tu) + \Theta(T\Theta(\mathfrak{X}, Tv), Tw, Tu) - \Theta([\mathfrak{X}, Tv]_{\mathfrak{g}}, Tw, Tu)) \\ & \stackrel{(8)+(1)+(3)}{=} -T(\Theta(\mathfrak{X}, [Tu, Tv, Tw]_{\mathfrak{g}}) + T\rho(\mathfrak{X})\Theta(Tu, Tv, Tw) \\ & \quad - T\rho(Tu, Tv)\Theta(\mathfrak{X}, Tw) - T\rho(Tv, Tw)\Theta(\mathfrak{X}, Tu) - T\rho(Tw, Tu)\Theta(\mathfrak{X}, Tv) \\ & \quad - T\Theta([\mathfrak{X}, Tw]_{\mathfrak{g}}, Tu, Tv) - T\Theta([\mathfrak{X}, Tu]_{\mathfrak{g}}, Tv, Tw) - T\Theta([\mathfrak{X}, Tv]_{\mathfrak{g}}, Tw, Tu)) \\ &= -T((\partial\Theta)(\mathfrak{X}, Tu, Tv, Tw)) = 0. \end{aligned}$$

Thus, we deduce that $\partial_{\Theta}\delta(\mathfrak{X}) = 0$. \square

Now, we give a cohomology of Θ -twisted O -operators on 3-Lie algebras.

Definition 4.3. *Let T be a Θ -twisted O -operator on a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ) . Define the set of n -cochains by*

$$\mathfrak{C}_{\Theta}^n(V; \mathfrak{g}) = \begin{cases} \mathfrak{C}_{3Lie}^n(V; \mathfrak{g}), & n \geq 1, \\ \mathfrak{g} \wedge \mathfrak{g}, & n = 0. \end{cases} \tag{33}$$

Define $D_\Theta : \mathfrak{C}_\Theta^n(V; \mathfrak{g}) \rightarrow \mathfrak{C}_\Theta^{n+1}(V; \mathfrak{g})$ by

$$D_\Theta = \begin{cases} \partial_\Theta, & n \geq 1, \\ \delta, & n = 0. \end{cases} \tag{34}$$

Denote the set of n -cocycles by $\mathfrak{Z}_\Theta^n(V; \mathfrak{g})$ and the set of n -coboundaries by $\mathfrak{B}_\Theta^n(V; \mathfrak{g})$. Denote by

$$\mathfrak{H}_\Theta^n(V; \mathfrak{g}) = \mathfrak{Z}_\Theta^n(V; \mathfrak{g}) / \mathfrak{B}_\Theta^n(V; \mathfrak{g}), \quad n \geq 0$$

the n^{th} cohomology group which will be taken to be the n^{th} cohomology group for the Θ -twisted \mathcal{O} -operator T .

Theorem 4.7. *Let T be a Θ -twisted \mathcal{O} -operator on a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\mathfrak{g})$ with respect to a representation (V, ρ) . Then we have*

$$d_T(f) = (-1)^{n-1} D_\Theta(f), \quad \forall f \in \text{Hom}(\underbrace{\wedge^2 V \otimes \cdots \otimes \wedge^2 V}_{n-1} \wedge V, \mathfrak{g}), \quad n = 1, 2, \dots \tag{35}$$

Proof. According to [40, Theorem 4.5], we have

$$\begin{aligned} & \frac{1}{2} l_3(T, T, f)(\mathfrak{u}_1, \dots, \mathfrak{u}_n, u_{n+1}) \\ &= (-1)^{n-1} \left\{ (-1)^{n+1} ([Tv_n, Tu_{n+1}, f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, u_n)]_\mathfrak{g} - T\rho(Tu_{n+1}, f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, u_n))v_n \right. \\ & \quad - T\rho(f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, u_n), Tv_n)u_{n+1}) \\ & \quad + (-1)^{n+1} ([Tu_{n+1}, f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n)]_\mathfrak{g} - T\rho(Tu_n, f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n))u_{n+1} \\ & \quad - T\rho(f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n), Tu_{n+1})u_n) + \sum_{j=1}^n (-1)^{j+1} ([Tu_j, Tv_j, f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, u_{n+1})]_\mathfrak{g} \\ & \quad - T\rho(Tv_j, f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, u_{n+1}))u_j - T\rho(f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, u_{n+1}), Tu_j)v_j) \\ & \quad + \sum_{1 \leq j < k \leq n} (-1)^j f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_{k-1}, (\rho(Tu_j, Tv_j)u_k + \rho(Tv_j, Tu_k)u_j + \rho(Tu_k, Tu_j)v_j) \wedge v_k \\ & \quad + u_k \wedge (\rho(Tu_j, Tv_j)v_k + \rho(Tv_j, Tv_k)u_j + \rho(Tv_k, Tu_j)v_j), \mathfrak{u}_{k+1}, \dots, \mathfrak{u}_n, u_{n+1}) \\ & \quad \left. + \sum_{j=1}^n (-1)^j f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, \rho(Tu_j, Tv_j)u_{n+1} + \rho(Tv_j, Tu_{n+1})u_j + \rho(Tu_{n+1}, Tu_j)v_j) \right\}. \end{aligned}$$

Here we observe that

$$\begin{aligned} & l_4(T, T, T, f)(\mathfrak{u}_1, \dots, \mathfrak{u}_n, u_{n+1}) \\ &= [\Theta, T]_{3\text{-Lie}}, T]_{3\text{-Lie}}, T]_{3\text{-Lie}}, f]_{3\text{-Lie}}(\mathfrak{u}_1, \dots, \mathfrak{u}_n, u_{n+1}) \\ &= [\Theta, T]_{3\text{-Lie}}, T]_{3\text{-Lie}}, T]_{3\text{-Lie}}(f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, u_n) \wedge v_n, u_{n+1}) \\ & \quad + [\Theta, T]_{3\text{-Lie}}, T]_{3\text{-Lie}}, T]_{3\text{-Lie}}(u_n \wedge f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n), u_{n+1}) \\ & \quad + \sum_{j=1}^n (-1)^{n-1} (-1)^{j-1} [\Theta, T]_{3\text{-Lie}}, T]_{3\text{-Lie}}, T]_{3\text{-Lie}}(\mathfrak{u}_j, f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, u_{n+1})) \\ & \quad - (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{j=1}^k (-1)^{j+1} \end{aligned}$$

$$\begin{aligned}
 & f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, [\Theta, T]_{3-Lie, T}]_{3-Lie, T}]_{3-Lie}(\mathfrak{u}_j, u_{k+1}) \wedge v_{k+1}, \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, u_{n+1}) \\
 & - (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{j=1}^k (-1)^{j+1} \\
 & f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, u_{k+1} \wedge [\Theta, T]_{3-Lie, T}]_{3-Lie, T}]_{3-Lie}(\mathfrak{u}_j, v_{k+1}), \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, u_{n+1}) \\
 & - (-1)^{n-1} \sum_{j=1}^n (-1)^{j+1} f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_i, \dots, \mathfrak{u}_n, [\Theta, T]_{3-Lie, T}]_{3-Lie, T}]_{3-Lie}(\mathfrak{u}_j, u_{n+1})) \\
 = & (-1)^{n-1} 6 \left\{ (-1)^{n+1} \left(-T\Theta(f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, u_n), Tv_n, Tu_{n+1}) - T\Theta(f(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n), Tu_{n+1}, Tu_n) \right) \right. \\
 & - \sum_{j=1}^n (-1)^{j+1} T\Theta(f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, u_{n+1}), Tu_j, Tv_j) \\
 & + \sum_{j=1}^n (-1)^j f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_n, \Theta(Tu_j, Tv_j, Tu_{n+1})) \\
 & \left. + \sum_{1 \leq j < k \leq n} (-1)^j f(\mathfrak{u}_1, \dots, \widehat{\mathfrak{u}}_j, \dots, \mathfrak{u}_{k-1}, \Theta(Tu_j, Tv_j, Tu_k) \wedge v_k + u_k \wedge \Theta(Tu_j, Tv_j, Tv_k), \mathfrak{u}_{k+1}, \dots, \mathfrak{u}_n, u_{n+1}) \right\}.
 \end{aligned}$$

Hence $d_T(f) = \frac{1}{2}l_3(T, T, f) + \frac{1}{6}l_4(T, T, T, f) = (-1)^{n-1}D_\Theta(f)$. The proof is finished. \square

5. Deformations of twisted \mathcal{O} -operators

In this section, we study infinitesimal and formal deformations of a Θ -twisted \mathcal{O} -operator. For deformations of Rota-Baxter and \mathcal{O} -operators, see [15, 16, 34, 41].

5.1. Infinitesimal deformations

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ be a 3-Lie algebra, (V, ρ) be a representation of \mathfrak{g} , and $\Theta \in \mathfrak{C}_{3Lie}^2(\mathfrak{g}; V)$ be a 2-cocycle in the Chevalley-Eilenberg cochain complex. Let $T : V \rightarrow \mathfrak{g}$ be a Θ -twisted \mathcal{O} -operator.

Definition 5.1. An infinitesimal deformation of T consists of a parametrized sum $T_t = T + tT_1$, for some $T_1 \in \text{Hom}(V, \mathfrak{g})$ such that T_t is a Θ -twisted \mathcal{O} -operator for all values of t . In this case, we say that T_1 generates an infinitesimal deformation of T .

Suppose that T_1 generates an infinitesimal deformation of T . Then we have

$$\begin{aligned}
 & [T_t u, T_t v, T_t w]_{\mathfrak{g}} \\
 & = T_t \left(\rho(T_t u, T_t v)w + \rho(T_t v, T_t w)u + \rho(T_t w, T_t u)v + \Theta(T_t u, T_t v, T_t w) \right),
 \end{aligned}$$

for $u, v, w \in V$. This is equivalent to the following conditions

$$\begin{aligned}
 & [Tu, Tv, T_1 w]_{\mathfrak{g}} + [Tu, T_1 v, Tw]_{\mathfrak{g}} + [T_1 u, Tv, Tw]_{\mathfrak{g}} \\
 & = T \left(\rho(Tu, T_1 v)w + \rho(T_1 u, Tv)w + \rho(Tv, T_1 w)u + \rho(T_1 v, Tw)u \right. \\
 & + \rho(Tw, T_1 u)v + \rho(T_1 w, Tu)v + \Theta(Tu, Tv, T_1 w) + \Theta(Tu, T_1 v, Tw) \\
 & \left. + \Theta(T_1 u, Tv, Tw) \right) \\
 & + T_1 \left(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \Theta(Tu, Tv, Tw) \right),
 \end{aligned} \tag{36}$$

$$\begin{aligned}
 & [Tu, T_1v, T_1w]_{\mathfrak{g}} + [T_1u, Tv, T_1w]_{\mathfrak{g}} + [T_1u, T_1v, Tw]_{\mathfrak{g}} \\
 & = T(\rho(T_1u, T_1v)w + \rho(T_1v, T_1w)u + \rho(T_1w, T_1u)v \\
 & + \Theta(Tu, T_1v, T_1w) + \Theta(T_1u, Tv, T_1w) + \Theta(T_1u, T_1v, Tw)) \\
 & + T_1(\rho(Tu, T_1v)w + \rho(T_1u, Tv)w + \rho(Tv, T_1w)u + \rho(T_1v, Tw)u \\
 & + \rho(Tw, T_1u)v + \rho(T_1w, Tu)v + \Theta(Tu, Tv, T_1w) + \Theta(Tu, T_1v, Tw) \\
 & + \Theta(T_1u, Tv, Tw)), \tag{37}
 \end{aligned}$$

$$\begin{aligned}
 & [T_1u, T_1v, T_1w]_{\mathfrak{g}} = T(\Theta(T_1u, T_1v, T_1w)) \\
 & + T_1(\rho(T_1u, T_1v)w + \rho(T_1v, T_1w)u + \rho(T_1w, T_1u)v \\
 & + \Theta(Tu, T_1v, T_1w) + \Theta(T_1u, Tv, T_1w) + \Theta(T_1u, T_1v, Tw)) \tag{38}
 \end{aligned}$$

and

$$T_1(\Theta(T_1u, T_1v, T_1w)) = 0. \tag{39}$$

Note that the identity (36) implies that T_1 is a 1-cocycle with respect to the cohomology of T . Hence, T_1 defines a cohomology class in $\mathcal{H}_{\Theta}^1(V; \mathfrak{g})$.

Definition 5.2. Two infinitesimal deformations $T_t = T + tT_1$ and $T'_t = T + tT'_1$ of a Θ -twisted \mathcal{O} -operator T are said to be equivalent if there exists an element $\mathfrak{X} \in \mathfrak{g} \wedge \mathfrak{g}$ such that the pair

$$(\phi_t = Id_{\mathfrak{g}} + t[\mathfrak{X}, -]_{\mathfrak{g}}, \psi_t = Id_V + t(\rho(\mathfrak{X})(-) + \Theta(\mathfrak{X}, T-))) \tag{40}$$

defines a morphism of Θ -twisted \mathcal{O} -operators from T_t to T'_t .

An infinitesimal deformation $T_t = T + tT_1$ of a Θ -twisted \mathcal{O} -operator is said to be trivial if T_t is equivalent to $T'_t = T$.

The condition that $\phi_t = Id_{\mathfrak{g}} + t[\mathfrak{X}, -]_{\mathfrak{g}}$ is a 3-Lie algebra morphism of $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ is equivalent to

$$\begin{cases} [z_1, [\mathfrak{X}, z_2]_{\mathfrak{g}}, [\mathfrak{X}, z_3]_{\mathfrak{g}}]_{\mathfrak{g}} + [[\mathfrak{X}, z_1]_{\mathfrak{g}}, z_2, [\mathfrak{X}, z_3]_{\mathfrak{g}}]_{\mathfrak{g}} \\ + [[\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}}, z_3]_{\mathfrak{g}} = 0, \\ [[\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}}, [\mathfrak{X}, z_3]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \text{ for all } z_1, z_2, z_3 \in \mathfrak{g}. \end{cases} \tag{41}$$

The condition $\psi_t(\rho(z_1, z_2)u) = \rho(\phi_t(z_1), \phi_t(z_2))\psi_t(u)$ implies that

$$\begin{cases} \Theta(\mathfrak{X}, T\rho(z_1, z_2)u) = \rho(z_1, z_2)\Theta(\mathfrak{X}, Tu), \\ (\rho(z_1, [\mathfrak{X}, z_2]_{\mathfrak{g}}) + \rho([\mathfrak{X}, z_1]_{\mathfrak{g}}, z_2))(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)) \\ + \rho([\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}})u = 0, \\ \rho([\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}})(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)) = 0, \end{cases} \tag{42}$$

Finally, conditions $\psi_t \circ \Theta = \Theta \circ (\phi_t \otimes \phi_t \otimes \phi_t)$ and $\phi_t \circ T_t = T'_t \circ \psi_t$ are respectively equivalent to

$$\begin{cases} \rho(\mathfrak{X})\Theta(z_1, z_2, z_3) + \Theta(\mathfrak{X}, T\Theta(z_1, z_2, z_3)) = \Theta([\mathfrak{X}, z_1]_{\mathfrak{g}}, z_2, z_3) \\ + \Theta(z_1, [\mathfrak{X}, z_2]_{\mathfrak{g}}, z_3) + \Theta(z_1, z_2, [\mathfrak{X}, z_3]_{\mathfrak{g}}), \\ \Theta(z_1, [\mathfrak{X}, z_2]_{\mathfrak{g}}, [\mathfrak{X}, z_3]_{\mathfrak{g}}) + \Theta([\mathfrak{X}, z_1]_{\mathfrak{g}}, z_2, [\mathfrak{X}, z_3]_{\mathfrak{g}}) \\ + \Theta([\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}}, z_3) = 0, \\ \Theta([\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}}, [\mathfrak{X}, z_3]_{\mathfrak{g}}) = 0, \end{cases} \tag{43}$$

$$\begin{cases} T_1u + [\mathfrak{X}, Tu]_{\mathfrak{g}} = T(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)) + T'_1u, \\ [\mathfrak{X}, T_1u]_{\mathfrak{g}} = T'_1(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)). \end{cases} \tag{44}$$

Note that the above identities hold for all $\mathfrak{X} \in \mathfrak{g} \wedge \mathfrak{g}$, $z_1, z_2, z_3 \in \mathfrak{g}$ and $u \in V$.

From the first condition of (44), we have

$$T_1u - T'_1u = T(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)) - [\mathfrak{X}, Tu]_{\mathfrak{g}} = D_{\Theta}(\mathfrak{X})(u).$$

Therefore, we get the following theorem.

Theorem 5.1. *Let $T_t = T + tT_1$ and $T'_t = T + tT'_1$ be two equivalent infinitesimal deformations of a Θ -twisted O -operator T . Then T_1 and T'_1 define the same cohomology class in $\mathcal{H}^1_{\Theta}(V; \mathfrak{g})$.*

5.2. Formal deformations

Now we consider a more general situation by using formal power series. Let \mathfrak{g} be a 3-Lie algebra, V be a \mathfrak{g} -module and Θ be a 2-cocycle in the Chevalley-Eilenberg cohomology of \mathfrak{g} with coefficients in V . Let $T : V \rightarrow \mathfrak{g}$ be a Θ -twisted O -operator.

Let $\mathbb{K}[[t]]$ be the power series ring in one variable t . For any \mathbb{K} -linear space V , denote by $V[[t]]$ the vector space of formal power series in t with coefficients in V . If $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ is a 3-Lie algebra over \mathbb{K} , then there is a 3-Lie algebra structure over the ring $\mathbb{K}[[t]]$ on $\mathfrak{g}[[t]]$ given by

$$\left[\sum_{i=0}^{+\infty} x_i t^i, \sum_{j=0}^{+\infty} y_j t^j, \sum_{k=0}^{+\infty} z_k t^k \right]_{\mathfrak{g}} = \sum_{s=0}^{+\infty} \sum_{i+j+k=s} [x_i, y_j, z_k]_{\mathfrak{g}} t^s, \quad \forall x_i, y_j, z_k \in \mathfrak{g}. \tag{45}$$

For any representation (V, ρ) of $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$, there is a natural representation of the 3-Lie algebra $\mathfrak{g}[[t]]$ on the $\mathbb{K}[[t]]$ -module $V[[t]]$, which is given by

$$\rho \left(\sum_{i=0}^{+\infty} x_i t^i, \sum_{j=0}^{+\infty} y_j t^j \right) \left(\sum_{k=0}^{+\infty} v_k t^k \right) = \sum_{s=0}^{+\infty} \sum_{i+j+k=s} \rho(x_i, y_j) v_k t^s, \quad \forall x_i, y_j \in \mathfrak{g}, v_k \in V. \tag{46}$$

Similarly, the 2-cocycle Θ can be extended to a 2-cocycle (denoted also by Θ) on the 3-Lie algebra $\mathfrak{g}[[t]]$ with coefficients in $V[[t]]$. Consider a power series

$$T_t = \sum_{i=0}^{+\infty} T_i t^i, \quad T_i \in Hom_{\mathbb{K}}(V; \mathfrak{g}), \tag{47}$$

that is, $T_t \in Hom_{\mathbb{K}}(V; \mathfrak{g})[[t]] = Hom_{\mathbb{K}}(V; \mathfrak{g}[[t]])$. Extend it to be a $\mathbb{K}[[t]]$ -module map from $V[[t]]$ to $\mathfrak{g}[[t]]$ which is still denoted by T_t .

Definition 5.3. *If $T_t = \sum_{i=0}^{+\infty} T_i t^i$ with $T_0 = T$ satisfies*

$$\begin{aligned} & [T_t u, T_t v, T_t w]_{\mathfrak{g}} \\ & = T_t \left(\rho(T_t u, T_t v) w + \rho(T_t v, T_t w) u + \rho(T_t w, T_t u) v + \Theta(T_t u, T_t v, T_t w) \right), \end{aligned} \tag{48}$$

we say that T_t is a formal deformation of the Θ -twisted O -operator T .

Recall that a formal deformation of a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ is a formal power series $\omega_t = \sum_{k=0}^{+\infty} \omega_k t^k$, where $\omega_k \in Hom(\wedge^3 \mathfrak{g}; \mathfrak{g})$ such that $\omega_0(x, y, z) = [x, y, z]_{\mathfrak{g}}$ for any $x, y, z \in \mathfrak{g}$ and ω_t defines a 3-Lie algebra structure over the ring $\mathbb{K}[[t]]$ on $\mathfrak{g}[[t]]$.

Based on the relationship between Θ -twisted O -operators and 3-Lie algebras, we have the following construction.

Proposition 5.2. Let $T_t = \sum_{i=0}^{+\infty} T_i t^i$ be a formal deformation of a Θ -twisted \mathcal{O} -operator T on the 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ) . Then $[\cdot, \cdot, \cdot]_{T_t}$ defined by

$$[u, v, w]_{T_t} = \sum_{s=0}^{+\infty} \left(\sum_{i+j=s} (\rho(T_i u, T_j v)w + \rho(T_i v, T_j w)u + \rho(T_i w, T_j u)v) + \sum_{i+j+k=s} \Theta(T_i u, T_j v, T_k w) \right) t^s$$

for all $u, v, w \in V$, is a formal deformation of the 3-Lie algebra $(V, [\cdot, \cdot, \cdot]_T)$ defined in (9).

By applying Eqs.(45)-(47) to expand Eq.(48) and collecting coefficients of t^s , we see that Eq.(48) is equivalent to the system of equations, for $s = 0, 1, 2, \dots$,

$$\begin{aligned} & \sum_{i+j+k=s} [T_i u, T_j v, T_k w]_{\mathfrak{g}} \\ &= \sum_{i+j+k=s} T_i (\rho(T_j u, T_k v)w + \rho(T_j v, T_k w)u + \rho(T_j w, T_k u)v) \\ &+ \sum_{i+j+k+m=s} T_i (\Theta(T_j u, T_k v, T_m w)). \end{aligned} \tag{49}$$

Note that (49) holds for $s = 0$ since $T_0 = T$ is a Θ -twisted \mathcal{O} -operator. For $s = 1$, we get

$$\begin{aligned} & [T_1 u, T_1 v, T_1 w]_{\mathfrak{g}} + [T_1 u, T_1 v, T_1 w]_{\mathfrak{g}} + [T_1 u, T_1 v, T_1 w]_{\mathfrak{g}} \\ &= T(\rho(T_1 u, T_1 v)w + \rho(T_1 v, T_1 w)u + \rho(T_1 w, T_1 u)v) \\ &+ \rho(T_1 w, T_1 u)v + \rho(T_1 v, T_1 u)v + \Theta(T_1 u, T_1 v, T_1 w) + \Theta(T_1 u, T_1 v, T_1 w) \\ &+ \Theta(T_1 u, T_1 v, T_1 w) \\ &+ T_1(\rho(T_1 u, T_1 v)w + \rho(T_1 v, T_1 w)u + \rho(T_1 w, T_1 u)v + \Theta(T_1 u, T_1 v, T_1 w)), \end{aligned}$$

which is exactly Eq. (36). This implies that $(D_{\Theta}(T_1))(u, v, w) = 0$. Hence the linear term T_1 is a 1-cocycle with respect to the cohomology of T . It is called the infinitesimal of the deformation T_t .

In the sequel, we discuss equivalent formal deformations.

Definition 5.4. Let $T_t = \sum_{i=0}^{+\infty} T_i t^i$ and $T'_t = \sum_{i=0}^{+\infty} T'_i t^i$ be two formal deformations of a Θ -twisted \mathcal{O} -operator $T = T_0 = T'_0$ on a 3-Lie algebra \mathfrak{g} with respect to a representation (V, ρ) . They are said to be equivalent if there exist an element $\mathfrak{X} \in \mathfrak{g} \wedge \mathfrak{g}$, $\phi_i \in \mathfrak{gl}(\mathfrak{g})$ and $\psi_i \in \mathfrak{gl}(V)$, $i \geq 2$, such that the pair

$$(\phi_t = Id_{\mathfrak{g}} + t[\mathfrak{X}, -]_{\mathfrak{g}} + \sum_{i=2}^{+\infty} \phi_i t^i, \psi_t = Id_V + t(\rho(\mathfrak{X})(-) + \Theta(\mathfrak{X}, T-)) + \sum_{i=2}^{+\infty} \psi_i t^i), \tag{50}$$

is a morphism of Θ -twisted \mathcal{O} -operators from T_t to T'_t .

Theorem 5.3. If two formal deformations of a Θ -twisted \mathcal{O} -operator T on a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ) are equivalent, then their infinitesimals are in the same cohomology class.

Proof. Let (ϕ_t, ψ_t) be the two maps defined by Eq.(50) which gives an equivalence between two deformations

$T_t = \sum_{i=0}^{+\infty} T_i t^i$ and $T'_t = \sum_{i=0}^{+\infty} T'_i t^i$ of a Θ -twisted \mathcal{O} -operator T . By $\phi_t \circ T_t = T'_t \circ \psi_t$, we have

$$T_1 u = T'_1 u + T(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)) - [\mathfrak{X}, Tu]_{\mathfrak{g}}$$

$$= T_1' u + (D_{\Theta}(\mathfrak{X}))(u), \quad \forall u \in V,$$

which implies that T_1 and T_1' are in the same cohomology class. \square

References

- [1] V. Abramov, *Super 3-Lie algebras induced by super Lie algebras* Adv. Appl. Clifford Algebras, **27** (1) (2017), 9–16.
- [2] V. Abramov, *Weil algebra, 3-Lie algebra and B.R.S. algebra*, In: Silvestrov, S., Malyarenko, A., Rancic, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics & Statistics, Vol **317**, Springer, Ch 1 (2020), 1–12.
- [3] V. Abramov and P. Lätt, *Classification of low dimensional 3-Lie superalgebras*, In: Silvestrov S., Rancic M. (eds), Engineering Mathematics II, Springer Proceedings in Mathematics & Statistics, Vol **179**, Springer, 1–12.
- [4] V. Abramov and P. Lätt, *Induced 3-Lie algebras, superalgebras and induced representations*, Proc. Est. Acad. Sci. **69** (2) (2020), 116–133.
- [5] J. Arnlind, A. Kitouni, A. Makhoulouf and S. Silvestrov, *Structure and cohomology of 3-Lie algebras induced by Lie algebras*, In Algebra, Geometry and Mathematical Physics (2014) (pp. 123–144). Springer Proceedings in Mathematics & Statistics, **85**, Springer, Berlin, Heidelberg.
- [6] J. Arnlind, A. Makhoulouf and S. Silvestrov, *Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras*, J. Math. Phys. **51**(4) (2010), 043515.
- [7] J. Arnlind, A. Makhoulouf and S. Silvestrov, *Construction of n-Lie algebras and n-ary Hom-Nambu-Lie algebras*, J. Math. Phys. **52** (2011), no. 12, 123502.
- [8] H. Awata, M. Li, D. Minic and T. Yoneya, *On the quantization of Nambu brackets*, J. High Energy Phys. **2**, Paper 13, 17 pp, (2001).
- [9] R. Bai, C. Bai and J. Wang, *Realizations of 3-Lie algebras*, J. Math. Phys. **51** (2010), 063505.
- [10] R. Bai, L. Guo, J. Li and Y. Wu, *Rota-Baxter 3-Lie algebras*, J. Math. Phys. **54** (2013) 064504.
- [11] C. Bai, L. Guo and Y. Sheng, *Bialgebras, the classical Yang-Baxter equation and Manin triples for 3-Lie algebras*, Adv. Theor. Math. Phys. **23** (2019) 27–74.
- [12] R. Bai, G. Song and Y. Zhang, *On classification of n-Lie algebras*, Front. Math. China **6**(4) (2011), 581–606.
- [13] L. Cai and Y. Sheng, *Hom-big brackets: theory and applications*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), 014, 18 pages.
- [14] J. M. Casas, J. L. Loday and T. Pirashvili, *Leibniz n-algebras*, Forum Math. **214** (2002), 189–207.
- [15] T. Chtioui, S. Mabrouk, and A. Makhoulouf, *Cohomology and deformations of O-operators on Hom-associative algebras*, J. Algebra **604** (2022), 727–759.
- [16] A. Das, *Deformations of associative Rota-Baxter operators*, J. Algebra **560** (2020): 144–180.
- [17] A. Das, *Twisted Rota-Baxter operators and Reynolds operators on Lie algebras and NS-Lie algebras*, J. Math. Phys. **62** (2021), no. 9, Paper No. 091701, 19 pp.
- [18] A. Das, *Cohomology and deformations of twisted Rota-Baxter operators and NS-algebras*, J. Homotopy Relat. Struct. **17** (2022), no. 2, 233–262.
- [19] A. Das and S. Guo, *Twisted relative Rota-Baxter operators on Leibniz algebras and NS-Leibniz algebra*, e-Print arXiv: 2102.09752 (2021).
- [20] J. Figueroa-O'Farrill, *Deformations of 3-algebras*, J. Math. Phys. **50** (2009) 113514
- [21] V. T. Filippov, *n-Lie algebras*, Sibirsk. Mat. Zh. **26** (1985), 126–140.
- [22] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math. (2) **79** (1964) 59–103.
- [23] E. Getzler, *Lie theory for nilpotent L_{∞} -algebras*, Ann. Math. (2) **170** (2009), 271–301.
- [24] S. Hou and Y. Sheng, *Generalized Reynolds operators on 3-Lie algebras and NS-3-Lie algebras*, Int. J. Geom. Methods Mod. Phys. **18** (2021), no. 14, Paper No. 2150223, 29 pp.
- [25] Sh. M. Kasymov, *On a theory of n-Lie algebras*, Algebra Logika **26** (1987), 277–297.
- [26] A. Kitouni, *On $(n + 1)$ -Lie induced by n-Lie algebras*, PhD Thesis University of Haute Alsace, Mulhouse, 2015.
- [27] C. Klimeik and T. Strobl, *WZW-Poisson manifolds*, J. Geom. Phys. **43** (2002), 341–344.
- [28] T. Lada and J. Stasheff, *Introduction to sh Lie algebras for physicists*, Internat. J. Theoret. Phys. **32** (1993), 1087–1103.
- [29] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. Algebra **23** (1995), 2147–2161.
- [30] A. Lazarev, Y. Sheng and R. Tang, *Deformations and homotopy theory of relative Rota-Baxter Lie algebras*, Comm. Math. Phys. **383** (2021), 595–631.
- [31] P. Leroux, *Construction of Nijenhuis operators and dendriform trialgebras*, Int. J. Math. Math. Sci. **49** (2004), 2595–2615.
- [32] Y. Nambu, *Generalized Hamiltonian dynamics*, Phys. Rev. D **7** (1973), 2405–2412.
- [33] A. Nijenhuis and R. Richardson, *Commutative algebra cohomology and deformations of Lie and associative algebras*, J. Algebra **9** (1968), 42–105.
- [34] R. Tang, C. Bai, L. Guo and Y. Sheng, *Deformations and their controlling cohomologies of O-operators*, Comm. Math. Phys. **368** (2) (2019), 665–700.
- [35] M. Rotkiewicz, *Cohomology ring of n-Lie algebras*, Extracta Math. **20** (2005), 219–232.
- [36] P. Severa and A. Weinstein, *Poisson geometry with a 3-form back? ground*, Progr. Theoret. Phys. Suppl. **144** (2001), 145–154.
- [37] M. Schlessinger and J. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. Pure Appl. Algebra **38** (1985), 313–322.
- [38] J. Stasheff, *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras*, Quantum groups (Leningrad, 1990), 120–137, Lecture Notes in Math., (1510), Springer, Berlin, (1992).
- [39] L. Takhtajan, *Higher order analog of Chevalley-Eilenberg complex and deformation theory of n-algebras*, St. Petersburg Math. J. **6** (1995), 429–438.

- [40] R. Tang, S. Hou and Y. Sheng, *Lie 3-algebras and deformations of relative Rota-Baxter operators on 3-Lie algebras*, J. Algebra **567** (2021), 37–62.
- [41] R. Tang, Y. Sheng, and Y. Zhou, *Deformations of relative Rota-Baxter operators on Leibniz algebras*, Int. J. Geom. Methods Mod. Phys. **17.12** (2020): 2050174-1606.
- [42] K. Uchino, *Quantum analogy of Poisson geometry, related dendriform algebras and Rota-Baxter operators*, Lett. Math. Phys. **85** (2-3) (2008), 91–109.
- [43] K. Uchino, *Twisting on associative algebras and Rota-Baxter type operators*, J. Noncommut. Geom. **4** (2010), 349–379.
- [44] Th. Voronov, *Higher derived brackets and homotopy algebras*, J. Pure Appl. Algebra **202** (2005), 133–153.